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Nonoscillation Theorems in Convex Sets

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1. INTRODUCTION

Let $p(z)$ be a regular function in a simply connected domain D which does not contain infinity. The differential equation

$$w''(z) + p(z)\, w(z) = 0 \tag{1}$$

is called *disconjugate* in D if no nontrivial solution of (1) has two zeros in D.

Several sufficient conditions for the disconjugacy of Eq. (1) are known. We mention here two of them. The first is the well-known theorem of Nehari [7]: If

$$|p(z)| \leqslant \pi^2/4, \qquad |z| < 1, \tag{2}$$

then Eq. (1) is disconjugate in $|z| < 1$.

The second criterion was proved by London [6]: If

$$\iint_{|z|<R} |p(z)|\, dx\, dy \leqslant \pi, \tag{3}$$

then Eq. (1) is disconjugate in $|z| < R$.

In the next two sections of this paper we consider Eq. (1) when $p(z)$ is regular in a convex set S. We obtain conditions similar to (2) and (3) which guarantee that no solution of Eq. (1) has n, $n \geqslant 2$, zeros in S.

The theorems about solutions of Eq. (1) have a function theoretic interpretation. Let $w_1(z)$ and $w_2(z)$ be linearly independent solutions of Eq. (1) and $f(z) = w_1(z)/w_2(z)$. The function $f(z)$ takes in n points of D the value b/a if and only if the nontrivial solution $w(z) = aw_1(z) - bw_2(z)$ of Eq. (1) vanishes in these n points. By the connection

$$p(z) = \tfrac{1}{2}\{f, z\},$$

where $\{f, z\}$ is the Schwarzian derivative of $f(z)$, the theorems of Sections 2 and 3 are equivalent to theorems about the valency of $f(z)$.

In the last section we consider the mth order differential equation

$$w^{(m)}(z) + p(z)\, w(z) = 0. \tag{4}$$

A sufficient condition for the solutions of Eq. (4) to have at most $n(m-1)$ zeros in a convex set S is obtained.

From now on, a solution will always mean a nontrivial solution.

2. The Area Integral

Let $w(z)$ be a solution of Eq. (1) in D. Assume that the bilinear transformation

$$\zeta = (az + b)/(cz + d) \qquad (ad \neq bc,\ \zeta = \xi + i\eta)$$

is regular in D and maps D onto a simply connected domain D_1. Then the function

$$w_1(\zeta) = (a - c\zeta)\, w(z(\zeta)) \tag{5}$$

satisfies in D_1 the differential equation

$$w_1''(\zeta) + p_1(\zeta)\, w_1(\zeta) = 0, \tag{6}$$

where

$$p_1(\zeta) = p(z(\zeta))(dz/d\zeta)^2.$$

The solutions $w(z)$ and $w_1(\zeta)$ vanish at corresponding points of D and D_1 respectively. Moreover,

$$\iint_D |\, p(z)|\, dx\, dy = \iint_{D_1} |\, p_1(\zeta)|\, d\xi\, d\eta.$$

This property is called by London [6] the *invariance of the area integral.*

The next lemma generalizes [6, Theorem 2].

LEMMA 1. *Let D be a domain bounded by two circular arcs and let α $(0 < \alpha < 2\pi)$ be the angle of intersection of these arcs. Let $p(z)$ be regular in $\bar{D}$. If there exists a solution of Eq.* (1) *which vanishes at the vertices of D, then*

$$\iint_D |\, p(z)|\, dx\, dy \geqslant \alpha.$$

Proof. We first assume that $\alpha < \pi$ and we prove that if there exists a solution $w(z)$ of Eq. (1) which vanishes at the vertices of D, then

$$\iint_D |\, p(z)|\, dx\, dy \geqslant \sin \alpha.$$

By the invariance of the area integral we may assume that the vertices of D are $z = -1, +1$, and one of the arcs of the boundary ∂D is the segment $[-1, 1]$. Hence, D is convex. Applying Grunsky's formula as in London's proof, we obtain for every $z' \in \bar{D}$

$$2Aw(z') = (z' - 1)(z' + 1) \iint_{\Delta} w''(z)\, dx\, dy,$$

where Δ is the triangle with vertices $-1, +1, z'$ and A is the area of Δ. Using Eq. (1), it follows that

$$2A \mid w(z')\mid \leqslant \mid z' - 1 \mid \mid z' + 1 \mid \iint_{\Delta} \mid p(z)\mid \cdot \mid w(z)\mid\, dx\, dy. \tag{7}$$

