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The Extremal Solutions of the Equation $Ly + p(x)y = 0$, II

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1. INTRODUCTION

In this paper we consider the differential equation

$$Ly + p(x)\, y = 0, \tag{1}$$

where $p(x)$ is a continuous function which does not change its sign and does not vanish identically on any interval and L is a disconjugate linear operator of order n on $[0, \infty)$. We assume that L is already written as a product of differential operators of first order

$$L_0 y = \rho_1 y,$$
$$L_i y = \rho_{i+1} D(L_{i-1} y), \qquad i = 1,..., n,$$

and

$$Ly = L_n y,$$

where $\rho_i > 0$ and $\rho_i \in C^{n-i+1}$ in $[0, \infty)$. $L_0 y,..., L_{n-1} y$ will be called the *quasi-derivatives* of $y(x)$.

Many equations which have been discussed in the literature are of form (1). Among them we mention

$$(ry'')'' + py = 0, \qquad \text{(Leighton and Nehari [4])},$$
$$(ry')'' + py = 0, \qquad (ry'')' + py = 0, \qquad \text{(Hanan [2])},$$
$$y''' + py' + qy = 0, \qquad p \leqslant 0, \qquad q - p' \geqslant 0\ (\leqslant 0), \qquad \text{(Lazer [3])},$$
$$y^{(4)} + py'' + qy = 0, \qquad p \leqslant 0, \qquad \text{(Pudei [6])},$$
$$y^{(n)} + p_1 y^{(n-q)} + p_2 y^{(n-2)} + p_n y = 0, \qquad p_2 \leqslant 0 \qquad \text{(Zettl [7])}.$$

In a previous paper [1] we investigated the kth conjugate point function, $\eta_k(a)$, for Eq. (1). $\eta_k(a)$ is defined as the infimum of the values b such that there exists a solution of Eq. (1) which vanishes at a and has at least $n + k - 1$

zeros in $[a, b]$. In this definition the zeros of $L_0 y$ (which are identical to those of $y(x)$) play a main role. The aim of this paper is to develop a similar theory in which the zeros of all n quasi-derivatives $L_0 y, \ldots, L_{n-1} y$ play the same role. By considering the zeros of all the quasi-derivatives we obtain sharper restrictions about possible zero distributions of solutions of Eq. (1).

Throughout, we shall always take the n quasi-derivatives in *cyclic order*, such that $L_0 y$ follows $L_{n-1} y$. We shall also arrange all the zeros of all quasi-derivatives from the left to the right such that ***common zeros of consecutive quasi-derivatives will be considered as multiple zeros*** but distinct subscripts will be used for zeros of nonconsecutive derivatives at the same point. Here $L_{n-1} y$ and $L_0 y$ are of course considered to be consecutive quasi-derivatives.

Let the zeros of the quasi-derivatives be $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r$. The number of consecutive quasi-derivatives of $y(x)$ which vanish at x_i will be called the multiplicity of the zero x_i and it will be denoted by $n(x_i, y)$. The total number of (not necessarily consecutive) quasi-derivatives of $y(x)$ which vanish at the point $x = c$ will be denoted by $\nu(c, y)$. (So $\nu(c, y) = \sum_{x_i = c} n(x_i, y)$). If it is clear which solution is considered, we replace $n(x_i, y)$ by $n(x_i)$.

For example, let $y(x)$ be a solution of the equation $L_6 y + py = 0$ and let its zeros be

$$\begin{aligned} (L_0 y)(a) &= (L_2 y)(a) = 0, \\ (L_0 y)(b) &= (L_4 y)(b) = (L_5 y)(b) = 0, \\ (L_2 y)(c) &= (L_3 y)(c) = 0, \qquad a < b < c. \end{aligned}$$

Here $x_1 = x_2 = a$, $x_3 = b$, $x_4 = c$ and the multiplicities of these zeros are $n(x_1) = 1, n(x_2) = 1, n(x_3) = 3(!), n(x_4) = 2$, and $\nu(a) = 2, \nu(b) = 3, \nu(c) = 2$.

Generally, the multiplicity of the zero at $x = c$ of a sufficiently smooth function $f(x)$ is the index i such that

$$f(c) = f'(c) = \cdots = f^{(i-1)}(c) = 0, \qquad f^{(i)}(c) \neq 0.$$

This definition is not applicable for counting the multiplicities of the zeros of $L_t y$, since generally $L_t y$ has only $n - t$ continuous derivatives. Therefore we define ***the multiplicity of the zero of $L_t y$ at $x = c$*** as the number of consecutive quasi-derivatives, counted from $L_t y$, which vanish at c, i.e., the index i such that

$$(L_t y)(c) = \cdots = (L_{t+i-1} y)(c) = 0, \qquad (L_{t+i} y)(c) \neq 0.$$

To simplify the notation, we agree that whenever $s \geqslant n$ $(s < 0)$, $L_s y$ is to be replaced by $L_{s-n} y (L_{s+n} y)$.

