JOURNAL OF DIFFERENTIAL EQUATIONS 29, 28–57 (1978)

Eigenvalue Problems for the Equation $Ly + \lambda p(x)\, y = 0$

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Received January 11, 1977

1. INTRODUCTION

Given the differential equation

$$Ly + \lambda p(x) y = 0, \tag{1}$$

where $p(x)$ is a continuous function of a constant sign which does not vanish identically on any interval and L is a linear differential operator of order n on $[a, b]$ which is factored into a product of operators of first order:

$$\begin{aligned} L_0 y &= \rho_0 y, \\ L_i y &= \rho_i (L_{i-1} y)', \qquad i = 1,..., n, \\ Ly &= L_n y. \end{aligned}$$

Here $\rho_i > 0$ and $\rho_i \in C^{n-i}$ on $[a, b]$. $L_0 y,..., L_{n-1} y$ are called the *quasi-derivatives* of $y(x)$.

We consider the following eigenvalue problem:

$$\begin{aligned} &Ly + \lambda p(x) y = 0, \\ &\quad (L_t y)(x_i) = 0, \qquad q_i \leqslant t \leqslant q_i + n_i - 1, \end{aligned}$$

where $a = x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r = b$, $\sum_{i=1}^{r} n_i = n$, $0 \leqslant q_i \leqslant q_i + n_i - 1 \leqslant n - 1$, and n_i is *even* whenever $a < x_i < b$. The reason for the last assumption will become clear later. To stress the difference between the boundary conditions imposed at points x_i, $a < x_i < b$, and those imposed at the endpoints a and b, we write the considered eigenvalue problem in the form

$$\begin{aligned} &Ly + \lambda p(x) y = 0, \\ &\quad (L_t y)(x_i) = 0, \qquad q_i \leqslant t \leqslant q_i + 2m_i - 1, \quad a < x_i < b, \\ &\quad (L_t y)(a) = 0, \qquad t \in \{i_1 ,..., i_{\nu(a)}\}, \\ &\quad (L_t y)(b) = 0, \qquad t \in \{j_1 ,..., j_{\nu(b)}\}, \end{aligned} \tag{2}$$

0022-0396/78/0291-0028$02.00/0

where $\nu(a) + \sum 2m_i + \nu(b) = n$, $0 \leqslant q_i \leqslant q_i + 2m_i - 1 \leqslant n - 1$, $\nu(a) \geqslant 1$, $\nu(b) \geqslant 1$, and $\{i_1, ..., i_{\nu(a)}\}$, $\{j_1, ..., j_{\nu(b)}\}$ are two arbitrary sets of integers from $\{0, ..., n-1\}$.

The following properties of the eigenvalues and the eigenfunctions of (2) are proved:

The eigenvalue problem (2) has an infinite number of real eigenvalues (Theorem 2). The set of all eigenvalues of (2) has no finite accumulation point and the sign of the real eigenvalues is determined by $(-1)^{\nu(b)}\lambda p(x) \leqslant 0$ (Corollary 2). To every nonzero eigenvalue there corresponds an essentially unique eigenfunction. Such an eigenfunction and its quasi-derivatives may have at points of (a, b), which are not given in (2), only simple zeros. At a point x_i where a sequence of boundary conditions is given in (2), at most one more consecutive quasi-derivative may vanish. No quasi-derivative, except those given in (2), may vanish at the endpoints a and b (Theorem 1).

If $\lambda = 0$ is an eigenvalue, then a finite number of linearly independent eigenfunctions correspond to it. If the independent eigenfunctions belonging to the eigenvalue $\lambda = 0$ are arranged in a suitable order, and the other eigenfunctions are arranged according to the magnitude of the corresponding eigenvalues, then the ith eigenfunction changes its sign exactly $i - 1$ times in (a, b) (Theorem 3). The changes of signs of two eigenfunctions, which belong to two consecutive nonzero eigenvalues, separate each other in (a, b) (Theorem 4). In every subinternal of $[a, b]$, the ith eigenfunction changes its sign for sufficiently large i (Theorem 2).

An interesting case is obtained when all the n boundary conditions are given at the endpoints of the interval:

$$\begin{aligned} Ly + \lambda p(x) y &= 0, \\ (L_i y)(a) &= 0, \qquad i \in \{i_1, ..., i_k\}, \\ (L_j y)(b) &= 0, \qquad j \in \{j_1, ..., j_{n-k}\}. \end{aligned} \tag{3}$$

The ith eigenfunction of (3) has exactly $i - 1$ simple zeros in (a, b) and all of its quasi-derivatives have only simple zeros in (a, b). In addition we prove that the eigenvalues of (3) decrease as the boundary conditions are posed on higher quasi-derivatives (Theorem 6).

Eigenvalue problems of type (3) were considered in several papers. When (3) is a selfadjoint problem, the existence of the eigenvalues is known and they are characterized by the max-min principle of Courant. Note, that multipoint boundary value problems cannot, in general, be selfadjoint in the usual meaning [11, Theorem 3]. Eigenfunctions of fourth order selfadjoint eigenvalue problems were discussed in [2, 3, 6, 10]. The max-min characterization of the first eigenvalue was used in numerous papers to obtain necessary conditions for disconjugacy.

Krein proved [8, 9] that the nonselfadjoint problem

$$\begin{aligned} &Ly + \lambda p(x) y = 0, \\ &y^{(i)}(a) = 0, \qquad i = 0,\ldots, k-1, \\ &y^{(j)}(b) = 0, \qquad j = 0,\ldots, n-k-1, \end{aligned} \tag{4}$$

is equivalent to an integral equation with an oscillating kernel. From this he deduced the existence of eigenfunctions and properties of their zeros. Karlin considered in [7] the operator L and boundary conditions more general than (3). He proved that if the corresponding Green's function exists, i.e. $\lambda = 0$ is not an eigenvalue, then $\pm G(x, t)$ is an oscillating kernel.

In this paper, (2) and (3) are investigated using methods of differential equations. As it turns out, it is essential to examine the case when $\lambda = 0$ is an eigenvalue, and thus Green's function does not exist, in order to enumerate the eigenfunctions and their zeros in (a, b). It is worth noting that rewriting Eq. (1) as

$$(L + \lambda_0 p) y + (\lambda - \lambda_0) p y = 0$$

does not solve the problem of the eigenvalue $\lambda = 0$ and the nonexistence of Green's function. Indeed, for small λ_0 the operator $L + \lambda_0 p$ is factorizable on $[a, b]$ [4, p. 95]. But the quasi-derivatives related to this factorization are different from those which correspond to L, and the boundary conditions of (2) cannot have been expressed simply by them.

2. Basic Lemmas

Let y be a solution of (1). As in [5], we arrange the n quasi-derivatives $L_0 y,\ldots, L_{n-1} y$ in a *cyclic order* so that $L_0 y$ follows $L_{n-1} y$. Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r$ be the zeros of $L_0 y,\ldots, L_{n-1} y$ in $[a, b]$, so that common zeros of consecutive quasi-derivatives will be considered as multiple zeros but distinct subscripts will be used for zeros of nonconsecutive derivatives at the same point. Here, common zeros of $L_{n-1} y$ and $L_0 y$ are denoted as a multiple zero. It is easy to see that $L_t y$ changes its sign at c if and only if an odd number of consecutive quasi-derivatives $L_t y, L_{t+1} y,\ldots$ vanish at c.

The number of consecutive quasi-derivatives of $y(x)$ which vanish at x_i will be denoted by $n(x_i, y)$, or if the considered solution y is obvious, by $n(x_i)$. The total number of (not necessarily consecutive) quasi-derivatives of $y(x)$ which vanish at a point c will be denoted by $\nu(c, y)$. Obviously, $\nu(c, y) = \sum_{x_i = c} n(x_i, y)$. For example, let $y(x)$ be a solution of the equation $L_6 y + p(x) y = 0$ and let the zeros of its quasi-derivatives be

$$\begin{aligned} &(L_0 y)(\alpha) = (L_2 y)(\alpha) = 0, \\ &(L_0 y)(\beta) = (L_4 y)(\beta) = (L_5 y)(\beta) = 0. \\ &(L_2 y)(\gamma) = (L_3 y)(\gamma) = 0, \qquad \alpha < \beta < \gamma. \end{aligned}$$

Here $x_1 = x_2 = \alpha$, $x_3 = \beta$, $x_4 = \gamma$, $n(x_1) = 1$, $n(x_2) = 1$, $n(x_3) = 3(!)$, $n(x_4) = 2$ and $\nu(\alpha, y) = 2$, $\nu(\beta, y) = 3$, $\nu(\gamma, y) = 2$.

Now we define for the interval $[a, b]$

$$I = \{i \mid x_i = a \text{ or } x_i = b \text{ or } a < x_i < b \text{ and } n(x_i) \text{ is even}\},$$
$$J = \{j \mid a < x_j < b \text{ and } n(x_j) \text{ is odd}\}$$

and

$$N(y) = \sum_{i\in I} n(x_i) + \sum_{j\in J} [n(x_j) - 1].$$

The following lemma, proved in [5], is the main tool in our investigation of Eq. (1).

LEMMA 1. *For $\lambda \neq 0$, every solution of equation* (1) *satisfies*

$$N(y) \leqslant n. \tag{5}$$

If $N(y) = n$, then $\nu(b, y)$ and $n - \nu(a, y)$ are both even when $\lambda p(x) \leqslant 0$ and both odd when $\lambda p(x) \geqslant 0$, $L_t y$ changes its sign exactly once between two consecutive zeros of $L_{t-1}y$ in $[a, b]$ and these are the only changes of sign of $L_t y$ in (a, b) which are not zeros of $L_{t-1}y$.

Inequality (5) generalizes Theorems 5.3 and 5.4 of [12].

In the sequel we occasionally apply the operator L to functions which satisfy the boundary conditions of (2) but are not necessarily solutions of (1). The next definitions and notations are needed to count the number of the zeros and the changes of signs of the quasi-derivatives of such functions.

Let $f \in C^n$ and let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r$ be the zeros of $L_0 f,\ldots, L_{n-1} f$ in $[a, b]$, so that common zeros of consecutive quasi-derivatives will be considered as multiple zeros but distinct subscripts will be used for zeros of nonconsecutive derivatives at the same point. (If a quasi-derivative vanishes on some subinterval of $[a, b]$ but not on whole $[a, b]$, we count this subinterval as an isolated zero.) The quasi-derivatives of f are not arranged, of course, in cyclic order, since there is no simple relation between $L_0 f$ and $L_{n-1} f$. For $1 \leqslant h \leqslant n$, we denote by $n_h(x_i, f)$ the number of consecutive quasi-derivatives among $L_0 f,\ldots, L_{h-1} f$ which vanish at x_i, disregarding the zeros of quasi-derivatives of higher order. For example, if $f \in C^4$ and it satisfies

$$(L_0 f)(\alpha) = (L_3 f)(\alpha) = (L_2 f)(\beta) = (L_3 f)(\beta) = 0, \qquad \alpha < \beta,$$

then $x_1 = x_2 = \alpha$, $x_3 = \beta$ and $n_4(x_1) = n_4(x_2) = 1$, $n_4(x_3) = 2$ but $n_3(x_1) = 1$, $n_3(x_3) = 1$.