Let $\mid w(z^*)\mid = \max_{z \in \bar{D}} \mid w(z)\mid > 0$, $z^* \in D$. z^* belongs either to the circular arc of ∂D or to the open segment $(-1, 1)$. In the first case, when z^* belongs to the circular arc, the triangle with vertices $-1, +1, z^*$ has a positive area A,

$$A = \tfrac{1}{2} \mid z^* - 1 \mid \mid z^* + 1 \mid \sin \gamma,$$

where γ is the angle between the segments $[-1, z^*]$ and $[1, z^*]$. By using (7) for $z' = z^*$ we obtain

$$\mid w(z^*)\mid \leqslant \frac{\mid z^* - 1 \mid \mid z^* + 1 \mid}{2A} \mid w(z^*)\mid \iint_{\Delta} \mid p(z)\mid\, dx\, dy,$$

or

$$\iint_{\Delta} \mid p(z)\mid\, dx\, dy \geqslant \sin \gamma.$$

It is easily seen that $\sin \gamma = \sin \alpha$, where α is the angle of intersection of the boundary arcs of D. Hence,

$$\iint_{D} \mid p(z)\mid\, dx\, dy > \iint_{\Delta} \mid p(z)\mid\, dx\, dy \geqslant \sin \alpha.$$

In the second case, when $z^* \in (-1, 1)$, we apply the transformation

$$\zeta = \frac{z - ih}{ihz - 1},$$

where ih is the midpoint of the circular arc of ∂D. This transformation maps D onto itself and exchanges the arcs of ∂D. By (5) it follows that

$$w_1(\zeta) = (1 - ih\zeta)\, w(z(\zeta))$$

satisfies Eq. (6) in $D_1 \equiv D$.

We now show that $\max_{\zeta\in\bar{D}_1} |w_1(\zeta)|$ cannot be attained on $(-1, 1)$. Indeed, for every $\zeta \in \bar{D}_1$,

$$|w_1(\zeta)| = |1 - ih\zeta|\,|w(z(\zeta))| \leqslant h\,|\zeta - (-i/h)|\,|w(z^*)|.$$

The disc with radius $(1 + h^2)^{1/2}$ and center $-i/h$ contains $(-1, 1)$, but it does not contain the circular arc of ∂D_1. The point $\zeta^* = \zeta(z^*)$ belongs to the circular arc of ∂D_1, therefore for every $\zeta \in (-1, 1)$ we have

$$|\zeta - (-i/h)| < |\zeta^* - (-i/h)|,$$

and consequently,

$$|w_1(\zeta)| \leqslant h\,|\zeta - (-i/h)|\,|w(z^*)| < h\,|\zeta^* - (-i/h)|\,|w(z(\zeta^*))| = |w_1(\zeta^*)|.$$

Hence $\max |w_1(\zeta)|$ is obtained only on the circular arc of ∂D_1. By the invariance of the area integral and by the proof of the first case it follows that

$$\iint_D |p(z)|\,dx\,dy = \iint_{D_1} |p_1(\zeta)|\,d\xi\,d\eta > \sin\alpha.$$

We consider now the general case where α may be bigger than π. We call a domain bounded by two circular arcs a *lens*. We assume again that there is a solution of Eq. (1) which vanishes at the vertices of the lens D. We divide D into n lenses $D_1, \ldots, D_n$ with common vertices such that the angle of intersection of the boundary arcs of each one is $\alpha/n < \pi$. Then

$$\iint_D |p(z)|\,dx\,dy = \sum_{i=1}^{n} \iint_{D_i} |p(z)|\,dx\,dy > n\sin(\alpha/n).$$

When $n \to \infty$ we obtain the required result.

We give now a new proof to London's theorem [6, Theorem 1].