When $0 \leqslant t \leqslant t + i - 1 \leqslant n - 1$, this definition of multiplicity coincides with the usual one, since $(L_t y)(c) = \cdots = (L_{t+i-1} y)(c) = 0$, if and only if $(L_t y)(c) = (d/dx)(L_t y)(c) = \cdots = (d^{i-1}/dx^{i-1})(L_t y)(c) = 0$. If $p(x) \neq 0$ and $p(x)$, $\rho_i(x) \in C^n$, the two definitions are equivalent for every $L_t y$. On the other hand assume $p(c) = p'(c) = 0$ (and say $p''(c) \neq 0$), and also assume that $(L_{n-2} y)(c) = (L_{n-1} y)(c) = 0$ but $(L_0 y)(c) \neq 0$. As $L_n y = -py$, we now obtain $(L_n y)(c) = (d/dx)(L_n y)(c) = 0$ and thus have $(d^i/dx^i)(L_{n-2} y)(c) = 0$, $i = 0, 1, 2, 3$, i.e., $L_{n-2} y$ has by the usual definition a zero of multiplicity 4. But as we assume $(L_0 y)(c) \neq 0$, we have by our definition for the zero x_i of $L_{n-2} y$ only the multiplicity 2 at $x_i = c$.

It seems that for Eq. (1) our definition of multiplicity of the zeros of $L_t y$ is the *natural one*. It is easy to show that this definition has the expected properties. For example, if $L_t y$ has at x_i a zero of multiplicity l, then $L_{t+1} y$ has there a zero of multiplicity $l - 1$. Our definition of multiplicity differs from the usual one by an even integer. Therefore, if $L_t y$ has a zero of odd multiplicity, its sign changes there. If $L_t y_l(x) \to L_t y_0(x)$ $(t = 0, \ldots, n-1)$ and $x_l \to x_0$ as $l \to \infty$, and if $n(x_l, y_l) \geqslant q$, then $n(x_0, y_0) \geqslant q$.

Nehari [5] defined the (first) focal point of the point a for Eq. (1) as the infimum of values b, for which there exists a solution $y(x)$ such that every quasi-derivative $L_0 y, \ldots, L_{n-1} y$ vanishes in $[a, b]$. We define *the kth focal point, $\zeta_k(a)$, as the infimum of values b for which there exists a solution $y(x)$ such that every quasi-derivative $L_0 y, \ldots, L_{n-1} y$ has at least k zeros is $[a, b]$*. If no such solution exists, we say that $\zeta_k(a)$ does not exist.

By a compactness argument it is easy to see that if $\zeta_k(a)$ exists then $\zeta_k(a) > a$ and there exists a solution $y(x)$ of Eq. (1) such that each quasi-derivative $L_0 y, \ldots, L_{n-1} y$ has at least k zeros in $[a, \zeta_k(a)]$. Such a solution will be called an *extremal solution for* $\zeta_k(a)$. By the definition it follows that every extremal solution has a quasi-derivative which vanishes at $\zeta_k(a)$.

At first sight it seems that in the definition of $\zeta_k(a)$ we impose $n \cdot k$ conditions on $y(x)$ and its derivatives, while in the definition of the kth conjugate point, $\eta_k(a)$, only $n + k - 1$ conditions were required. But these $n \cdot k$ conditions are not independent, since $L_t y$ vanishes between any two zeros of $L_{t-1} y$. Indeed, assume that $y(x)$ satisfies the $n + k - 1$ equations

$$(L_0 y)(x_i) = 0, \qquad i = 1, 2, \ldots, n + k - 1,$$

where $a = x_1 < x_2 < \cdots < x_{n+k-1} = b$. It is easy to see that every quasi-derivative of $y(x)$ has at least k different zeros in $[a, b)$, hence $\zeta_k(a) < b$.

The same argument proves that if $\eta_k(a)$ exists, then $\zeta_k(a) < \eta_k(a)$. Indeed, an extremal solution $y(x)$ for $\eta_k(a)$ has exactly $n + k - 1$ zeros in $[a, \eta_k(a)]$, and $L_t y$ $(t = 0, \ldots, n-1)$ has at least $(n + k - 1) - t$ zeros. At least k of these $k + (n - 1 - t)$ zeros are in $[a, \eta_k(a))$, and the inequality follows.

In the rest of the paper we investigate the properties of $\zeta_k(a)$ as a function of a and the properties of the extremal solutions. It will be seen that there is a full parallelism between the behavior of $\zeta_k(a)$ and $\eta_k(a)$. Moreover, the tools which were used for $\eta_k(a)$ are applicable to $\zeta_k(a)$ after suitable modifications.

2. The Main Lemma

Let $y(x)$ be a solution of Eq. (1) and let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r$ be the zeros of the quasi-derivatives of $y(x)$. Let $a = x_1$ and $b = x_r$ be the leftmost and rightmost of these zeros, respectively. For the solution $y(x)$ we define

$$I = \{i \mid x_i = a \text{ or } x_i = b \text{ or } n(x_i) \text{ is even}\},$$

$$J = \{j \mid a < x_j < b \text{ and } n(x_j) \text{ is odd}\},$$

$$N(y) = \sum_{i \in I} n(x_i) + \sum_{j \in J} [n(x_j) - 1].$$

Lemma 1. *Every solution $y(x)$ of Eq. (1) satisfies $N(y) \leqslant n$. If $y(x)$ satisfies the equality $N(y) = n$ then $\nu(b, y)$ and $n - \nu(a, y)$ are both even (odd) when $p(x) \leqslant 0$ ($p(x) \geqslant 0$).*

Proof. We decompose I and J into disjoint subsets

$$I_t = \{i \in I \mid (L_{t-1}y)(x_i) \neq 0 \text{ and } (L_t y)(x_i) = 0\},$$
$$J_t = \{j \in J \mid (L_{t-1}y)(x_j) \neq 0 \text{ and } (L_t y)(x_j) = 0\},$$

$t = 0, \ldots, n - 1$. Of course, every one of the indices $1, \ldots, r$ belongs to one and only one of the sets I_t or J_t ($t = 0, \ldots, n - 1$). Moreover, between two consecutive zeros of $L_{t-1}y$ there is at least one zero x_j, $j \in J_t$.