We define in the interval $[a, b]$

$$I(h) = \{i \mid x_i = a \text{ or } x_i = b \text{ or } a < x_i < b \text{ and } n_h(x_i) \text{ is even}\},$$
$$J(h) = \{j \mid a < x_j < b \text{ and } n_h(x_j) \text{ is odd}\},$$

and

$$N_h(f) = \sum_{i\in I(h)} n_h(x_i, f) + \sum_{j\in J(h)} [n_h(x_j, f) - 1].$$

Let $S(f)$ denote the number of changes of sign of f in (a, b).

LEMMA 2. *Let $f \in C^n$ and let $L_h f \not\equiv 0$ on $[a, b]$, $i \leqslant h \leqslant n$. Then*

$$S(L_h f) \geqslant S(f) + N_h(f) - h. \tag{6}$$

Proof. The proof of Lemma 2 is analogous to that of Lemma 1. The differences follow from the fact that $n_h(x_i)$ is not the exact multiplicity of the zero x_i and so a quasi-derivative may change sign at x_i regardless of the parity of $n_h(x_i)$.

We decompose $I(h)$ and $J(h)$ into disjoint subsets

$$I_t(h) = \{i \in I(h) \mid (L_t f)(x_i) = 0 \text{ but } (L_{t-1} f)(x_i) \neq 0\},$$
$$J_t(h) = \{j \in J(h) \mid (L_t f)(x_j) = 0 \text{ but } (L_{t-1} f)(x_j) \neq 0\},$$

$t = 0,..., h-1$. The set $\{x_i \mid i \in I_t(h) \cup J_t(h)\}$, $1 \leqslant t \leqslant h-1$, consists of all the zeros x_i, $a \leqslant x_i \leqslant b$, of $L_t f$ such that $(L_{t-1} f)(x_i) \neq 0$, while

$$\{x_i \mid i \in I_0(h) \cup J_0(h)\}$$

consists of all the zeros of $L_0 f = \rho_0 f$.

Let x_i be a zero of $L_t f$. Consider the number of consecutive quasi-derivatives among $L_t f,..., L_{h-1} f$ which vanish at x_i. We sum these numbers for all the zeros of $L_t f$ and we denote the result by γ_t. γ_t may be smaller than the total number of zeros of $L_t f$ in $[a, b]$, including multiplicities, since its definition does not take into account the full multiplicity of the zeros of $L_t f$, but only the vanishing of $L_t f,..., L_{h-1} f$.

Assume that there are $m_1,..., m_q$ consecutive quasi-derivatives, starting with $L_{t-1} f$, which vanish respectively at q distinct points of $[a, b]$. Here we count only the quasi-derivatives, $L_{t-1} f,..., L_{h-1} f$. Then $\gamma_{t-1} = m_1 + \cdots + m_q$. At the same points $m_1 - 1,..., m_q - 1$ consecutive quasi-derivatives, starting with $L_t f$, vanish respectively and therefore

$$\gamma_t = (m_1 - 1) + \cdots + (m_q - 1) + \sum_{I_t(h)} n_h(x_i) + \sum_{J_t(h)} n_h(x_j). \tag{7}$$

Between the q different zeros of $L_{t-1} f$, $L_t f$ has at least $q - 1$ changes of sign.

Assume first that the multiplicities of the zeros of $L_t f$ at these changes of sign are not greater than $h - t$. Then the number of the consecutive quasi-derivatives among $L_t f, \ldots, L_{h-1} f$, which vanish at these points is odd and in $J_t(h)$ there are at least $q - 1$ indices. Hence using (7), we obtain that

$$\gamma_t \geqslant m_1 + \cdots + m_q + \sum_{I_t(h)} n_h(x_i) + \sum_{J_t(h)} [n_h(x_j) - 1] - 1$$

or

$$\gamma_t \geqslant \gamma_{t-1} + \sum_{I_t(h)} n_h(x_i) + \sum_{J_t(h)} [n_h(x_j) - 1] - 1, \qquad t = 1, \ldots, h-1. \tag{8}$$

We have

$$\gamma_0 = \sum_{I_0(h)} n_h(x_i) + \sum_{J_0(h)} n_h(x_j).$$

If the multiplicities of the zeros of $L_0 f$ at the points where it changes its sign are not greater than h, then the number of consecutive quasi-derivatives among $L_0 f, \ldots, L_{h-1} f$ which vanish at these points is odd and in $J_0(h)$ there are exactly $S(f)$ indices. Hence

$$\gamma_0 = \sum_{I_0(h)} n_h(x_i) + \sum_{J_0(h)} [n_h(x_j) - 1] + S(f). \tag{9}$$

Adding the inequalities (8) and using (9), we obtain

$$\gamma_{h-1} \geqslant S(f) + \sum_{t=0}^{h-1} \left\{ \sum_{I_t(h)} n_h(x_i) + \sum_{J_t(h)} [n_h(x_j) - 1] \right\} - (h - 1)$$

or

$$\gamma_{h-1} \geqslant S(f) + N_h(f) - (h - 1).$$

γ_{h-1} is, by definition, the number of distinct zeros of $L_{h-1} f$ in $[a, b]$. Between any two of them $L_h f$ changes its sign, hence the number of changes of sign of $L_h f$, which are not zeros of $L_{h-1} f$ is at least $S(f) + N_h(f) - h$.

If x_k is a point where $L_0 f$ changes its sign but more than h consecutive quasi-derivatives vanish there, then $n_h(x_k) = h$. Similarly, if between two consecutive zeros of $L_{t-1} f$, $L_t f$ changes its sign at a point x_k but more than $h - t$ quasi-derivatives, starting with $L_t f$ vanish at x_k, then $n_h(x_k) = h - t$. In both cases $n_h(x_k)$ may be even and $L_h f$ may change its sign. Now $k \notin J(h)$, but $L_h f$ changes its sign at a point which is a zero of $L_{h-1} f$. Hence also in this case $L_h f$ changes its sign in (a, b) at least $S(f) + N_h(f) - h$ times. Therefore (6) is proved.

COROLLARY 1. *If $f \in C^n$, $Lf \not\equiv 0$ and f satisfies the boundary conditions of* (2), *then*

$$S(Lf) \geqslant S(f).$$

Moreover, $S(Lf)$ is not smaller than the number of zeros of f in $[a, b]$ which are not given in the boundary conditions (2).

Proof. If f satisfies the boundary conditions of (2), then $N_n(f) \geqslant n$ and the inequality follows by (6). Let f have a zero of multiplicity m which is not given in (2). The even part of m is added to the term $N_n(f)$ in (6) and if m is odd, one is added to $S(f)$. Thus the right hand side of (6) is not smaller than the number of the zeros of f which are not given in the boundary conditions.

When y is a solution of (1), $S(Ly) = S(y)$, hence (6) implies $N_n(y) \leqslant n$. However, this inequality is weaker than (5), since $n_n(x_i) \leqslant n(x_i)$ and $N_n(y) \leqslant N(y)$, because $n(x_i)$ counts zeros in cyclic order.

3. Existence of Eigenvalues

We apply Lemma 1 to prove some properties of eigenvalues of (2), if such eigenvalues exist.

Let $y(x)$ be an eigenfunction of (2) corresponding to the nonzero real eigenvalue λ. By the boundary conditions of (2), $N(y) \geqslant n$ and by Lemma 1, $N(y) \leqslant n$. Therefore $N(y) = n$, so $\lambda p(x) \leqslant 0$ ($\lambda p(x) \geqslant 0$) when $\nu(b)$ is even (odd), i.e. $(-1)^{\nu(b)} \lambda p(x) \leqslant 0$.

Let $y_0, \ldots, y_{n-1}$, be a set of n linearly independent solutions of (1), e.g. the set determined by $(L_i y_j)(a) = \delta_{ij}$, $i, j = 0, \ldots, n-1$. The eigenvalues of (2) are the zeros of the function $\Delta(\lambda)$ defined by the $n \times n$ determinant

$$\Delta(\lambda) = \begin{vmatrix} (L_{i_1} y_0)(a), & \ldots, & (L_{q_i} y_0)(x_i), & \ldots, & (L_{j_{\nu(b)}} y_0)(b) \\ (L_{i_1} y_1)(a), & \ldots, & (L_{q_i} y_1)(x_i), & \ldots, & (L_{j_{\nu(b)}} y_1)(b) \\ \vdots & & \vdots & & \vdots \\ (L_{i_1} y_{n-1})(a), & \ldots, & (L_{q_i} y_{n-1})(x_i), & \ldots, & (L_{j_{\nu(b)}} y_{n-1})(b) \end{vmatrix}.$$

$y_0(x), \ldots, y_{n-1}(x)$ are entire analytic functions of λ and therefore $\Delta(\lambda)$ is also an entire function. Now it follows that the eigenvalues of (2), if they exist, have no finite accumulation point. For, if the eigenvalues, which are zeros of $\Delta(\lambda)$, had a finite accumulation point, the entire function $\Delta(\lambda)$ would be identically zero and every λ would be an eigenvalue. But (2) cannot have both positive and negative eigenvalues. Thus we have proved

Corollary 2. *The set of all eigenvalues of* (2) *has no finite accumulation point. The sign of the nonzero real eigenvalurs of* (2) *is determined by* $(-1)^{\nu(b)} \lambda p(x) \leqslant 0$.

Henceforth λ will be always real.

To establish the existence of eigenvalues of (2), we consider a solution of (1) which satisfies only $n - 1$ of the boundary conditions of (2). So we delete one of the boundary conditions of (2) posed at one of the endpoints, e.g. the condition $(L_{j_1} y)(b) = 0$.

LEMMA 3. *Given the $n-1$ boundary conditions*

$$\begin{aligned} (L_t y)(x_i) &= 0, \qquad q_i \leqslant t \leqslant q_i + 2m_i - 1, \quad a < x_i < b, \\ (L_t y)(a) &= 0, \qquad t \in \{i_1, i_2, ..., i_{\nu(a)}\}, \\ (L_t y)(b) &= 0, \qquad t \in \{j_2, j_3, ..., j_{\nu(b)}\}. \end{aligned} \tag{10}$$

(a) *For $\lambda \neq 0$, Eq.* (1) *has an essentially unique solution $y(x, \lambda)$ satisfying* (10).

(b) *Two consecutive quasi-derivatives of $y(x, \lambda)$ cannot have zeros at the same point of (a, b) in addition to the zeros posed in* (10). *Moreovert at most one of the quasi derivatives $L_0 y(x, \lambda), ..., L_{n-1} y(x, \lambda)$ can have a zero at most at one of the two endpoints a, b in addition to the zeros posed in* (10).

(c) *$L_t y(x, \lambda)$ is an entire analytic function of λ. The points where $L_t y(x, \lambda)$ changes its sign, considered as functions of λ, are continuous.*

Proof. First we note that for $\lambda = 0$, equation (1) may have several independent solutions satisfying (10). These solutions will be studied later.