THEOREM 1. *Let $p(z)$ be regular in $|z| < R$. If*

$$\iint_{|z|<R} |p(z)|\,dx\,dy \leqslant \pi$$

then Eq. (1) *is disconjugate in* $|z| < R$.

Proof. Assume that there is a solution of Eq. (1), which has two zeros in $|z| < R$. There is a circle in $|z| < R$ which passes through the zeros. Denote the disc bounded by this circle by D. D is a lens with angle π and its vertices are zeros of a solution of Eq. (1). Since $p(z)$ is regular in $\bar{D}$, we have by Lemma 1,

$$\iint_{|z|<R} |p(z)| \, dx \, dy > \iint_D |p(z)| \, dx \, dy \geqslant \pi,$$

contradicting the assumption.

Remark. In the same way one can show that if $p(z)$ is regular in a half plane D and $\iint_D |p(z)| \, dx \, dy \leqslant \pi$, then Eq. (1) is disconjugate in D. This conclusion may be deduced also by conformal mapping of the half plane on a disc.

Similar methods may be used to derive disconjugacy criteria for Eq. (1) in other domains.

EXAMPLE 1. Let D be a cone in the complex plane and let $\alpha \leqslant \pi$, be the angle at its vertex. If $p(z)$ is regular in D and

$$\iint_D |p(z)| \, dx \, dy \leqslant \alpha,$$

then Eq. (1) is disconjugate in D.

EXAMPLE 2. Let D be a triangle with angles α, β, γ. If $p(z)$ is regular in D and

$$\iint_D |p(z)| \, dx \, dy \leqslant \min(\alpha, \beta, \gamma),$$

then Eq. (1) is disconjugate in D.

To prove both propositions we assume that there exists a solution of Eq. (1) which vanishes at two points, P and Q of D. It is easy to show by geometric arguments that there exist lenses with vertices P and Q such that the closure of the lens is included in the domain and the angle of intersection of the boundary arcs is α and $\min(\alpha, \beta, \gamma)$, respectively.

We turn now to the problem of obtaining disconjugacy criteria for Eq. (1) in an arbitrary bounded convex set. We require the following definition.

DEFINITION. Let S be a bounded convex planar set. The width of S in direction θ is the distance apart of the support lines perpendicular to this direction, and it is denoted by ω_θ. The least of these widths is called the ***minimal width of*** S and it is denoted by ω. [1]

THEOREM 2. *Let S be a bounded convex set of width ω and diameter d and let $p(z)$ be regular in S. If*

$$\iint_D |p(z)|\,dx\,dy \leqslant \arctan(\omega/d),$$

then Eq. (1) *is disconjugate in D.*

Proof. Assume that there is a solution which vanishes at two points, P and Q, of S. We shall construct in S a lens with vertices P and Q and angle $\arctan(\omega/d)$ at least.

Let l be the straight line through P and Q and let the perpendicular bisector of the segment $\overline{PQ}$ meet ∂S at R_1 and R_2. We denote the two support lines of S which are parallel to l by l_1 and l_2. ∂S and l_1 have at least one common point. Let F_1 be the projection of this point (or one of them) on l, and let ω_1 be the distance apart of l_1 and l. Similarly we define F_2 and ω_2 by applying l_2 (Fig. 1).

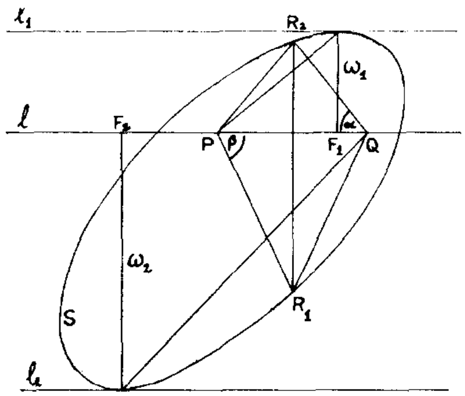

FIG. 1. Construction of a lens inside a convex set.