Let $L_{t-1}y$ have at the q points $a \leqslant z_1 < z_2 < \cdots < z_q \leqslant b$ zeros of multiplicities $m_1, \ldots, m_q$, respectively, i.e., m_1 consecutive quasi-derivatives, beginning with $L_{t-1}y$ vanish at z_1, etc. $L_t y$ has at the same points zeros of multiplicities $m_1 - 1, \ldots, m_q - 1$ and additional $\sum_{l \in I_t \cup J_t} n(x_l)$ zeros. Hence the total number of zeros of $L_t y$ in $[a, b]$ is

$$\gamma_t = (m_1 - 1) + \cdots + (m_q - 1) + \sum_{l \in I_t} n(x_l) + \sum_{l \in J_t} n(x_l).$$

In each one of the $q - 1$ intervals $(z_1, z_2), \ldots, (z_{q-1}, z_q)$, $L_t y$ has at least one zero of odd multiplicity x_j, $j \in J_t$, It follows that

$$\gamma_t \geqslant m_1 + \cdots + m_q + \sum_{i \in I_t} n(x_i) + \sum_{j \in J_t} [n(x_j) - 1] - 1,$$

$$\gamma_t \geqslant \gamma_{t-1} + \sum_{I_t} n(x_i) + \sum_{J_t} [n(x_j) - 1] - 1. \tag{2}$$

By adding $n - 1$ such inequalities we have

$$\gamma_{n-1} \geqslant \gamma_0 + \sum_{t=1}^{n-1} \left\{ \sum_{I_t} n(x_i) + \sum_{J_t} [n(x_j) - 1] \right\} - (n - 1).$$

Passing from $L_{n-1}y$ to $L_0 y$ we get

$$\gamma_0 \geqslant \gamma_{n-1} + \sum_{I_0} n(x_i) + \sum_{J_0} [n(x_j) - 1] - 1$$

and by adding

$$n \geqslant \sum_{t=0}^{n-1} \left\{ \sum_{I_t} n(x_i) + \sum_{J_t} [n(x_j) - 1] \right\},$$

i.e.,

$$n \geqslant \sum_{I} n(x_i) + \sum_{J} [n(x_j) - 1].$$

This proves the first proposition of the lemma.

Now, let us assume that $N(y) = n$. Denote the rightmost zero of $L_t y$ in $[a, b]$ by β_t, $t = 0,..., n - 1$. As before, we assume that $L_{t-1}y$ has q different zeros in $[a, b]$. If $N(y) = n$, then $L_t y$ has in (a, b) exactly $q - 1$ zeros x_j, $j \in J_t$, one between every two consecutive zeros of $L_{t-1}y$. Otherwise, inequality (2) would be strict and $N(y) < n$. Especially, if $\beta_{t-1} < b$, $L_t y$ has no zero of odd multiplicity in (β_{t-1}, b) and $L_t y$ does not change its sign in this interval.

Suppose $(L_{t-1}y)(b) \neq 0$. Then $\beta_{t-1} < b$ and $L_t y$ does not change its sign in (β_{t-1}, b). From

$$(L_{t-1}y)(b) = \int_{\beta_{t-1}}^{b} \frac{(L_t y)(x)}{\rho_{t+1}(x)} \, dx$$

it follows that $\operatorname{sgn}(L_{t-1}y) = \operatorname{sgn}(L_t y)$ in a left neighborhood of b. If $(L_{t-1}y)(b) = 0$ then

$$(L_{t-1}y)(b - \epsilon) = - \int_{b-\epsilon}^{b} \frac{(L_t y)(x)}{\rho_{t+1}(x)} \, dx, \qquad (\epsilon > 0),$$

and $\operatorname{sgn}(L_{t-1}y) = -\operatorname{sgn}(L_t y)$ in a left neighborhood of b. It follows that $\operatorname{sgn}(L_n y) = (-1)^{\nu(b)} \operatorname{sgn}(L_0 y)$ to the left of b. But $L_n y = -(p(x)/\rho_1(x)) L_0 y$, hence $(-1)^{\nu(b)}$ is positive (negative) if $p(x) \leqslant 0$ ($p(x) \geqslant 0$). Similarly we obtain, when $N(y) = n$, that $\operatorname{sgn}(L_n y) = (-1)^{n-\nu(a)} \operatorname{sgn}(L_0 y)$ to the right of a. This completes the proof.