(a) Every linear homogeneous differential equation of n-th order has a nontrivial solution which satisfies $n-1$ given homogeneous boundary conditions. For equation (1) and boundary conditions (10), this solution is unique. Every solution of (1) and (10) satisfies $N(y) \geqslant n-1$. If there are such two independent solutions y_1, y_2, we can choose a linear combination $c_1 y_1 + c_2 y_2$, so that one of its quasi-derivatives has an additional zero at one of the endpoints. Then $N(c_1 y_1 + c_2 y_2) = n$ and the additional zero can be chosen so that the parity of $\nu(b, c_1 y_1 + c_2 y_2)$ contradicts Lemma 1.

(b) If two consecutive quasi-derivatives of $y(x, \lambda)$ have zeros at the same point of $[a, b]$ or $L_0 y(x, \lambda), ..., L_{n-1} y(x, \lambda)$ have two zeros at the endpoints a or b, in addition to the zeros posed in (10), then

$$N(y(x, \lambda)) \geqslant 2 + \Big[\nu(a) + \sum 2m_i + (\nu(b) - 1)\Big] = n + 1$$

would contradict Lemma 1. Therefore at the points of (a, b) which are not given in (10), all the quasi-derivatives may have only simple zeros and at the points x_i, given by (10), at most one more consecutive quasi-derivative may vanish. For example, let $(L_t y)(x_i) = 0$, $q_i \leqslant t \leqslant q_i + 2m_i - 1$, be a sequence of boundary conditions of (10). Now, if $L_{q_i + 2m_i} y(x, \lambda)$ vanishes at x_i for some λ, it has a simple zero at x_i. On the other hand, if $L_{q_i - 1} y(x, \lambda)$ vanishes at x_i, it has at x_i a zero of multiplicity $2m_i + 1$.

(c) We first show that

$$y(x, \lambda) = \begin{vmatrix} (L_{i_1} y_0)(a), & ..., & (L_{j_2} y_0)(b), & ..., & (L_{j_{\nu(b)}} y_0)(b), & y_0(x) \\ \vdots & & \vdots & & \vdots & \vdots \\ (L_{i_1} y_{n-1})(a), ..., & & (L_{j_2} y_{n-1})(b), ..., & & (L_{j_{\nu(b)}} y_{n-1})(b), & y_{n-1}(x) \end{vmatrix}.$$

Of course, the determinant is a solution of (1) which satisfies (10). We have only to show that for $\lambda \neq 0$ it is not the trivial solution. If it were identically zero, the minor consisting of the first $n - 1$ columns would be of rank less than $n - 1$. This would contradict the uniqueness proved before, of the solution which satisfies (10). So the solution $y(x, \lambda)$ may be the trivial one only for $\lambda = 0$. $y(x, \lambda)$ is an entire function of λ for every x, since $y_0, \ldots, y_{n-1}$ are such functions. By the intermediate value property it follows that the points where $L_t y(x, \lambda)$ changes its sign, in particular the simple zeros, are continuous functions of λ.

Reexamining the proof, one sees that the main point is that every solution of (10) satisfies $N(y) \geqslant n - 1$. So any other boundary condition of (2) at a or at b could be deleted, obtaining thus a similar solution $\hat{y}(x, \lambda)$ for which Lemma 3 holds.

As $L_{j_1} y(b, \lambda) = \pm \Delta(\lambda)$, it follows that $\lambda \neq 0$ is an eigenvalue and that $y(x, \lambda)$ is an eigenfunction for those values λ for which $L_{j_1} y(x, \lambda)$ vanishes at b, and all the nonzero eigenvalues and the corresponding eigenfunctions are obtained from $L_{j_1} y(b, \lambda) = 0$. Using this remark and Lemma 3, we deduce the following theorem:

THEOREM 1. *To every nonzero eigenvalue of* (2) *corresponds a unique eigenfunction. An eigenfunction y and its quasi-derivatives may have at points of* (a, b), *for which no boundary conditions are posed in* (2), *only simple zeros. At points* x_i *of* (a, b), *for which boundary conditions are posed in* (2), *at most one consecutive derivative of y can have a zero, in addition to the zeros posed in* (2). *The only quasi-derivatives of y which vanish at the endpoints a and b are those determined in* (2). *Between any two consecutive zeros of* L_{t-1} *in* $[a, b]$, L_t *changes its sign exactly once, and these are the only zeros of* L_t *in* $[a, b]$ *which are not given in* (2).

The next lemma describes the behavior of the zeros of $y(x, \lambda)$ as $\lambda \to \infty$.

LEMMA 4. *In every subinterval of* $[a, b]$, $y(x, \lambda)$ *and all its quasi-derivatives change their signs if* $|\lambda|$ *is large enough and its sign is determined by* $\lambda(-1)^{\nu(b)} p(x) \leqslant 0$. *If* $\nu(b) > 1$, *the assertion holds for large values of* λ *of both signs.*

Proof. Without loss of generality it may be assumed that $\lambda > 0$ and $(-1)^{\nu(b)} p(x) \leqslant 0$, because equation (1) can be written as $Ly + (-\lambda)(-p(x))y = 0$.

We shall prove our lemma for a larger family of solutions $y_\lambda(x)$, $0 < \lambda < \infty$, which have the following property:

Property P. If $p(x) \leqslant 0$ then for every $\lambda > 0$, one of the quasi-derivatives of $y_\lambda(x)$ vanishes at b and if $(-1)^n p(x) \leqslant 0$ then for every $\lambda > 0$, one of the quasi-derivatives of $y_\lambda(x)$ vanishes at a.

Obviously, if $p(x) \geqslant 0$ and n is even, every family of solutions has property P.

The solutions $y(x, \lambda)$ have property P. At least one quasi-derivative of $y(x, \lambda)$

vanishes at a since $\nu(a) \geqslant 1$. If $p(x) \leqslant 0$ then $(-1)^{\nu(b)}p(x) \leqslant 0$ implies that $\nu(b)$ is even, and since $\nu(b) \geqslant 1$, we have $\nu(b) \geqslant 2$. Hence $\nu(b) - 1 \geqslant 1$ and by (10), at least one quasi-derivative of $y(x, \lambda)$ vanishes at b. If $\nu(b) > 1$, $y(x, \lambda)$ has property P regardless of the sign of $\lambda(-1)^{\nu(b)}p(x)$ and the lemma holds whenever $|\lambda| \to \infty$.

The origin of property P will be clarified in the course of the proof.

First we note that it is enough to show that $y_\lambda(x)$ changes its sign on every given interval for λ sufficiently large. For its quasi-derivatives the same holds true by Rolle's theorem.

Assume on the contrary that $y_{\lambda_i}(x) \geqslant 0$ in $[\alpha, \beta] \subset [a, b]$, for a sequence $\lambda_i \to \infty$. In the sequel the index i will be deleted. We normalize $y_\lambda(x)$ at $x = \alpha$ by

$$\sum_{i=0}^{n-1} (L_i y_\lambda)^2(\alpha) = 1. \tag{11}$$

If $y_\lambda(x) = \sum_{i=0}^{n-1} c_i(\lambda)\, y_i(x)$, where $y_0, \ldots, y_{n-1}$ are the n solutions of (1) defined by $(L_i y_j)(\alpha) = \delta_{ij}$, this normalization is equivalent to $\sum_{i=0}^{n-1} c_i^2(\lambda) = 1$.

Integrating the equality $Ly_\lambda = \rho_n(L_{n-1}y_\lambda)'$ between α and x we have

$$(L_{n-1}y_\lambda)(x) = \int_\alpha^x \frac{(Ly_\lambda)(t_n)}{\rho_n(t_n)}\, dt_n + (L_{n-1}y_\lambda)(\alpha).$$

Repeating this process $n - q$ times and substituting $Ly_\lambda = -\lambda p y_\lambda$, we obtain

$$(L_q y_\lambda)(x) + \lambda \int_\alpha^x \frac{dt_{q+1}}{\rho_{q+1}(t_{q+1})} \int_\alpha^{t_{q+1}} \frac{dt_{q+2}}{\rho_{q+2}(t_{q+2})} \cdots \int_\alpha^{t_{n-1}} \frac{p(t_n)}{\rho_n(t_n)}\, y_\lambda(t_n)\, dt_n$$
$$= \underbrace{\sum_{i=q}^{n-1} (L_i y_\lambda)(\alpha) \int_\alpha^x \frac{dt_{q+1}}{\rho_{q+1}(t_{q+1})} \cdots \int_\alpha^{t_{i-1}} \frac{dt_i}{\rho_i(t_i)}}_{i-q \text{ integrations}}. \tag{12}$$

We estimate the absolute value of the right hand side.

$$\left| \sum_{i=q}^{n-1} (L_i y_\lambda)(\alpha) \int_\alpha^x \frac{dt_{q+1}}{\rho_{q+1}} \int_\alpha^{t_{q+1}} \frac{dt_{q+2}}{\rho_{q+2}} \cdots \int_\alpha^{t_{i-1}} \frac{dt_i}{\rho_i} \right|$$
$$\leqslant \left\{ \sum_{i=q}^{n-1} (L_i y_\lambda)^2(\alpha) \right\}^{1/2} \left\{ \sum_{i=q}^{n-1} \left[\int_\alpha^x \frac{dt_{q+1}}{\rho_{q+1}} \cdots \int_\alpha^{t_{i-1}} \frac{dt_i}{\rho_i} \right]^2 \right\}^{1/2} \tag{13}$$

Since the ρ_i's are continuous and positive in $[a, b]$, $m \leqslant 1/\rho_i \leqslant M$. The right hand side increases if $1/\rho_i(t_i)$ is replaced by M, the interval $[\alpha, x]$ is replaced by

$[a, b]$ and the summation is extended for $0 \leqslant i \leqslant n-1$. Hence, using (11), (12) and (13), we obtain

$$\left| (L_q y_\lambda)(x) + \lambda \int_\alpha^x \frac{dt_{q+1}}{\rho_{q+1}(t_{q+1})} \int_\alpha^{t_{q+1}} \frac{dt_{q+2}}{\rho_{q+2}(t_{q+2})} \cdots \int_\alpha^{t_{n-1}} \frac{p(t_n)}{\rho_n(t_n)} y_\lambda(t_n)\, dt_n \right|$$
$$\leqslant 1 \cdot \left\{ \sum_{i=0}^{n-1} \left[M^{i-q} \frac{(b-a)^{i-q}}{(i-q)!} \right]^2 \right\}^{1/2} = A, \tag{14}$$

for every $\alpha \leqslant x \leqslant \beta$, $0 \leqslant q \leqslant n-1$ and every $\lambda > 0$.