We now construct a lens with vertices P and Q which is included in the quadrangle PQR_1R_2 and has an angle $\alpha + \beta$ where

$$\alpha \geqslant \arctan(\omega_1/PF_1) \geqslant \arctan(\omega_1/d)$$
$$\beta \geqslant \arctan(\omega_2/QF_2) \geqslant \arctan(\omega_2/d).$$

Here PF_1, $QF_2 \leqslant d$, since they are projections of segments which are included in S. Since $\arctan(x) + \arctan(y) \geqslant \arctan(x + y)$ for $x, y \geqslant 0$,

$$\alpha + \beta \geqslant \arctan(\omega_1/d) + \arctan(\omega_2/d) \geqslant \arctan(\omega_1 + \omega_2)/d$$
$$\geqslant \arctan(\omega_\theta/d) \geqslant \arctan(\omega/d),$$

where ω_θ denotes the width of S in direction θ perpendicular to l. This lens satisfies the assumption of Lemma 1, therefore

$$\iint_S |p(z)|\, dx\, dy > \iint_{\text{lens}} |p(z)|\, dx\, dy \geqslant \arctan(\omega/d),$$

which contradicts the assumption of the theorem.

Remark. It is not necessary for the convex set S to be a domain or a closed domain.

Now we use the same method to obtain restrictions on the area integral which will ensure that no solution of Eq. (1) has n zeros.

THEOREM 3. *Let $p(z)$ be regular in $|z| < R$. If*

$$\iint_{|z|<R} |p(z)|\, dx\, dy \leqslant (\pi/6)n,$$

then no solution of Eq. (1) *has n zeros.*

Proof. Assume that there is a solution of Eq. (1) with n zeros $P_1, \ldots, P_n$ in $|z| < R$. We show now that it is possible to construct a set of disjoint lenses such that the closure of each lens is in $|z| < R$, their vertices are $P_1, \ldots, P_n$ and the sum of their angles is $(\pi/6)n$.

Let $P \in \{P_1, \ldots, P_n\}$. We denote the nearest of the remaining $n - 1$ points (or one of them) by P'. For short we denote $(P')'$ by P''. We want to draw a lens with vertices P and P' such that any two open lenses which have the vertex P (or P') in common will be disjoint. Two cases are to be considered.

I. $P'' \neq P$. In this case we construct two lenses with vertices P, P' and P', P'', respectively, each of angle $(\pi/6)$, such that the closure of each one is included in $|z| < R$. Since $P \neq P''$ we have by definition

$$d(P'', P') \leqslant d(P', P) \leqslant d(P, P'')$$

and therefore the angle $\sphericalangle PP'P'' \geqslant \pi/3$. Hence two disjoint lenses of angle $\pi/6$ may be drawn inside the angle $\sphericalangle PP'P''$. If there exist a point Q such that $Q' = P$ (or $Q \neq P'$ such that $Q' = P'$) then we may construct a pair of disjoint lenses with vertices P, P' and Q, P (or P, P' and Q, P') similarly.

II. $P'' = P$. In this case, we first draw a single lens with angle $\pi/3 = 2 \cdot (\pi/6)$ and vertices P $(=P'')$ and P'. If there exists a point $Q \neq P'$ such that $Q' = P$, we want to draw a lens of angle $\pi/6$, and vertices Q, P which is disjoint from the previous lens. By definition

$$d(P', P) \leqslant d(P, Q) \leqslant d(Q, P')$$

and therefore

$$\sphericalangle P'QP \leqslant \sphericalangle PP'Q \leqslant \sphericalangle QPP'.$$

If $\sphericalangle QPP' \geqslant \pi/2$, two disjoint lenses with vertices P, P' and P, Q and with angles $\pi/3$ and $\pi/6$, respectively, may be drawn inside this angle. If $\sphericalangle QPP' < \pi/2$, we replace these two lenses by three disjoint lenses such that the sum of their angles is $\pi/2 = 3 \cdot (\pi/6)$. Let the angles of the triangle $PP'Q$ be $\alpha \leqslant \beta \leqslant \gamma < \pi/2$. We construct inside this triangle three lenses whose vertices are at P, P' and Q; their angles are $\frac{1}{2}(\alpha + \beta - \gamma)$, $\frac{1}{2}(\beta + \gamma - \alpha)$, $\frac{1}{2}(\gamma + \alpha - \beta) > 0$, and the segments $\overline{PP'}$, $\overline{P'Q}$, $\overline{QP}$ are parts of their boundaries. These three lenses are disjoint because the sum of angles of two of them is, for example

$$\tfrac{1}{2}(\alpha + \beta - \gamma) + \tfrac{1}{2}(\beta + \gamma - \alpha) = \beta = \sphericalangle PQP'.$$

The sum of their three angles is of course $\pi/2$.