COROLLARY. *Every solution $y(x)$ of Eq.* (1) *satisfies $\nu(a) + \nu(b) \leqslant n$. When equality $\nu(a) + \nu(b) = n$ occurs, then $\nu(b)$ and $n - \nu(a)$ are both even (odd) if $p(x) \leqslant 0$ ($p(x) \geqslant 0$).*

This corollary follows from Lemma 1 since $\nu(a) + \nu(b)$ is included in the sum $\sum_I n(x_i)$. The corollary was proved by Nehari [5, Theorems 5.3, 5.4].

3. EXTREMAL SOLUTIONS

Now we turn to the extremal solutions of Eq. (1) for $\zeta_k(a)$.

LEMMA 2. *Every extremal solution $y(x)$ for $\zeta_k(a)$ satisfies $N(y) = n$.*

Proof. It will be proved that if $N(y) < n$, then there exists another solution of Eq. (1) such that each of its quasi-derivatives has at least k zeros in $[a, \zeta_k(a))$, contradicting the definition of $\zeta_k(a)$.

First, let us assume that $p(x) \leqslant 0$. The leftmost zero of the quasi-derivatives of $y(x)$ in $[a, \zeta_k(a)]$ will be denoted by a'. We shall see later that $a' = a$. We look for a solution of Eq. (1) which satisfies certain boundary conditions at the points where the quasi-derivatives of $y(x)$ vanish. At x_i, $i \in I$, $x_i \neq \zeta_k(a)$, the $n(x_i, y)$ conditions are

$$(L_t u)(x_i) = 0, \qquad s \leqslant t \leqslant s + n(x_i) - 1, \qquad i \in I_s, \qquad x_i \neq \zeta_k(a), \tag{3}$$

and at x_j, $j \in J$, we have the $n(x_j, y) - 1$ conditions

$$(L_t u)(x_j) = 0, \qquad s \leqslant t \leqslant s + n(x_j) - 2, \qquad j \in J_s, \tag{4}$$

where $s = 0, 1, \ldots, n - 1$. At $\zeta_k(a)$ the boundary conditions are given separately. Let l be one of the indices such that $(L_l y)(\zeta_k(a)) = 0$ but $(L_{l+1} y)(\zeta_k(a)) \neq 0$. For every $t \neq l$ such that $(L_t y)(\zeta_k(a)) = 0$ we require

$$(L_t u)(\zeta_k(a)) = 0 \qquad [t \neq l, (L_t y)(\zeta_k(a)) = 0] \tag{5}$$

and for l,

$$(L_l u)(\zeta_k(a)) = 1, \tag{6}$$

i.e., $\nu(\zeta_k(a), y)$ conditions at $\zeta_k(a)$.

Equations (3)–(6) give $N(y)$ boundary conditions. We add $n - N(y)$ ($\geqslant 1$) more conditions. If $\nu(\zeta_k(a), y)$ is odd, we add $n - N(y)$ conditions at a'. There exist $n - N(y)$ indices t_i such that $(L_{t_i} y)(a') \neq 0$, because $n - N(y) < n - \nu(a')$. For these $n - N(y)$ indices, we let

$$(L_{t_i} u)(a') = 0, \qquad 1 \leqslant i \leqslant n - N(y) \qquad [(L_{t_i} y)(a') \neq 0]. \tag{7}$$

If $\nu(\zeta_k(a), y)$ is even, one condition will be added at $\zeta_k(a)$ and $n - N(y) - 1$ conditions at a':

$$(L_{t_0}u)(\zeta_k(a)) = 0 \qquad [(L_{t_0}y)(\zeta_k(a)) \neq 0], \tag{7'}$$

$$(L_{t_i}u)(a') = 0, \qquad 1 \leqslant i \leqslant n - N(y) - 1 \qquad [(L_{t_i}y)(a') \neq 0]. \tag{7''}$$

For the differential equation (Eq. (1)) we now consider the boundary value problem given by (3) to (7) or by (3) to (6), (7'), (7''), respectively. There are n conditions in this system of equations. Now the *corresponding homogeneous system*, for which condition (6) is thus replaced by

$$(L_t u)(\zeta_k(a)) = 0, \tag{6'}$$

has only the trivial solution. For if $v(x) \not\equiv 0$ solves the homogeneous system, then $N(v) \geqslant n$ and hence, by Lemma 1, $N(v) = n$ while $\nu(\zeta_k(a), v)$ is odd. But this contradicts, again by Lemma 1, the assumption $p(x) \leqslant 0$. Therefore the given *nonhomogeneous* system has a *unique* solution, which is denoted by $\bar{y}(x)$.

We now count the zeros of the solution $y_1(x) = y(x) + \alpha\bar{y}(x)$. At a point x_i, $i \in I$, $x_i \neq \zeta_k(a)$, $y(x)$ has *exactly* $n(x_i)$ vanishing consecutive quasi-derivatives and $\bar{y}(x)$ has *at least* $n(x_i)$ vanishing consecutive quasi-derivatives. Therefore x_i is a zero of multiplicity $n(x_i)$ of $y_1(x)$ for every α.