There exists an index q_0 such that the two terms inside the absolute value of (14) have the same sign for $x = \beta$. If $p(x) \geqslant 0$, the choice is $q_0 = 0$, since $y_\lambda(x)$, $(L_0 y_\lambda)(x)$, $p(x)$ and λ are non-negative for $\alpha \leqslant x \leqslant \beta$. If $p(x) \leqslant 0$ then the integral in (14) is negative for $\alpha < x \leqslant \beta$ and for all $0 \leqslant q \leqslant n-1$. One of the quasi-derivatives $(L_q y_\lambda)(\beta)$ must also be negative. If not, then from $(L_q y_\lambda)(\beta) \geqslant 0$, $q = 0,\dots, n-1$ and $p(x) \leqslant 0$ it follows like in Lemma (2.1) of [10] that $(L_q y_\lambda)(x) > 0$ in (β, ∞), $q = 0,\dots, n-1$. But when $p(x) \leqslant 0$, one of the quasi-derivatives of $y_\lambda(x)$ vanishes at b by property P. This shows that one of the quasi-derivatives, $(L_{q_0} y_\lambda)(\beta)$, is negative as the integral. (When $\beta = b$, we may choose the quasi-derivative which vanishes at b to be $L_{q_0} y_\lambda$.) Thus it is proved that there exists a q_0 such that the two terms in (14) are of the same sign at $x = \beta$. For this q_0

$$|(L_{q_0} y_\lambda)(\beta)| + \lambda \left| \int_\alpha^\beta \frac{dt_{q_0+1}}{\rho_{q_0+1}} \int_\alpha^{t_{q_0+1}} \frac{dt_{q_0+2}}{\rho_{q_0+2}} \cdots \int_\alpha^{t_{n-1}} \frac{p(t_n)}{\rho_n(t_n)} y_\lambda(t_n)\, dt_n \right| \leqslant A \tag{15}$$

for every $\lambda > 0$. Note, that up to this point we have used only part of property P, that is, if $p(x) \leqslant 0$, then a quasi-derivative of $y_\lambda(x)$ vanishes at b.

Now we prove similar inequalities for every $0 \leqslant q \leqslant n-1$. If the domain of integration in the integral in (15) is changed from the simplex

$$\alpha \leqslant t_n \leqslant t_{n-1} \leqslant \cdots \leqslant t_{q_0+1} \leqslant \beta$$

into the smaller domain $\alpha \leqslant t_n \leqslant \delta \leqslant t_{n-1} \leqslant \cdots \leqslant t_{q_0+1} \leqslant \beta$, where δ is a fixed number, $\alpha < \delta < \beta$, we obtain (since the integrand has a constant sign) that

$$\lambda \left| \int_\delta^\beta \frac{dt_{q_0+1}}{\rho_{q_0+1}} \int_\delta^{t_{q_0+1}} \frac{dt_{q_0+2}}{\rho_{q_0+2}} \cdots \int_\delta^{t_{n-2}} \frac{dt_{n-1}}{\rho_{n-1}} \int_\alpha^\delta \frac{p(t_n)}{\rho_n(t_n)} y_\lambda(t_n)\, dt_n \right| \leqslant A$$

for every $\lambda > 0$. Since $1/\rho_i \geqslant m$,

$$\lambda \cdot m^{n-q_0-1} \frac{(\beta-\delta)^{n-q_0-1}}{(n-q_0-1)!} \left| \int_\alpha^\delta \frac{p(t_n)}{\rho_n(t_n)} y_\lambda(t_n)\, dt_n \right| \leqslant A$$

or

$$\lambda \left| \int_\alpha^\delta \frac{p(t)}{\rho_n(t)} y_\lambda(t)\, dt \right| \leqslant B \tag{16}$$

for each $\lambda > 0$. From (16) it follows by q integrations that

$$\lambda \left| \int_\alpha^x \frac{dt_{q+1}}{\rho_{q+1}} \int_\alpha^{t_{q+1}} \frac{dt_{q+2}}{\rho_{q+2}} \cdots \int_\alpha^{t_{n-1}} \frac{p(t_n)}{\rho_n(t_n)} y_\lambda(t_n)\, dt_n \right|$$

is bounded by an absolute constant for every $\lambda > 0$ and for every x satisfying $\alpha \leqslant x \leqslant \delta$ and $0 \leqslant q \leqslant n-1$. Therefore the same holds for the second term in (14):

$$|(L_q y_\lambda)(x)| \leqslant C \tag{17}$$

for $\lambda > 0$, $\alpha \leqslant x \leqslant \delta$ and $0 \leqslant q \leqslant n-1$.

Using (16) and (17), we shall prove that the $n-1$ quasi-derivatives $L_0 y_\lambda, \ldots, L_{n-2} y_\lambda$ tend uniformly to zero in $[\alpha, \delta]$ for a certain sequence $\lambda \to \infty$. Let $0 \leqslant q \leqslant n-2$. By (17) we have that

$$|(L_q y_\lambda)'| = \left| \frac{1}{\rho_{q+1}} L_{q+1} y_\lambda \right| \leqslant MC. \tag{18}$$

By (17) and (18), $\{L_q y_\lambda \mid \lambda > 0\}$ is a family of uniformly bounded and equicontinuous functions. Hence, by Ascoli's theorem, it contains a uniformly convergent subsequence. Let $\lambda_i \to \infty$ be a subsequence such that the $n-1$ quasi-derivatives $L_0 y_{\lambda_i}, \ldots, L_{n-2} y_{\lambda_i}$ converge uniformly on $[\alpha, \delta]$. Since $y_{\lambda_i}(x) \geqslant 0$ on $[\alpha, \delta]$, (16) implies that $y_{\lambda_i}(x)$ converges uniformly to 0 when $\lambda_i \to \infty$. If the uniform limit of $L_q y_{\lambda_i}(x)$, $1 \leqslant q \leqslant n-2$, would be nonzero on some subinterval of $[\alpha, \delta]$, we would obtain by q integrations a contradiction to the convergence of $y_{\lambda_i}(x)$ to 0.

The quasi-derivative $L_{n-1} y_\lambda$, on the other hand, converges pointwise to zero on (α, δ), and not necessarily uniformly. This difference follows from the fact that its derivative $(L_{n-1} y_\lambda)' = -\lambda p y_\lambda / \rho_n$ is not necessarily bounded as $\lambda \to \infty$. If $|L_{n-1} y_{\lambda_i}|$ is bounded from below on some fixed subinterval of $[\alpha, \delta]$ by a positive constant, we would obtain by an integration a contradiction to the convergence of $(L_{n-2} y_{\lambda_i})(x)$ to 0. Thus $L_{n-1} y_{\lambda_i}(x)$ tends to zero in a dense subset of $[\alpha, \delta]$. But $(L_{n-1} y_{\lambda_i})' = -\lambda p y_{\lambda_i} / \rho_n$ has a fixed sign $[\alpha, \delta]$, which means that $L_{n-1} y_{\lambda_i}$ is monotone there and $|L_{n-1} y_{\lambda_i}|$ may have at most two local maxima, at α and at δ. Therefore $L_{n-1} y_{\lambda_i}$ tends pointwisely to zero in (α, δ) (see Remark 2).

Consider the function

$$r(y, x) = \left\{ \sum_{i=0}^{n-1} (L_i y)^2(x) \right\}^{1/2}.$$

The normalization (11) of $y_\lambda(x)$ at α means $r(y_\lambda, \alpha) = 1$, and the former parts of the proof imply that $r(y_\lambda, x)$ converges pointwisely to zero in (α, δ). Therefore $r(y_\lambda, x)/r(y_\lambda, \alpha) \to 0$ as $\lambda \to \infty$ for every $\alpha < x < \delta$.

The last limit was obtained for $y_\lambda(x)$ normalized at α. But the quotient is homogeneous and independent of the normalization, therefore the limit is the same one for all the multiples of y_λ. If y_λ is normalized at $x = s$, $\alpha \leqslant s < \delta$, we obtain some multiple of $y_\lambda(x)$, say $c(\lambda)\, y_\lambda(x)$. This solution, as the previous one, satisfies

$$r(c(\lambda) y_\lambda\,, t)/r(c(\lambda) y_\lambda\,, s) = r(y_\lambda\,, t)/r(y_\lambda\,, s) \to 0 \tag{19}$$

as $\lambda \to \infty$ for every $\alpha \leqslant s < t < \delta$.

Now we shall use the other part of property P, which has no yet been used, to show that (19) is impossible. This will be done through reversing the order of the points by means of the transformation $\tilde{x} = -x$. Equation (1) is transformed into

$$\tilde{L}\tilde{y} + (-1)^n \lambda \tilde{p}(\tilde{x})\tilde{y} = 0 \tag{20}$$

in the interval $[-b, -a]$. Here $\tilde{y}(\tilde{x}) = y(x)$, $(\tilde{L}_q\tilde{y})(\tilde{x}) = (-1)^q(L_q y)(x)$, $\tilde{p}(\tilde{x}) = p(x)$ and $r(\tilde{y}, \tilde{x}) = r(y, x)$. The points $a \leqslant \alpha < s < t < \delta < \beta \leqslant b$ are transformed into $-b \leqslant -\beta < -\delta < \tilde{t} < \tilde{s} < -\alpha \leqslant -a$, where $\tilde{t} = -t$, $\tilde{s} = -s$.

By property P either $(-1)^n\tilde{p}(\tilde{x}) \geqslant 0$ or one of the quasi-derivatives $(\tilde{L}_q\tilde{y})(\tilde{x})$ vanishes at the right endpoint $\tilde{x} = -a$ of $[-b, -a]$. These properties of equation (20) are the same as those which has been used before for equation (1). Therefore, if $\tilde{y}_\lambda(\tilde{x})$ is normalized at $\tilde{x} = \tilde{t}(=-t)$, the solution $d(\lambda)\tilde{y}_\lambda(\tilde{x})$ so obtained satisfies, as in (19),

$$r(d(\lambda)\tilde{y}_\lambda\,, \tilde{s})/r(d(\lambda)\tilde{y}_\lambda\,, \tilde{t}) = r(\tilde{y}_\lambda\,, \tilde{s})/r(\tilde{y}_\lambda\,, \tilde{t}) \to 0 \tag{21}$$

when $\lambda \to \infty$ for every $-\delta < \tilde{t} < \tilde{s} < -\alpha$. Since $r(\tilde{y}_\lambda\,, \tilde{s}) = r(y_\lambda\,, s)$ and $r(\tilde{y}_\lambda\,, \tilde{t}) = r(y_\lambda\,, t)$, (21) contradicts (19).

This completes the proof of Lemma 3.

Remark 1. From the proof of Lemma 3 follows that property P excludes families of solutions which converge monotonously to zero or to infinity. Property P is necessary, in general. Take the equation $y^{(3)} + \mu^3 y = 0$, $\mu > 0$. It is necessary to assume that a quasi-derivative of $y_\lambda(x)$ vanishes at the left endpoint, because the solution $y(x) = \exp(-\mu x) - \exp(-\mu x/2)\cos(3^{1/2}\mu x/2)$ vanishes at $x = 0$ but it is positive on $(-\infty, 0)$ for every $\mu > 0$. When $p(x) \leqslant 0$ and n is even, it is necessary to assume that quasi-derivatives of $y_\lambda(x)$ vanish at both endpoints. For example, the equation $y^{(4)} - \mu^4 y = 0$ has the solution $y = \sinh(\mu x) - \sin(\mu x)$ with a triple zero at $x = 0$ but $y^{(i)}(x) \neq 0$, $i = 0,\dots, 3$ in $(-\infty, 0) \cup (0, \infty)$ for every $\mu \neq 0$.

Remark 2. The uniform convergence of $L_0 y_\lambda\,,\dots, L_{n-2}y_\lambda$ on $[\alpha, \delta]$ and the pointwise convergence of $L_{n-1}y_\lambda$ on (α, δ) is illustrated by the equation $y^{(3)} + \mu^3 y = 0$ and its solution $y = (1 + \mu^2 + \mu^4)^{-1/2}\exp(-\mu x)$, which is normalized at $x = 0$. Here, $y, y' \to 0$ uniformly on $[0, \infty]$ and $y'' \to 0$ pointwisely on $(0, \infty)$ when $\mu \to \infty$.