It is easy to show by using geometric argument that any two lenses (even those which do not have a common vertex) are disjoint. The closures of the lenses are in $|z| < R$, they do not cover this disc and the sum of their angles is $(\pi/6)n$. Hence, by Lemma 1,

$$\iint_{|z|<R} |p(z)|\, dx\, dy > \sum_{\text{lenses}} \iint |p(z)|\, dx\, dy \geqslant (\pi/6)n,$$

which contradicts the assumption.

For $2 \leqslant n \leqslant 6$ the result is trivial by Theorem 1.

THEOREM 4. *Let S be a bounded convex set in the complex plane with diameter d and width ω. If $p(z)$ is regular in S and*

$$\iint_S |p(z)|\, dx\, dy \leqslant \tfrac{1}{2} \arctan(\omega/d) \cdot n, \qquad n \geqslant 2,$$

then no solution of (1) *has n zeros in S.*

Proof. The proof of this theorem is similar to the proof of Theorem 3. Let $P_1, \ldots, P_n$ be zeros of a solution of Eq. (1) in S. We have shown in Theorem 2 that given any two points of S, there exists a lens of angle $\arctan(\omega/d)$ which is included in S and such that the two points are its vertices. Therefore it is possible to replace in the proof of Theorem 3 the angle $\pi/6$ by $\frac{1}{2}\arctan(\omega/d)$. Since

$$\tfrac{1}{2}\arctan(\omega/d) \leqslant \tfrac{1}{2}\arctan(1) = \pi/8 < \pi/6,$$

the so-constructed lenses are disjoint and the sum of their angles is $\frac{1}{2}\arctan(\omega/d)\cdot n$.

We may apply the same idea and obtain the following result to *unbounded* convex sets.

THEOREM 5. *Let S_i, $i = 1, 2,\dots$ be a sequence of bounded convex sets each of diameter d_i and width ω_i, such that $S_i \subset S_{i+1}$ and $\arctan(\omega_i/d_i) \geqslant C$, for every i. Let $\bigcup_{i=1}^{\infty} S_i = S$. If $p(z)$ is regular in S and*

$$\iint_S |p(z)|\,dx\,dy \leqslant \tfrac{1}{2}C\cdot n$$

then no solution of Eq. (1) *has n zeros in S.*

3. A BOUND FOR $p(z)$

In this section we obtain relations between the length L of the boundary of a convex set S, and the number of zeros in S of a solution of Eq. (1).

LEMMA 2. *Let L be the length of the boundary of a convex planar set S, and let $P_1,\dots, P_n$ be n given points of S. There exist two points P_{i_1}, P_{i_2} such that the length of the segment $\overline{P_{i_1}P_{i_2}}$ is less than*

$$(\pi/12^{1/2})^{1/2}\cdot L/\pi\cdot 1/(n^{1/2}-1).$$

Proof. Let a be the minimal distance between the pairs of points. We draw n discs of radius $\frac{1}{2}a$ such that $P_1,\dots, P_n$ are their centers. These discs are disjoint and they are included in the convex set $S(\frac{1}{2}a) = \{z;\ d(z, S) \leqslant \frac{1}{2}a\}$. If we denote the area of S by A, then the area of $S(\frac{1}{2}a)$ is $A + L(\frac{1}{2}a) + \pi(\frac{1}{2}a)^2$. By [2, p. 67] we have

$$n\pi(\tfrac{1}{2}a)^2 < (\pi/12^{1/2})[A + L(\tfrac{1}{2}a) + \pi(\tfrac{1}{2}a)^2],$$

and the result follows by solving this inequality.