At a point x_j, $j \in J$, $\bar{y}(x)$ has a zero of multiplicity $n(x_j) - 1$ or $n(x_j)$ but not of greater multiplicity. Otherwise, $\bar{y}(x)$ would satisfy $N(\bar{y}) > n$, again contradicting Lemma 1. If x_j, $j \in J$, is a zero of multiplicity $n(x_j)$ for $\bar{y}(x)$, then it is a zero of multiplicity $n(x_j)$ for $y_1(x)$ too. Next assume that $\bar{y}(x)$ has at x_j, $j \in J$, exactly $n(x_j) - 1$ consecutive vanishing quasi-derivatives

$$(L_t \bar{y})(x_j) = 0, \qquad t_j \leqslant t \leqslant t_j + n(x_j) - 2, \tag{8}$$

while $y(x)$ has $n(x_j)$ such quasi-derivatives

$$(L_t y)(x_j) = 0, \qquad t_j \leqslant t \leqslant t_j + n(x_j) - 1. \tag{9}$$

At x_j, $L_{t_j}y_1$ has a zero of multiplicity $n(x_j) - 1$ for every α. Recall that if $(L_{t-1}u)(c) = 0$ then $L_{t-1}u = \int_c^x (L_t u/\rho_{t+1})\,dx$ $(t = 1,\ldots, n-1)$ and $L_{n-1}u = \int_c^x (p(x)\,u/\rho_{n+1})\,dx$. If $(L_{t-1}u)(c) = (L_{t-1}v)(c) = 0$ then by L'Hopital's rule

$$\lim_{x\to c} \frac{L_{t-1}u}{L_{t-1}v} = \lim_{x\to c} \frac{\int_c^x (L_t u/\rho_{t+1})\,dx}{\int_c^x (L_t v/\rho_{t+1})\,dx} = \lim_{x\to c} \frac{L_t u}{L_t v}\,. \tag{10}$$

We set $u = y$, $v = \bar{y}$ and apply (10) successively for t, $t_j \leqslant t \leqslant t_j + n(x_j) - 1$. It follows then by (8) and (9) that $\lim_{x\to x_j}(L_{t_j}y/L_{t_j}\bar{y}) = 0$. Since the denominator has a zero of even order, the numerator of odd order, it follows that $L_{t_j}y/L_{t_j}\bar{y}$ changes its sign at x_j. Therefore, for sufficiently small $|\alpha|$, the quasi-derivative

$$L_{t_j}y_1 = L_{t_j}y + \alpha L_{t_j}\bar{y} = L_{t_j}\bar{y}(L_{t_j}y/L_{t_j}\bar{y} + \alpha)$$

changes its sign in a deleted neighborhood of x_j. Now, as $L_{t_j}y_1$ vanishes at x_j and in a given neighborhood of x_j, $L_{t_j+1}y_1$, too, has a zero in the deleted neighborhood of x_j. By a repeated use of Rolle's theorem, we obtain that every $L_t y_1$, $t_j \leqslant t \leqslant t_j + n(x_j) - 1$, changes its sign in a deleted neighborhood of x_j. Evidently, all of them change their sign on the same side of x_j. All of these zeros are simple for small α. Otherwise, as $\alpha \to 0$ we would find that $L_{t_j}y = \lim_{\alpha\to 0} L_{t_j}y_1$ has at x_j a zero of multiplicity greater than $n(x_j)$. This describes the behavior of the zeros of y_1 near x_j, $j \in J$.

By Eqs. (5) and (6) the same reasoning holds in the neighborhood of $\zeta_k(a)$. By a suitable choice of the sign of α, we obtain that the appropriate quasi-derivatives of $y_1(x)$ change their signs on the left side of $\zeta_k(a)$, in $[a, \zeta_k(a))$.

The quasi-derivatives of $y_1(x)$ have no zeros other than those at $x_i (i \in I)$, $x_j (j \in J)$ and simple zeros in the given neighborhoods of x_j and $\zeta_k(a)$, if α is sufficiently small. By a simple count, $L_t y_1$ ($t = 0,..., n-1$) has in $[a, \zeta_k(a)]$ the same number of zeros as $L_t y$. But this process splits a zero of one of the quasi-derivatives of $y(x)$ at the endpoint $\zeta_k(a)$ and shifts one zero into $[a, \zeta_k(a))$, such that $y_1(x)$ has $\nu(\zeta_k(a), y) - 1$ quasi-derivatives vanishing at $\zeta_k(a)$ and one $L_t y_1$ has a simple zero in a left neighborhood of $\zeta_k(a)$. Therefore $N(y_1) < N(y) < n$. By a finite number of such steps, we shift all the zeros of $y(x)$ and its quasi-derivatives at $\zeta_k(a)$ into the interior of $[a, \zeta_k(a)]$. So we obtain a solution such that its quasi-derivatives have in $[a, \zeta_k(a))$ the same number of zeros as the quasi-derivatives of $y(x)$ in $[a, \zeta_k(a)]$. This solution contradicts the definition of $\zeta_k(a)$, and completes the proof.

For $p(x) \geqslant 0$, the proof is similar.

LEMMA 3. *Let $y(x)$ be an extremal solution for $\zeta_k(a)$. In the interval $(a', \zeta_k(a))$, $L_t y$ ($t = 0,..., n-1$) has one and only one zero between every two consecutive zeros of $L_{t-1}y$ (which is thus of odd multiplicity) and has all its other zeros at the multiple zeros of $L_{t-1}y$.*

Proof. If $L_{t-1}y$ has q different zeros in $[a, \zeta_k(a)]$, then in J_t there are exactly $q - 1$ indices, else inequality (2) would be strict and $N(y) < n$. We thus have exactly one zero of odd multiplicity of $L_t y$ between every two consecutive zeros of $L_{t-1}y$ and further zeros of odd multiplicity of $L_t y$ can only occur at the multiple zeros of $L_{t-1}y$.