Now we are able to prove the existence of eigenvalues of (2).

THEOREM 2. *The boundary value problem* (2) *has an infinite sequence of real eigenvalues. Given an arbitrary subinterval of* $[a, b]$, *then an eigenfunction which belongs to a sufficiently large eigenvalue, changes its sign in that subinterval.*

Proof. If (2) has eigenvalues, then by Corollary 2 they satisfy

$$(-1)^{\nu(b)}\lambda p(x) \leqslant 0.$$

For convenience we may assume that $(-1)^{\nu(b)}p(x) \leqslant 0$ and look only for positive eigenvalues. This is possible since equation (1) may be written as $Ly + (-\lambda)(-p(x))y = 0$. So let λ vary in $(0, \infty)$.

The eigenfunctions of (2) are functions $y(x, \lambda)$ for values of λ for which $L_{j_1}y(x, \lambda)$ vanishes at b. When $\lambda \to \infty$, then, by Lemma 4, the number of the zeros of $L_0y(x, \lambda),..., L_{n-1}y(x, \lambda)$ in $[a, b]$, $\sum_I n(x_i) + \sum_J n(x_j)$, tends to infinity. On the other hand, $N(y(x, \lambda)) = \sum_I n(x_i) + \sum_J [n(x_j) - 1] \leqslant n$ for every λ. Therefore, the number of the zeros x_j of $L_0y(x, \lambda),..., L_{n-1}y(x, \lambda)$ such that $n(x_j) = 1$, tends to infinity. By using Rolle's theorem and equation (1) we obtain that the number of the zeros of each quasi-derivative tends to infinity.

We prove that the number of the zeros of a quasi-derivative $L_jy(x, \lambda)$ in (a, b) which are not given in (10), can vary as λ varies in $(0, \infty)$ only when a simple zero enters (a, b) or leaves it through the endpoint b.

Let us review what are the possible zeros of $L_jy(x, \lambda)$ in (a, b).

A zero of $L_jy(x, \lambda)$ in (a, b) is determined in (10) if the condition

$$(L_ty)(x_i) = 0, \qquad q_i \leqslant t \leqslant q_i + 2m_i - 1, \qquad a < x_i < b \tag{22}$$

appears in (10) and if $q_i \leqslant j \leqslant q_i + 2m_i - 1$. In this case, $L_jy(x, \lambda)$ has at x_i a zero of multiplicity $q_i + 2m_i - j$ for every λ.

Two consecutive quasi-derivatives of $y(x, \lambda)$ cannot have zeros at the same point of (a, b) in addition to the zeros posed in (10). Therefore at a point which is not given in (10), $L_jy(x, \lambda)$ may have only a simple zero. At a point x_i which is given in (10), at most one more quasi-derivative may vanish. If for $\lambda = \lambda_0$,

$$L_ty(x_i, \lambda_0) = 0, \qquad q_i \leqslant t \leqslant q_i + 2m_i, \tag{23}$$

then $L_jy(x, \lambda_0)$ has at x_i a zero exactly of multiplicity $q_i + 2m_i - j + 1$.

If $j = q_i - 1$ and if

$$(L_ty)(x_i, \lambda_0) = 0, \qquad q_i - 1 \leqslant t \leqslant q_i + 2m_i - 1,$$

then $L_jy(x, \lambda_0) = L_{q_i-1}y(x, \lambda_0)$ has at x_i a zero exactly of multiplicity $2m_i + 1$.

First we show that two zeros of $L_jy(x, \lambda)$ in $[a, b]$ which are not given in (10), cannot meet as λ varies. Assume on the contrary that when $\lambda \to \lambda_0$, two zeros which are not given in (10) meet at c, $a \leqslant c \leqslant b$. If c is not given in (10) then $L_jy(c, \lambda_0) = L_{j+1}y(c, \lambda_0) = 0$, and this is impossible by Lemma 3. If c appears in

(10), then it is easy to show by Rolle's theorem that the number of consecutive quasi-derivatives of $y(x, \lambda_0)$ which vanish at c is greater by 2 from the number of those given in (10). Again this is impossible by Lemma 3. Note that if the conditions given in (10) are, for example, $(L_t y)(x_i) = 0$, $j \leqslant t \leqslant n-2$, we have to use the cyclic count of the zeros.

Now we prove that the zeros of $L_j y(x, \lambda)$ in (a, b) which are not given in (10) are continuous functions of λ and they can be extended as long as they do not meet one of the endpoints.

Let $x(\lambda)$, $a < x(\lambda) < b$ be a zero of $L_j y(x \lambda)$, which is not given in (10). If $x(\lambda_0)$ is different of the points given in (10), then it is a simple zero. By the implicit functions theorem, $x(\lambda)$ is a differentiable function of λ defined in some neighborhood of λ_0.

Now we assume that $x(\lambda_0)$ coincides with a zero x_i given in (10). That is, condition (22) is given at x_i and $q_i \leqslant j \leqslant q_i + 2m_i - 1$ and in addition, $y(x, \lambda_0)$ satisfies (23) at $x_i = x(\lambda_0)$. By Lemma 3, no additional consecutive quasi-derivative of $y(x, \lambda_0)$ can vanish at x_i. Hence $L_{q_i} y(x, \lambda_0)$ has at $x_i = x(\lambda_0)$ a zero exactly of multiplicity $2m_i + 1$, i.e. $L_{q_i} y(x, \lambda_0)$ changes its sign. $L_{q_i} y(x, \lambda)$ depends continuously on λ, so by the intermediate value property $L_{q_i} y(x, \lambda)$ changes its sign in a given neighborhood of $x(\lambda_0)$, when λ is near λ_0. If for $\lambda \neq \lambda_0$ the change of sign of $L_{q_i} y(x, \lambda)$ coincides with x_i, the additional zero of $L_j y(x, \lambda)$ is constant, hence continuous. If the change of sign of $L_{q_i} y(x, \lambda)$ is different from x_i, then by Rolle's therorem each $L_t y(x, \lambda)$, $q_i \leqslant t \leqslant q_i + 2m_i$, changes its sign near x_i. In particular, $L_j y(x, \lambda)$ has a zero in a given neighborhood of $x_i = x(\lambda_0)$ when λ is near λ_0. By Lemma 3, this zero is a simple one. Moreover, in a small neighborhood of $x_i = x(\lambda_0)$ there is a unique such zero, since, as we proved before, two zeros which are not given in (10) cannot coincide as $\lambda \to \lambda_0$.

Thus we have proved that if $a < x(\lambda_0) < b$, then $x(\lambda)$, the zero of $L_j y(x, \lambda)$ which is not given in (10), is defined in a neighborhood of λ_0 and it is continuous there.

Since two zeros of $L_j y(x, \lambda)$, which are not given in (10), do not meet, $x(\lambda)$ can be extended uniquely as λ varies as long as it does not meet one of the endpoints. For the same reason, the number of zeros of $L_j y(x, \lambda)$ in (a, b), which are not given in (10), can vary only when one simple zero enters (a, b) or leaves it through one of the endpoints.

If a simple zero of $L_j y(x, \lambda)$ meets a when $\lambda \to \lambda_0 \neq 0$, then exactly one more quasi-derivative of $y(x, \lambda_0)$ vanishes at a, either if $(L_j y)(a) = 0$ appears in (10) or not. But then $y(x, \lambda_0)$ satisfies $\nu(a, y) = \nu(a) + 1$, $\nu(b, y) = \nu(b) - 1$ and $N(y(x, \lambda_0)) = n$, contradicting Lemma 1. Hence a zero of $L_j y(x, \lambda)$ which is not given in (10) can appear in (a, b) or disappear from it as λ varies only when one simple zero enters (a, b) or leaves it through the endpoint b.

This conclusion holds in particular for $L_{j_1} y(x, \lambda)$. But if a zero of $L_{j_1} y(x, \lambda)$ meets b, then $y(x, \lambda)$ becomes an eigenfunction of (2).

For a fixed λ_0, $L_{j_1} y(x, \lambda_0)$ has a certain number of zeros in (a, b) which are not given in (10), and when $\lambda \to \infty$, their number increases indefinitely. Therefore an infinite sequence of zeros of $L_{j_1} y(x, \lambda)$ enter (a, b) through b when $\lambda \to \infty$ (and a part of them perhaps leaves (a, b)). So the existence of an infinite sequence of eigenfunctions of (2) is proved.

The assertion about the zeros of the eigenfunctions which belong to sufficiently great eigenvalues follows from the property of the zeros of $y(x, \lambda)$, which was proved in Lemma 4. In particular, if $x(\lambda)$ is one of the zeros of $L_j y(x, \lambda)$ which is not given in (10), then $x(\lambda) \to a$ when $\lambda \to \infty$. Now the proof of the theorem is completed.

From our former considerations it is clear why the number of the boundary conditions in (2), posed at points x_i, $a < x_i < b$, was required to be even. Nevertheless, the proofs of Lemma 4 and Theorem 2 may be applied also for different boundary conditions. For example, if one zero of odd multiplicity is imposed on a quasi-derivative at one of the points of (a, b), the corresponding boundary value problem has infinitely many eigenvalues of both signs. So assume that one of the boundary conditions is

$$(L_t y)(x_l) = 0, \qquad q_l \leqslant t \leqslant q_l + n_l - 1, \qquad a < x_l < b,$$

where n_l is odd while the other boundary conditions are like those in (2). Let $y(x, \lambda)$ be the solution of equation (1) which satisfies $n - 1$ out of the n boundary conditions, omitting the condition $(L_{q_l} y)(x_l) = 0$. Since $n_l - 1$ is even, we have

$$N(y(x, \lambda)) \geqslant \nu(a) + \sum_{i \neq l} n_i + (n_l - 1) + \nu(b) = n - 1.$$

Therefore $y(x, \lambda)$ is determined up to a multiplicative constant. When $\lambda \to \infty$ or $\lambda \to -\infty$, simple zeros of $L_{q_l}(x, \lambda)$ enter (a, b) through the suitable endpoint and traverse the interval toward the opposite endpoint. An eigenfunction is formed when a simple zero of $L_{q_l} y(x, \lambda)$ meets x_l.

Examining the proofs of Lemma 4 and Theorem 2 it is seen that a similar conclusion may be achieved when the given equation depends on λ in a more complicated way. Let $p(x, \lambda)$ be continuous for $a \leqslant x \leqslant b$, $\Lambda_1 < \lambda < \Lambda_2$ and analytic for $\Lambda_1 < \lambda < \Lambda_2$. Let $(-1)^{\nu(b)} p(x, \lambda) \leqslant 0$ and assume that $p(x, \lambda)$ tends unformly to infinity on some subinterval of $[a, b]$ if $\lambda \to \Lambda_2^-$. Consider the boundary problem which consists of the equation

$$Ly + p(x, \lambda) y = 0$$

and the boundary conditions of (2). Then this problem has an infinite sequence of eigenvalues, with no accumulation point in (Λ_1, Λ_2).