For small values of n it is easy to find better bounds. For example, if three points are given in a convex domain S, then there are two of them such that their distance is smaller than $\frac{1}{3}L$.

THEOREM 6. *Let S be a convex set in the complex plane and let L be the length of its boundary. If $p(z)$ is regular in S and*

$$|p(z)|\,(L/\pi)^2 \leqslant \pi\, 12^{1/2}(n^{1/2}-1)^2, \qquad z \in S, \quad n \geqslant 2 \tag{8}$$

then no solution of Eq. (1) *has n zeros in S.*

Proof. If $|p(z)| \leqslant \pi^2 M$, then it follows by the well-known result of Nehari [7] that any two zeros of a solution of Eq. (1) satisfy $|z_1 - z_2| \geqslant M^{-1/2}$. In our case, assumption (8) yields

$$|z_1 - z_2| \geqslant (\pi/12^{1/2})^{1/2} \cdot \frac{L}{\pi} \cdot 1/(n^{1/2}-1). \tag{9}$$

Now, if there are n zeros of a solution of Eq. (1) in S, then by Lemma 2 there are two zeros which violate (9).

By the remark following Lemma 2, in a convex domain this criteria may be replaced for $n = 3$ by $|p(z)|\,(L/\pi)^2 \leqslant 9$.

4. EQUATIONS OF mTH ORDER

In order to prove a similar theorem for equations of mth order we need the following lemma.

LEMMA 3. *Let L be the length of the boundary of a convex planar set S, and let $P_1, \ldots, P_n$ be n given points of S. The n points $P_1, \ldots, P_n$ can be connected by a polygonal line which consists of segments $\overline{P_{i_k}P_{i_{k+1}}}$ and is of total length less than*

$$L((n/\pi)^{1/2} + 3/4\pi + 1/2).$$

Proof. The proof is based on an idea of L. Few, who considered the same problem for a unit square [3].

Let ω be the minimal width of S and let it be attained in the direction θ. We divide S into q strips of equal width by $q+1$ segments parallel to θ. The length of no segment is greater than ω and the spacing of every pair of adjacent segments is less than $L/2q$. Let δ_i be the distance of P_i from the nearest segment.

We draw two paths through the n points and estimate their lengths, counting twice any arc traversed in both directions. The first path consists of the following parts:

(1) The $q+1$ segments parallel to θ.

(2) Twice the segment from P_i, $1 \leqslant i \leqslant n$, to the nearest segment parallel to θ.

(3) Arcs on the boundary of S, connecting the right- and left-hand endpoints of the segments alternately. Suitably chosen, their total length will not be greater than $\frac{1}{2}L$.

The length of this path is L_1 and

$$L_1 < (q+1)\omega + \tfrac{1}{2}L + 2\sum_{i=1}^{n} \delta_i .$$

To draw the second path, we consider q segments in the middle of the previous q strips. The distance between such a new segment and the two nearest old segments is less than $L/4q$. The distance of P_i from the nearest new segment is $(L/4q) - \delta_i$. Hence the length of the second path is L_2 and

$$L_2 < q\omega + \tfrac{1}{2}L + 2\sum_{i=1}^{n}\left(\frac{L}{4q} - \delta_i\right).$$

Since $\omega \leqslant L/\pi$, we have

$$L_1 + L_2 < (2q+1)\frac{L}{\pi} + L + \frac{nL}{2q} = L\left(\frac{2q}{\pi} + \frac{n}{2q} + \frac{1}{\pi} + 1\right).$$

To minimize the right-hand side, we take q as the nearest integer to $(n\pi/4)^{1/2}$; namely

$$q + \theta = (n\pi/4)^{1/2}, \qquad |\theta| \leqslant \tfrac{1}{2}.$$

Then

$$L_1 + L_2 < L\left[\frac{4}{\pi}(q+\theta) + \frac{2\theta^2}{\pi q} + \frac{1}{\pi} + 1\right] \leqslant L\left[\left(\frac{4n}{\pi}\right)^{1/2} + \frac{1}{2\pi} + \frac{1}{\pi} + 1\right],$$

and the length of one of the paths is less than

$$L((n/\pi)^{1/2} + (3/4\pi) + (1/2)).$$

If we replace the parts of this path connecting pairs of points by the segments between them, which are included in S, we obtain the required polygonal line.