Next we prove that if $a' < x_l < \zeta_k(a)$ then $n(x_l)$ is odd. This means that $L_t y$ has no zero of even multiplicity between consecutive zeros of $L_{t-1}y$

and in (β_{t-1}, b), but only at those points where $L_{t-1}y$ has a multiple zero of odd multiplicity. Suppose that there exist i_0, t_0 such that $i_0 \in I_{t_0}$ and $x_{i_0} \neq a'$, $\zeta_k(a)$. Then $n(x_{i_0})$ is even and $n(x_{i_0}) \geqslant 2$. For Eq. (1) we consider the following boundary conditions at the points where the quasi-derivatives of $y(x)$ vanish:

$$\begin{aligned}
&(L_t u)(x_i) = 0, \qquad s \leqslant t \leqslant s + n(x_i) - 1, \quad i \in I_s, \quad i \neq i_0,\\
&(L_t u)(x_{i_0}) = 0, \qquad t_0 \leqslant t \leqslant t_0 + n(x_{i_0}) - 3,\\
&(L_t u)(x_j) = 0, \qquad s \leqslant t \leqslant s + n(x_j) - 2, \quad j \in J_s,
\end{aligned}$$

where $s = 0,\ldots, n-1$. This system consists of $n-2$ equations, therefore it has a solution $\bar{y}(x)$ which is linearly independent of $y(x)$.

Three different assumptions about $\bar{y}(x)$ will be checked separately.

(a) Suppose $\bar{y}(x)$ has exactly $n(x_{i_0}) - 2$ consecutive quasi-derivatives vanishing at x_{i_0}. Then the solution $y_1(x) = y(x) + \alpha\bar{y}(x)$ also has exactly $n(x_{i_0}) - 2$ zeros at x_{i_0}. But as $y(x)$ has exactly $n(x_{i_0})$ consecutive vanishing quasi-derivatives at x_{i_0}, we have $\lim_{x \to x_{t_0}}(L_{t_0}y/L_{t_0}\bar{y}) = 0$, and $L_{t_0}y/L_{t_0}\bar{y}$ does not change its sign at x_{i_0}. Therefore for α sufficiently small and of suitable sign the quasi-derivative $L_{t_0}y_1 = L_{t_0}y + \alpha L_{t_0}\bar{y} = L_{t_0}\bar{y}(L_{t_0}y/L_{t_0}\bar{y} + \alpha)$ has simple zeros on the two sides of x_{i_0}. Since $n(x_{i_0}) - 2$ consecutive quasi-derivatives vanish at x_{i_0}, by Rolle's theorem the following quasi-derivatives also have zeros in a given neighborhood of x_{i_0}, one the two sides of x_{i_0}. For small α, all these zeros are simple. At a zero x_j, $j \in J_{t_j}$, $L_{t_j}y_1$ has, by a similar reasoning, a zero of multiplicity $n(x_j) - 1$ and an additional simple zero in a given neighborhood of x_j. For small α, $L_t y_1$ $(t = 0,\ldots, n-1)$ have no other zeros except those prescribed by the boundary conditions and the simple zeros in their neighborhoods. By a simple count it is seen that the quasi-derivatives of $y_1(x)$ have in $[a, \zeta_k(a)]$ the same number of zeros as the corresponding quasi-derivatives of $y(x)$. But as the zero at x_{i_0} splits into a zero of multiplicity $n(x_{i_0}) - 2$ and two simple zeros, we have $N(y_1) < N(y) = n$. This contradicts Lemma 2.

(b) If exactly $n(x_{i_0}) - 1$ consecutive quasi-derivatives of $\bar{y}(x)$ vanish at x_{i_0}, we split the zero of $L_{t_0}y$ at x_i, as in Lemma 2. We obtain that for suitable α, $y_1(x) = y(x) + \alpha\bar{y}(x)$ and its quasi-derivatives have, in $[a, \zeta_k(a)]$, the same number of zeros as $y(x)$ and its corresponding quasi-derivatives and $N(y_1) < n$.

(c) If $\bar{y}(x)$ has at least $n(x_{i_0})$ zeros at x_{i_0}, then there exists a linear combination of $y(x)$ and $\bar{y}(x)$ such that $N(c_1 y + c_2\bar{y}) > n$, again a contradiction.

LEMMA 4. *Let $y(x)$ be an extremal solution for $\zeta_k(a)$. Then at least one of its queasi-derivatives has exactly k zeros in $[a, \zeta_k(a)]$.*

COROLLARY. $\zeta_k(a) < \zeta_{k+1}(a)$.