4. The Zeros of the Eigenfunctions

All the boundary conditions imposed on $L_0 y$ in (a, b) are of the form

$$(L_t y)(x_i) = 0 \qquad 0 \leqslant t \leqslant 2m_i - 1. \tag{24}$$

At the points of (a, b), which are not given in (2), $y(x, \lambda) = L_0 y(x, \lambda)/\rho_0(x)$ may have only simple zeros. When a simple zero of $y(x, \lambda)$ meets a point x_i, where (24) holds, $y(x, \lambda)$ has a zero exactly of multiplicity $2m_i + 1$, i.e., $y(x, \lambda)$ still changes its sign. The same thing happens when a simple zero of $y(x, \lambda)$ meets x_i, where

$$(L_t y)(x_i) = 0, \qquad 1 \leqslant t \leqslant 2m_i,$$

holds. Here a simple zero is replaced by a zero of multiplicity $2m_i + 1$. These facts suggest to inquire the number of the sign changes of the eigenfunctions in (a, b). This will be our next aim.

The sign changes of the other quasi-derivatives have no such role. For, if a simple zero of $L_j y(x, \lambda)$ meets, when $\lambda \to \lambda_0$, the point x_i, where the boundary condition (22) is posed in (2), the type of the zero x_i of $L_j y(x, \lambda_0)$ depends on the parity of $q_i + 2m_i - j$.

As it was shown, to each nonzero eigenvalue λ there belongs an essentially unique eigenfunction which is given by $y(x, \lambda)$. In contrast, several independent eigenfunctions may belong to the eigenvalue $\lambda = 0$, if $\lambda = 0$ is an eigenvalue. To obtain a complete picture of the eigenfunctions and their zeros, we first investigate the problem for the eigenvalue $\lambda = 0$.

For $\lambda = 0$, a system of n independent solutions $y_0(x),..., y_{n-1}(x)$ of $Ly = 0$ is given by

$$y_0(x) = 1/\rho_0(x), \qquad y_1(x) = 1/\rho_0(x) \int_a^x dt_1/\rho_1(t_1),...,$$

$$y_{n-1}(x) = 1/\rho_0(x) \int_a^x dt_1/\rho_1(t_1) \int_a^{t_1} dt_2/\rho_2(t_2) \cdots \int_a^{t_{n-2}} dt_{n-1}/\rho_{n-1}(t_{n-1}).$$

The first r solutions and their linear combinations $\sum_{i=0}^{r-1} a_i y_i$ satisfy the equation $L_r y = 0$.

If $u = \sum_{i=0}^{r-1} a_i y_i$, $a_{r-1} \neq 0$ and $v = \sum_{i=0}^{r-1} b_i y_i$, $b_{r-1} \neq 0$, are two independent eigenfunctions corresponding to the eigenvalue $\lambda = 0$, then $b_{r-1} u - a_{r-1} v$ is also an eigenfunction and this eigenfunction is a linear combination of $y_0, ..., y_{r-2}$. Therefore, if the maximal number of independent eigenfunctions belonging to the eigenvalue $\lambda = 0$ is l, then these eigenfunctions may be chosen as

$$u_i(x) = \sum_{j=0}^{r_i-1} a_{ji} y_i(x), \qquad a_{r_i-1,i} \neq 0, \qquad i = 1,..., l,$$
$$1 \leqslant r_1 < r_2 < \cdots < r_l \leqslant n. \tag{25}$$

Here r_i, $1 \leqslant i \leqslant l$, are determined uniquely but the eigenfunctions u_i are not determined uniquely. To $u_i(x)$ an arbitrary linear combination of $u_0(x),..., u_{i-1}(x)$ can be added.

In the next lemma, the numbers $l, r_1,..., r_l$ appearing in the representation of the eigenfunctions of $\lambda = 0$ in the form (25) are determined in terms of the boundary conditions in (2).

LEMMA 5. *A necessary and sufficient condition that $\lambda = 0$ is an eigenvalue to which there correspond exactly l independent eigenfunctions and that there is a set of l eigenfunctions of the form* (25), *is that for every $r_i \leqslant q < r_{i+1}$, $0 \leqslant i \leqslant l$ (where $i = 0$ means $0 < q < r_1$ and $i = l$ means $r_l \leqslant q < n$), at least $q - i$ boundary conditions are posed on $L_0 y,..., L_{q-1} y$ and exactly $r_i - i$ conditions are posed on $L_0 y,..., L_{r_i-1} y$.*

Equivalently, at least $q - r_i$ conditions are posed on the $q - r_i$ quasi-derivatives $L_{r_i} y,..., L_{q-1} y$, when $r_i < q < r_{i+1}$ and exactly $r_i - r_{i-1} - 1$ conditions are posed on the $r_i - r_{i-1}$ quasi-derivatives $L_{r_{i-1}} y,..., L_{r_i-1} y$.

Proof. Necessary. Assume that $u_1,..., u_l$ given in (25) form a maximal independent set of eigenfunctions which correspond to $\lambda = 0$. Let $r_i \leqslant q < r_{i+1}$ and assume that at most $q - i - 1$ boundary conditions are posed on $L_0 y,..., L_{q-1} y$. Then there exist at least $i + 1$ independent q-tuples $(a_{0,j},..., a_{q-1,j})$ such that the solutions $v_j = \sum_{k=0}^{q-1} a_{kj} y_k$, $j = 1,..., i + 1$, satisfy these $q - i - 1$ conditions. v_j satisfies $L_q v_j \equiv 0$, so it satisfies any homogeneous boundary condition posed on $L_q y,..., L_{n-1} y$ and it is an eigenfunction. Thus we have found $i + 1$ independent eigenfunctions which are linear combinations of $y_0,..., y_{q-1}$, $q < r_{i+1}$. But according to (25), there are i independent eigenfunctions which are linear combinations of $y_0,..., y_{r_{i+1}-2}$. This contradiction proves the first part of the assertion.

For $q = r_i - 1$ we conclude that at least $(r_i - 1) - (i - 1) = r_i - i$ conditions are posed on $L_0 y,..., L_{r_i-2} y$. $u_i(x)$, $1 \leqslant i \leqslant l$, satisfies $L_{r_i-1} u_i = a_{r_i-1,i} \neq 0$, therefore, no boundary condition is posed on $L_{r_i-1} y$. To complete the proof of necessity it is enough to show that at most $r_i - i$ conditions are posed on $L_0 y, .., L_{r_i-2} y$. We show this by induction on i.

Since no condition is posed on $L_{r_i-1} y$, no sequence of consecutive boundary conditions of (24) splits when we consider only the quasi-derivatives $L_0 y,..., L_{r_i-2} y$, and for every x_t, $a < x_j < b$, such sequence is of even length.

For $i = 1$, assume on the contrary that at least r_1 conditions are posed on $L_0 y,..., L_{r_1-2} y$. By Lemma 2, we have

$$S(L_{r_1-1} u_1) \geqslant S(u_1) + N_{r_1-1}(u_1) - (r_1 - 1) \geqslant 0 + r_1 - (r_1 - 1) = 1,$$

contradicting $L_{r_1-1} u_1 = a_{r_1-1,1} \neq 0$. Thus exactly $r_1 - 1$ conditions are posed of $L_0 y,..., L_{r_1-1} y$.

Assume that we have proved that exactly $r_{i-1} - (i - 1)$ conditions are posed on $L_0 y, \ldots, L_{r_{i-1}-1} y$ and assume that at least $(r_i - i) + 1$ conditions are posed on $L_0 y, \ldots, L_{r_i-2} y$. Then at least $r_i - r_{i-1}$ conditions are posed on $L_{r_{i-1}} y, \ldots, L_{r_i-2} y$. Consider now $L_{r_i-1} y$ as a factorizable differential operator of order $r_i - r_{i-1} - 1$ applied to $L_{r_{i-1}} y$. By Lemma 2,

$$S(L_{r_i-1} u_i) \geqslant S(L_{r_{i-1}} u_i) + (r_i - r_{i-1}) - (r_i - r_{i-1} - 1) \geqslant 1$$

contradicting $L_{r_i-1} u_i \neq 0$. This completes the proof of the necessity.

Sufficient. Assume that the boundary conditions of (2) satisfy the assumptions of the lemma. Then exactly $r_i - i$ conditions are posed on $L_0 y, \ldots, L_{r_i-1} y$. Therefore there exist at least i independent r_i-tuples $(a_{0,j}, \ldots, a_{r_i-1,j})$, $j = 1, \ldots, i$, so that $v_j = \sum_{k=0}^{r_i-1} a_{kj} y_k$ satisfies these $r_i - i$ homogeneous conditions. Since $L_{r_i} v_j \equiv \cdots \equiv L_{n-1} v_j \equiv 0$, each of the solutions $v_1, \ldots, v_i$ is an eigenfunction. If $a_{r_i-1,j} = 0$ for $j = 1, \ldots, i$, then we have obtained at least i eigenfunctions which are linear combinations of $y_0, \ldots, y_{r_i-2}$ and when we arrange them as in (25), the i-th of them is $\sum_{k=0}^{q-1} a_{k,i} y_k$, $a_{q-1,i} \neq 0$, $q < r_i$. Then, by the necessary condition which has been already proved, exactly $q - i$ conditions are posed on $L_0 y, \ldots, L_{q-1} y$. But since $q < r_i$, by the assumption of the lemma at least $q - (i - 1)$ condition are posed on $L_0 y, \ldots, L_{q-1} y$. Therefore for some j, say $j = i$, we have $a_{r_i-1,i} \neq 0$, so we obtain that there exists a system of eigenfunctions of the form (25).

We note that from Lemma 5 follows that the number l of independent eigenfunctions corresponding to $\lambda = 0$ and the structure of these eigenfunctions (i.e., the numbers $r_1, \ldots, r_l$) depend only on the boundary conditions in (2) and not on the points x_i or the operator L.

COROLLARY 3. *$\lambda = 0$ is not an eigenvalue of (2) if and only if for every $q = 1, \ldots, n$ at least q boundary conditions are imposed on the q quasi-derivatives $L_0 y, \ldots, L_{q-1} y$.*

Corollary 3 is known. $\lambda = 0$ is not an eigenvalue of (2) if and only if the Hermite–Birkhoff interpolation by the solutions $y_0, \ldots, y_{n-1}$ of $Ly = 0$ at the points given in (2) is unique. Thus Corollary 3 is equivalent to Theorem 2 of [1].

EXAMPLE. $\lambda = 0$ is an eigenvalue of

$$y^{(8)} + \lambda y = 0,$$
$$y'(\pm 1) = y^{(5)}(\pm 1) = y^{(6)}(\pm 1) = y^{(7)}(\pm 1) = 0.$$

Here $l = 3$, $r_1 = 1$, $r_2 = 4$ and $r_3 = 5$. As can easily be seen, the corresponding eigenfunctions are 1, $x^3 - 3x$, $x^4 - 2x^2$.

LEMMA 6. *The number of the zeros of u_i, $1 \leqslant i \leqslant l$ in (a, b) and the zeros of $L_0 u_i, \ldots, L_{r_i-1} u_i$ at the endpoints, which are not included in the boundary condi-*

tions of (2), *is at most* $i - 1$. *Moreover*, u_i *can be chosen so t at it will change its sign exactly* $i - 1$ *times at* $i - 1$ *given points of* (a, b).