Theorem 7. *Let S be a convex set and let L be the length of its boundary. If $p(z)$ is regular in S and*

$$|p(z)|\,(L/\pi)^m \leqslant \frac{m!}{[3\pi(m-1)]^{m/2}}\, n^{m/2}, \qquad z \in S, \quad n \geqslant 1$$

then every solution of

$$w^{(m)}(z) + p(z)\,w(z) = 0 \tag{4}$$

has at most $n(m-1)$ zeros in S.

Proof. Assume that there is a solution of Eq. (1), with $n(m-1)+1$ zeros in S. We want to create n sets, each set consisting of m zeros, including multiplicities. We consider Eq. (4) in the convex hull of each set of m points and we use the diameter of this convex set of find a bound to $\max|p(z)|$.

By Lemma 3 there exists a polygonal line of total length less than

$$L\left[\left(\frac{n(m-1)+1}{\pi}\right)^{1/2}+\frac{3}{4\pi}+\frac{1}{2}\right],$$

which passes through the zeros and consists of at most $n(m-1)$ segments (since some of the zeros may coincide). We may assume that the length of the polygonal line is less than

$$L\left(\frac{3n(m-1)}{\pi}\right)^{1/2}.$$

For $n=2$, $m=3$ this may be proved directly and for $n(m-1)\geqslant 5$ this is a conclusion of the previous estimate.

We now start counting the zeros along the polygonal line according to their multiplicity and we construct sets of m zeros, so that the last point of one set is the first point of the next set. In this process at least n sets of m zeros are constructed. (If the last point of a set is a zero of multiplicity greater than one, some zeros may be not used at all).

The diameter of the convex hull of the ith set d_i is not greater than the length of the segments connecting the points of this set. Hence

$$\sum_{i=1}^{n} d_i < L(3n(m-1)/\pi)^{1/2}.$$

By [4], there is in the convex hull of the ith set a point z_i such that

$$(1/m!)\,|p(z_i)|\,d_i^m \geqslant 1.$$

Let $z_0\in S$ be such that $|p(z_0)|=\max|p(z_i)|$, $i=1,\ldots,n$, then

$$n\cdot(1/m!)\,|p(z_0)|\geqslant\sum_{i=1}^{n}1/d_i^m\geqslant\left[n\Big/\left(\sum d_i/n\right)^m\right]>[n/(L/\pi)^m(3\pi(m-1)/n)^{m/2}],$$

contradicting the assumption of the theorem.

In a similar way it is possible to show that no solution of

$$w^{(m)}+p_l(z)\,w^{(m-l)}+\cdots+p_m(z)w=0,\qquad 2\leqslant l\leqslant m$$

has more than $n(m-1)$ zeros in S if

$$\sum_{k=l}^{m} \frac{1}{k!} \, | p_k(z) | \left(\frac{L}{\pi}\right)^k [3\pi(m-1)]^{k/2} \, n^{(k-l)/2} \leqslant n^{l/2}.$$

We note that the bound for the number of zeros in S of any solution is an integer multiple of $(m-1)$. The following example shows that such a bound is natural. Let u, v be two independent solutions of Eq. (1), then it is easily checked that $u^{m-1}, u^{m-2}v, \ldots, v^{m-1}$ are m independent solutions of an equation of the form [5]

$$y^{(m)} + p_2(z)\, y^{(m-2)} + \cdots + p_m(z)\, y = 0. \tag{10}$$

Since

$$c_1 u^{m-1} + c_2 u^{m-2} v + \cdots + c_m v^{m-1} = \prod_{i=1}^{m-1} (a_i u + b_i v),$$

every solution of Eq. (10) is a product of $m-1$ solutions of Eq. (1). Now if a solution of Eq. (10) has k zeros, $k > n(m-1)$, then one of the solutions of Eq. (1), say w, has at least $(n+1)$ zeros and the solution $y = w^{m-1}$ of Eq. (10) has at least $(n+1)(m-1)$ zeros.

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