Proof of Lemma 4. Suppose that every quasi-derivative of $y(x)$ has at least $k + 1$ zeros in $[a, \zeta_k(a)]$. At least one quasi-derivative has a multiple zero at $\zeta_k(a)$. Else, if all the quasi-derivatives vanishing at $\zeta_k(a)$ had simple zeros there, each quasi-derivative would have at least k zeros in $[a, \zeta_k(a))$. Let x_r be one of the multiple zeros at the right endpoint of the interval, $\zeta_k(a)$. For Eq. (1) we consider the $n - 2$ boundary conditions

$$(L_t u)(x_j) = 0, \qquad s \leqslant t \leqslant s + n(x_j) - 2, \quad j \in J_s,$$

$$(L_t u)(x_i) = 0, \qquad s \leqslant t \leqslant s + n(x_i) - 1, \quad i \in I_s, \quad i \neq r,$$

where $s = 0,..., n - 1$ and

$$(L_t u)(x_r) = 0, \qquad t_r \leqslant t \leqslant t_r + n(x_r) - 3.$$

This system has a solution $\bar{y}(x)$ linearly independent of $y(x)$.

Assume that exactly $n(x_r, y) - 2$ consecutive quasi-derivatives of $\bar{y}(x)$ vanish at x_r, explicitly $L_t y$, $t_r \leqslant t \leqslant t_r + n(x_r) - 3$. Then the same quasi-derivatives of $y_1(x) = y(x) + \alpha\bar{y}(x)$ vanish at x_r. As in the proof of Lemma 3, we obtain that for suitable α, $L_t y_1$ $(t_r \leqslant t \leqslant t_r + n(x_r) - 2)$ has two simple zeros, one to the right and the other to the left of $\zeta_k(a)$. One of them is clearly in $[a, \zeta_k(a)]$. Recall that each quasi-derivative of $y(x)$ has at least $k + 1$ zeros in $[a, \zeta_k(a)]$. A simple count shows that every quasi-derivative of $y_1(x)$ has at least k zeros in $[a, \zeta_k(a)]$ and $N(y_1) < N(y) = n$ because of the splitting of the multiple zero at x_r. This contradicts Lemma 2.

If $n(x_{r_l} \bar{y}) > n(x_r, y) - 2$, we obtain a contradiction as in the proof of Lemma 3.

4. PROPERTIES OF $\zeta_k(a)$

The following lemma is required for the proof of Theorem 1.

LEMMA 5. *For every $\epsilon > 0$ there exists a solution $y_\epsilon(x)$ such that each of its quasi-derivatives has at least k simple zeros in $[a, \zeta_k(a) + \epsilon)$ and $N(y_\epsilon) < n$. The same proposition holds for the interval $(a - \epsilon, \zeta_k(a)]$.*

Proof. Let $y(x)$ be an extremal solution for $\zeta_k(a)$, and if $L_t y$, $t = 0,...,$

$n - 1$, have multiple zeros in $[a, \zeta_k(a)]$, let x_l be the rightmost of them. If $x_l = \zeta_k(a)$, we consider the system

$$\begin{aligned}(L_t u)(x_i) &= 0, \qquad s \leqslant t \leqslant s + n(x_i) - 1, \quad i \in I_s, \quad i \neq l,\\ (L_t u)(x_j) &= 0, \qquad s \leqslant t \leqslant s + n(x_j) - 2, \quad j \in J_s,\end{aligned}$$

where $s = 0,..., n - 1$, and

$$(L_t u)(x_l) = 0, \qquad t_l \leqslant t \leqslant t_l + n(x_l) - 3.$$

In this system there are $n - 2$ equations, and it has a solution $\bar{y}(x)$ linearly independent of $y(x)$. $n(x_l, \bar{y}) \geqslant n(x_l, y)$ is impossible, since otherwise there would be constants c_1, c_2 such that $N(c_1 y + c_2 \bar{y}) > n$. If $n(x_l, \bar{y}) = n(x_l, y) - 1$, then, recalling that $x_l = \zeta_k(a)$, we obtain as in the proof of Lemma 2 that for suitable α, the quasi-derivatives of $y_1 = y + \alpha\bar{y}$ have in $[a, \zeta_k(a)]$ as many zeros as the quasi-derivatives of $y(x)$ and $N(y_1) < N(y) = n$. This contradicts Lemma 2, therefore $y(x)$ has at x_l a zero exactly of order $n(x_l) - 2$. As in the proof of Lemma 3, we obtain a solution y_1 with a zero of multiplicity $n(x_l) - 2$ at x_l and two simple zeros near x_l. The quasi-derivatives of $y_1(x)$ have in $[a, \zeta_k(a) + \epsilon)$ the same number of zeros as the quasi-derivatives of $y(x)$ have in $[a, \zeta_k(a)]$. But by splitting the multiple zero at x_l the number of simple zeros increases in this process and $N(y_1) < n$ in $[a, \zeta_k(a) + \epsilon)$.