Proof. By Lemma 2 we have

$$S(L_{r_i-1}u_i) \geqslant S(u_i) + N_{r_i-1}(u_i) - (r_i - 1).$$

There are exactly $r_i - i$ boundary conditions imposed on $L_0 y, \ldots, L_{r_i-1} y$ in (2). As no boundary condition is imposed on $L_{r_i-1}y$, no sequence of consecutive conditions is split if we consider only boundary conditions imposed on $L_0 y, \ldots, L_{r_i-1}y$. If the number of the zeros of u_i in (a, b) and the zeros of $L_0 u_i, \ldots, L_{r_i-2}u_i$ at the endpoints, which are not included in (2), is at least i, then

$$S(u_i) + N_{r_i-1}(u_i) \geqslant i + (r_i - i).$$

Note that if u_i has a zero of multiplicity m, the even part of m contributes to $N_{r_i-1}(u_i)$ and 1 is added to $S(u_i)$, if m is odd. Therefore $S(L_{r_i-1}u_i) \geqslant 1$, contradicting $L_{r_i-1}u_i = \text{const} \neq 0$.

The ith eigenfunction, u_i, is not determined uniquely and an arbitrary linear combination of $u_1, \ldots, u_{i-1}$ can be added to it. Therefore we can determine u_i so that it will vanish at $i - 1$ given points of (a, b), distinct from the points $x_1, \ldots, x_r$ of (2). By what we have proved before, these $i - 1$ points are simple zeros of u_i and they are the only points in (a, b) where u_i changes its sign.

We choose the eigenfunctions $u_1, \ldots, u_l$ which belong to the eigenvalue $\lambda = 0$ as in Lemma 6, i.e. u_i, $1 \leqslant i \leqslant l$, changes its sign exactly $i - 1$ times in (a, b), and we arrange the other eigenfunctions according to the magnitude of the corresponding eigenvalues. This is possible since the eigenvalues have no finite accumulation point and they all have the same sign. The sequence of eigenfunctions will be denoted by $u_1, u_2, \ldots$ and the corresponding eigenvalues by $\lambda_1, \lambda_2, \ldots$. According to this notation, if l independent eigenfunctions belong to the eigenvalue $\lambda = 0$, then

$$0 = \lambda_1 = \cdots = \lambda_l < |\lambda_{l+1}| < \cdots.$$

In Lemma 7 and Theorem 3 we determine how many changes of sign an eigenfunction has in (a, b).

LEMMA 7. *If* u_i, u_{i+1} *are two consecutive eigenfunctions which correspond to non-zero eigenvalues, then*

$$S(u_{i+1}) = S(u_i) + 1, \qquad i > l. \tag{26}$$

Proof. For convenience we assume that the eigenvalues are nonnegative. First we show that if two eigenfunctions u, v, belong respectively to two nonzero eigenvalues λ, μ, such that $0 < \mu < \lambda$, then u changes its sign in (a, b) more times than v does, i.e.,

$$S(v) < S(u). \tag{27}$$

To prove this, we utilize an idea used by Bochenek [3] for the boundary value problem

$$L_4 y + \lambda p y = 0,$$
$$y(a) = y'(a) = y(b) = y'(b) = 0.$$

Let us choose a linear combination $f = c_1 v + c_2 u$ such that one of its quasi-derivatives, which is not given in (2), vanishes at one of the endpoints. We shall prove that

$$S(v) \leqslant S(f) < S(u). \tag{28}$$

The function of f satisfies $N_n(f) \geqslant n + 1$, and hence, by Lemma 2, $S(Lf) \geqslant S(f) + 1$. But $Lf = -\lambda p[(\mu/\lambda)c_1 v + c_2 u]$, therefore the function $f_1 = (\mu/\lambda)c_1 v + c_2 u$ satisfies $S(f_1) \geqslant S(f) + 1$. For f_1 we have $N_n(f_1) \geqslant n$ and hence, using again Lemma 2, $S(Lf_1) \geqslant S(f_1)$. Repeating this process m times, we obtain

$$S((\mu/\lambda)^m c_1 v + c_2 u) \geqslant \cdots \geqslant S((\mu/\lambda)c_1 v + c_2 u) \geqslant S(f) + 1.$$

When $m \to \infty$, $f_m = (\mu/\lambda)^m c_1 v + c_2 u \to c_2 u$, therefore $S(f_m) \geqslant S(u)$ for m sufficiently large. When $m \to \infty$, a zero of f_m cannot tend to one of the endpoints and two zeros of odd multiplicities of f cannot coincide, because each of these cases would imply that $N(u) = N(\lim_{m\to\infty} f_m) > n$, contradicting Lemma 1. Therefore for sufficiently large m, f_m changes its sign in (a, b) the same number of times as the limit function $c_2 u$ does. Hence

$$S(u) = S(f_m) \geqslant S(f) + 1.$$

This proves half of (28).

To prove that $S(v) \leqslant S(f)$, let $g_m = c_1 v + (\mu/\lambda)^m c_2 u$. g_m satisfies $N_n(g_m) \geqslant n$, hence $S(Lg_m) \geqslant S(g_m)$, where $Lg_m = -\mu p[c_1 v + (\mu/\lambda)^{m-1} c_2 u]$. Repeating m times this process, we obtain

$$S(c_1 v + (\mu/\lambda)^m c_2 u) \leqslant S(c_1 v + (\mu/\lambda)^{m-1} c_2 u) \leqslant \cdots \leqslant S(f).$$

By the previous argument, for sufficiently large m, $S(g_m) = S(v)$, and the proof of (28) and (27) is completed.

As shown above, the eigenfunctions of (2) are functions $y(x, \lambda)$ for values of λ for which $L_{j_1} y(b, \lambda) = 0$.

Now we show that if we pass from a nonzero eigenvalue λ_i, to the following one, λ_{i+1}, the number of the sign changes in (a, b) of the corresponding eigenfunctions increases exactly by one. That is

$$S(y(x, \lambda_{i+1})) = S(y(x, \lambda_i)) + 1. \tag{29}$$

If $j_1 = 0$, then the number of sign changes of $y(x, \lambda_{i+1}) = L_0 y(x, \lambda_{i+1})/\rho_0(x)$ in (a, b) differs at most by one from the number of sign changes of $y(x, \lambda_i)$. For, when a simple zero of $L_0 y(x, \lambda)$ meets a point x_i of (a, b), where the boundary condition (24) determines a zero of even multiplicity, then $S(y(x, \lambda))$ does not vary. And when λ meets an eigenvalue, then, as we have proved in Theorem 2, at most one simple zero of $L_j y(x, \lambda) = L_0 y(x, \lambda)$ from (a, b) can meet b. Hence $|(y(x, \lambda_{i+1})) - S(y(x, \lambda_i))| \leqslant 1$. As $\lambda_{i+1} > \lambda_i > 0$, $S(y(x, \lambda_{i+1})) > S(y(x, \lambda_i))$, and so, when $j_1 = 0$, (29) is proved.

Assume now that $j_1 > 0$. If $S(y(x, \lambda_{i+1})) \geqslant S(y(x, \lambda_i)) + 2$, then between the eigenvalues λ_i, λ_{i+1}, there are two values λ', λ'', $0 < \lambda_i < \lambda' < \lambda'' < \lambda_{i+1}$, such that $y(b, \lambda') = y(b, \lambda'') = 0$. Because only one simple zero of $y(x, \lambda)$ can meet b at once, and this cannot happen for λ_i (or for λ_{i+1}), since the vanishing of both $L_0 y(b, \lambda_i)$ and $L_{j_1} y(b, \lambda_i)$ would imply $N(y(x, \lambda_i)) \geqslant n + 1$. Now, $y(x, \lambda')$, $y(x, \lambda'')$ are also eigenfunctions of another boundary value problem, in which $(L_{j_1} y)(b) = 0$ is replaced by $(L_0 y)(b) = 0$. Therefore, by (27), $S(y(x, \lambda'')) > S(y(x, \lambda'))$. As $N(y(x, \lambda')) = n$, by Lemma 1 the quasi-derivative $L_t y(x, \lambda')$ changes its sign exactly once between two consecutive zeros of $L_{t-1} y(x, \lambda')$ and all the zeros of $L_t y(x, \lambda')$, which are not given in the suitable boundary value problem, are obtained so. Hence, the zeros of the quasi-derivatives of $y(x, \lambda')$ are exactly the fixed zeros given in the boundary conditions and those, whose existence may be deduced by Rolle's theorem, i.e. the minimal number of zeros that is possible. The same holds also for $y(x, \lambda'')$. Apply now Rolle's theorem to each of the quasi derivatives for λ' and λ''. Since $S(y(x, \lambda'')) > S(y(x, \lambda'))$, the location of the zeros of the quasi-derivatives which was described before, implies that also $L_{j_1} y(x, \lambda'')$ has in (a, b) more zeros which are not given in (10) than $L_j y_1(x, \lambda')$ has. This can happen only if a simple zero of $L_{j_1} y(x, \lambda)$ enters (a, b) through b for some λ, $\lambda' \leqslant \lambda \leqslant \lambda''$. But then (2) has an eigenvalue in $[\lambda', \lambda''] \subset (\lambda_i, \lambda_{i+1})$, which is impossible, since λ_i, λ_{i+1} are consecutive eigenvalues of (2). This contradiction completes the proof of (29). It also proves that when λ increases through an eigenvalue of (2), precisely one simple zero of $L_{j_1} y(x, \lambda)$ enters (a, b) through the endpoint b.

THEOREM 3. *The ith eigenfunction, u_i, changes its sign exactly $i - 1$ times in (a, b). That is,*

$$S(u_i) = i - 1, \qquad i = 1, 2, \ldots . \tag{30}$$

Proof. The l independent eigenfunctions $u_1, \ldots, u_l$, which belong to the eigenvalue $\lambda = 0$, were chosen (using Lemma 6) to satisfy (30). To complete the proof of our theorem, it is enough, by (26), to show that the eigenfunction u_{l+1}, which belongs to the first nonzero eigenvalue λ_{l+1}, $l \geqslant 0$, changes its sign exactly l times in (a, b).

First we shall prove that each eigenfunction $y(x)$ which corresponds to a nonzero eigenvalue λ, in particular u_{l+1}, changes its sign at least l times in (a, b).

By Lemma 5, there are in (2) exactly $r_l - l$ boundary conditions imposed on $L_0 y, \dots, L_{r_l - 1} y$, therefore $n - r_l + l$ conditions are imposed on the other $n - r_l$ quasi derivatives $L_{r_l} y, \dots, L_{n-1} y$. Consider $L_n y$ as an operator of order $n - r_l$ applied to $L_{r_l} y$. No boundary condition is imposed on $L_{r_l - 1} y$, so no sequence of consecutive boundary conditions in (2) is split if we consider only boundary conditions posed on $L_{r_l} y, \dots, L_{n-1} y$. Therefore, by Lemma 2,

$$S(Ly) \geqslant S(L_{r_l} y) + (n - r_l + l) - (n - r_l) \geqslant l.$$

As $\lambda p y = -Ly$, it follows that $y(x)$ changes its sign at least l times in (a, b).