If the rightmost multiple zero x_l is inside $[a, \zeta_k(a)]$, then the construction of $\bar{y}(x)$ is more complicated. If $a' < x_l < \zeta_k(a)$, then by Lemma 3, $l \in J$ and $n(x_l)$ is odd, hence $n(x_l) \geqslant 3$. We define $\bar{y}(x)$ as a solution of the system

$$\begin{aligned}(L_t u)(x_i) &= 0, \qquad s \leqslant t \leqslant s + n(x_i) - 1, \quad i \in I_s, \quad i \neq r,\\ (L_t u)(x_r) &= 0, \qquad t_r \leqslant t \leqslant t_r + n(x_r) - 2,\end{aligned}$$

where x_r is one of the zeros at $\zeta_k(a)$, and so $n(x_n) = 1$,

$$\begin{aligned}(L_t u)(x_j) &= 0, \qquad s \leqslant t \leqslant s + n(x_j) - 2, \quad j \in J_s, \quad j \neq l,\\ (L_t u)(x_l) &= 0, \qquad t_l \leqslant t \leqslant t_l + n(x_l) - 3,\end{aligned}$$

and $s = 0,..., n - 1$. In this system there are

$$\sum_{i \neq r} n(x_i) + [n(x_r) - 1] + \sum_{j \neq l} [n(x_j) - 1] + [n(x_l) - 2] = N(y) - 2 = n - 2$$

equations and it therefore has a solution $\bar{y}(x)$ linearly independent of $y(x)$. Now, the proof goes on as before. If $x_l = a'$, we define $\bar{y}(x)$ similarly.

By a finite number of such steps we obtain a solution $y_\epsilon(x)$ such that each of the quasi-derivatives $L_0 y_\epsilon, ..., L_{n-1} y_\epsilon$ has at least k simple zeros in

$[a, \zeta_k(a) + \epsilon)$ and $N(y_\epsilon) < n$. If $L_t y$ $(t = 0, ..., n - 1)$ have only simple zeros, i.e., x_l does not exist, but $N(y) = \nu(a') + \nu(\zeta_k(a)) = n$, we obtain by the same method a solution with only simple zeros and $N(y_\epsilon) < n$.

For the interval $(a - \epsilon, \zeta_k(a)]$ the proof is similar.

THEOREM 1. *$\zeta_k(a)$ is a strictly increasing continuous function of a which is defined on an interval of the form* $[0, b)$, $0 \leqslant b \leqslant \infty$.

Proof. We recall that in the definition of $\zeta_k(a)$, we considered all the solutions $y(x)$ such that each of its quasi-derivatives has k zeros, and not only those solutions $y(x)$ with a quasi-derivative vanishing at a. Therefore, if $\zeta_k(a)$ exists, ζ_k is defined also on $[0, a]$.

To prove the continuity of ζ_k, we use Lemma 5. Let $y(x)$ be a solution such that each one of its quasi-derivatives has at least k simple zeros in $[a, \zeta_k(a) + \epsilon)$. Let $v(x)$ be the solution of Eq. (1) which has the initial values $(L_t v)(c) = (L_t y)(a)$, $t = 0, ..., n - 1$. The solutions of Eq. (1) are continuously dependent on the initial conditions. Therefore, if $| a - c |$ is sufficiently small, $L_t y$ and $L_t v$ are close and $L_t v$ has simple zeros near the simple zeros of $L_t y$. Moreover, all these zeros are in $[c, \zeta_k(a) + \epsilon)$. Thus $\zeta_k(c)$ exists for $| a - c | < \delta_1$ and $\zeta_k(c) < \zeta_k(a) + \epsilon$. Here the same $\delta_1 = \delta_1(\epsilon)$ is good for every a in a given compact interval. By interchanging the roles of a and c, we get $\zeta_k(a) < \zeta_k(c) + \epsilon$ when $| c - a | < \delta_2$. These inequalities prove that ζ_k is defined in some neighborhood of a and it is continuous there.

Now, we also use the second half of Lemma 5. For every $\epsilon > 0$ there exists a solution $y_\epsilon(x)$ such that each of its quasi-derivatives has at least k simple zeros in $(a - \epsilon, \zeta_k(a)]$ and $N(y_\epsilon) < n$. Thus $\zeta_k(a - \epsilon)$ exists and $\zeta_k(a - \epsilon) \leqslant \zeta_k(a)$. By the inequality $N(y_\epsilon) < n$ and by Lemma 2, $\zeta_k(a - \epsilon) < \zeta_k(a)$. Now ζ_k is a continuous function which is strictly increasing in some left neighborhood of each point. Therefore it is strictly increasing in the whole interval.

This completes the proof of the theorem.

We remark that the monotonity of ζ_k implies that for every extremal solution for $\zeta_k(a)$, one of the quasi-derivatives vanishes at a. Otherwise if the first zero of the quasi-derivatives is a', $a' > a$, then by definition we would obtain $\zeta_k(a') \leqslant \zeta_k(a)$, contradicting the strict monotonity.

Using this remark and the lemmas, we can summarize the properties of extremal solutions in the following theorem.

THEOREM 2. *Every extremal solution $y(x)$ of Eq.* (1) *for $\zeta_k(a)$ has the following properties.*

(1) *One of the quasi-derivatives of $y(x)$ has* exactly *k zeros in $[a, \zeta_k(a)]$.*

(2) *One of the quasi-derivatives vanishes of* y *vanishes at* a *and one of the quasi-derivatives vanishes at* $\zeta_k(a)$.

(3) $\nu(\zeta_k(a))$ *and* $n - \nu(a)$ *are both even (odd) when* $p(x) \leqslant 0$ ($p(x) \geqslant 0$).

(4) *Between two consecutive zeros of* $L_{t-1}y$ *there is exactly one zero of odd multiplicity of* $L_t y$. *For each* x_l, $a < x_l < \zeta_k(a)$, $n(x_l, y)$ *is odd.*

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