Now we prove that for sufficiently small ϵ, $y(x, \epsilon)$ changes its sign at most l times in (a, b). This result will be used to investigate the changes of sign of $y(x, \lambda)$ for λ in $(0, \lambda_{l+1}]$. $y(x, \lambda)$ is an entire analytic function of λ, $y(x, \lambda) = y(x, 0) + \sum_{i=1}^{\infty} q_i(x) \lambda^i$. If $\lambda = 0$ is a zero of multiplicity m of $y(x, \lambda)$, $m \geqslant 1$, then $y(x, 0) \equiv 0$. But $\lambda^{-m} y(x, \lambda)$ is a solution of (1) for $\lambda \neq 0$ and as $\lambda \to 0$, $v(x) = \lim_{\lambda \to 0} \lambda^{-m} y(x, \lambda)$ is a nontrivial solution of $Ly = 0$ which satisfies the $n - 1$ boundary conditions of (10). If $\lambda = 0$ is not a zero of $y(x, \lambda)$, i.e. $m = 0$, we take $v(x) = y(x, 0)$.

Assume on the contrary that there exists a sequence $\epsilon_i \to 0$ so that $S(y(x, \epsilon_i)) \geqslant l + 1$. Then $v(x) = \lim_{i \to \infty} \epsilon_i^{-m} y(x, \epsilon_i)$ has in $[a, b]$ at least $l + 1$ zeros more than we required in (10). Since $Lv \equiv 0$, let $L_{q-1} v$, $1 \leqslant q \leqslant n$, be the last nonzero quasi-derivative, i.e. $L_{q-1} v \equiv \text{const} \neq 0$, $L_q v \equiv \cdots Lv \equiv 0$. Let $r_i \leqslant q < r_{i+1}$ for some $0 \leqslant i \leqslant l$, where $i = 0$ means $0 < q < r_1$ and $i = l$ means $r_l \leqslant q < n$. Since $r_i \leqslant q < r_{i+1}$, at least $q - i$ boundary conditions are posed in (2) on $L_0 y, \dots, L_{q-1} y$.

If $j_1 \geqslant q$, then $L_0 v, \dots, L_{q-1} v$ satisfy these $q - i$ conditions of (2), which are also conditions of (10). Moreover, since $L_{q-1} v \neq 0$, all the $q - i$ conditions are posed in fact on $L_0 v, \dots, L_{q-2} v$. But $v(x)$ does not necessarily satisfy the condition $(L_{j_1} y)(b) = 0$. If $j_1 \leqslant q - 1$, we delete this condition and thus obtain that $L_0 v, \dots, L_{q-2} v$ satisfy at least $q - i - 1$ of the boundary conditions of (10).

Since $L_{q-1} v \neq 0$, no condition is posed in (2) on $L_{q-1} y$ in (a, b) (If $q - 1 = j_1$, one condition is given at b). Therefore, when we consider only $L_0 y, \dots, L_{q-2} y$, no sequence of consecutive conditions at a point of (a, b) is split. So at any point of (a, b), $L_0 v, \dots, L_{q-2} v$ satisfy a sequence of conditions of even length. By the $q - i - 1$ conditions of (10) which are satisfied by $L_0 v, \dots, L_{q-2} v$ and the additional $l + 1$ zeros which v has in $[a, b]$, we obtain that

$$S(v) + N_{q-1}(v) \geqslant (l + 1) + (q - i - 1)$$

and

$$\begin{aligned} S(L_{q-1} v) &\geqslant S(v) + N_{q-1}(v) - (q - 1) \\ &\geqslant (l + 1) + (q - i - 1) - (q - 1) = (l - i) + 1 \geqslant 1, \quad (31) \end{aligned}$$

contradicting $L_{q-1} v \neq 0$.

We have proved that for sufficiently small ϵ, $y(x, \epsilon)$ changes its sign at most l times in (a, b) and $u_{l+1}(x) = y(x, \lambda_{l+1})$ changes its sign at least l times in (a, b). To investigate the behavior of the zeros of $y(x, \lambda)$ for $0 < \lambda \leqslant \lambda_{l+1}$, we distinguish between two cases:

If $S(y(x, \epsilon)) \leqslant l - 1$ for sufficiently small ϵ, then $S(y(x, \lambda_{l+1})) = l$. This is proved using the same argument as in the proof of (29), since two changes of sign cannot be added to $y(x, \lambda)$ for $0 < \lambda \leqslant \lambda_{l+1}$.

If $S(y(x, \epsilon)) = l$ for sufficiently small ϵ, i.e. the number of sign changes is as large as possible, then $L_t y(x, \epsilon)$ changes its sign exactly once between two consecutive zeros of $L_{t-1} y(x, \epsilon)$, and except these, $L_t y(x, \epsilon)$ has no additional zeros which are not given in (10). Otherwise, if $L_t y(x, \epsilon)$ has an additional zero, which was not taken into account in the proof of Lemma 2, we would obtain a contradiction as in (31).

Now, if $S(y(x, \lambda))$ increases for $\epsilon < \lambda \leqslant \lambda_{l+1}$, then a simple zero of $y(x, \lambda)$ enters (a, b) through b for some λ, $\epsilon < \lambda < \lambda_{l+1}$. Let this simple zero meet b for λ', $\epsilon < \lambda' < \lambda_{l+1}$. Then $y(x, \lambda')$ has one more zero than the number given in (10) than $y(x, \epsilon)$ has. Apply now Rolle's theorem for $L_t y(x, \lambda)$ for $\lambda = \epsilon$ and $\lambda = \lambda'$. The location of the zeros of $L_t y(x, \epsilon)$ which was described before, implies that also $L_{j_1} y(x, \lambda')$ has more zeros in (a, b) which do not appear in (10) then $L_{j_1} y(x, \epsilon)$ has. This means that a simple zero of $L_{j_1} y(x, \lambda)$ enters (a, b) for some λ in $(\epsilon, \lambda'] \subset (0, \lambda_{l+1})$, which is impossible since (2) has no eigenvalues in $(0, \lambda_{l+1})$. This last contradiction proves that also in this case $u_{l+1}(x) = y(x, \lambda_{l+1})$ changes its sign exactly l times in (a, b) and the proof of the theorem is thus completed.

Remark. The eigenfunction u_i, $i > l$, satisfies $N(u_i) = n$. So by Lemma 1, $L_t u_i$ changes its sign exactly once between two consecutive zeros of $L_{t-1} u_i$ in $[a, b]$ and these are the only zeros of $L_t u_i$ in $[a, b]$ which are not given in (2). u_i has in (a, b) exactly $i - 1$ zeros which are not given in (2). It follows thus, that the number of the zeros of each quasi-derivative of u_i, which are not given in (2) is determined only by the indices of the quasi-derivatives given in (2) and it does not depend on the operator L, the coefficient $p(x)$ and the points $x_1 \leqslant \cdots \leqslant x_r$.

COROLLARY 4. *For every $c_1, \ldots, c_i$, not all zero, we have*

$$S(c_1 u_1 + \cdots + c_i u_i) \leqslant S(u_i) = i - 1, \qquad i = 1, 2, \ldots. \tag{32}$$

This is proved by applying the argument, which was used to prove (28), to a combination of several eigenfunctions.

THEOREM 4. *Let u_i and u_{i+1} be the eigenfunctions corresponding to the nonzero* *eigenvalues* λ_i *and* λ_{i+1} *respectively and let* $\alpha_1, \dots, \alpha_{i-1}$ *and* $\beta_i, \dots, \beta_i$ *be the points in* (a, b) *where* u_i *and* u_{i+1} *change their signs. Then*

$$\beta_1 < \alpha_1 < \beta_2 < \cdots < \beta_{i-1} < \alpha_{i-1} < \beta_i .$$

Proof. The boundary conditions imposed in (2) on $u_i = (L_0 u_i)/\rho_0$ and on $u_{i+1} = (L_0 u_{i+1})/\rho_0$ in (a, b) are of the form

$$(L_t y)(x_i) = 0, \qquad 0 \leqslant t \leqslant 2m_i - 1, \quad a < x_i < b. \tag{24}$$

Therefore u_i and u_{i+1} have in (a, b) the same zeros of even multiplicities and they may have different zeros only of odd multiplicities, i.e. changes of sign. Let α, β be two consecutive zeros of odd multiplicities of u_i and assume that u_{i+1} has no zeros of odd multiplicities in $[\alpha, \beta]$. Let $\alpha \leqslant x_1 < \cdots < x_k \leqslant \beta$ be the points where boundary conditions are posed on $L_0 y$ in (2). If we take $f(x) = u_i(x)(x - x_1)^{-2m_1} \cdots (x - x_k)^{-2m_k}$ and $g(x) = u_{i+1}(x)(x - x_1)^{-2m_1} \cdots (x - x_k)^{-2m_k}$, then $g(x) \neq 0$ in $[\alpha, \beta]$, $f(x) \neq 0$ in (α, β) and $f(\alpha) = f(\beta) = 0$. Hence, by Lemma 1.1 of [10], there exists a linear combination $c_1 f + c_2 g$ which has a double zero in (α, β). At the same point two consecutive quasi-derivatives of $c_1 u_i + c_2 u_{i+1}$, which are not given in (2), vanish.

If u_i, u_{i+1} have zeros of odd multiplicities at the same point, they satisfy $(L_t u_i)(c) = (L_t u_{i+1})(c) = 0$, $0 \leqslant t \leqslant 2m$. If we choose $c_1 u_i + c_2 u_{i+1} = (L_{2m+1} u_{i+1})(c) u_i - (L_{2m+1} u_i)(c) u_{i+1}$, once again two quasi-derivatives of $c_1 u_i + c_2 u_{i+1}$, which are not given in (2), vanish at c.

So, $N_n(c_1 u_i + c_2 u_{i+1}) \geqslant n + 2$ and, by Lemma 2, $S(L(c_1 u_i + c_2 u_{i+1})) \geqslant S(c_1 u_i + c_2 u_{i+1}) + 2$. If we repeat the proof of (28), we obtain

$$S(u_i) \leqslant S(c_1 u_i + c_2 u_{i+1}) \leqslant S(u_{i+1}) - 2.$$

This contradicts (26), therefore between two changes of sign of u_i in (a, b) there must be a change of sign of u_{i+1}. The same argument shows that between two changes of sign of u_{i+1} in (a, b) there must be a change of sign of u_i and the proof of the theorem is completed.

A similar separation property may be proved for the zeros of the quasi-derivatives $L_t u_i$ and $L_t u_{i+1}$ which are not given in (2).

The method by which (28) was proved will be used also in the proof of the next theorem.

THEOREM 5. *The nonzero real eigenvalues of* (2) *are simple zeros of the determinant* $\Delta(\lambda)$.

Proof. The determinant $\Delta(\lambda)$ is connected with the solution $y(x, \lambda)$ by $\Delta(\lambda) = \pm L_{j_1} y(b, \lambda)$. If $\lambda \neq 0$ is a multiple zero of $\Delta(\lambda)$ then $L_{j_1} y(b, \lambda) = (\partial/\partial\lambda) L_{j_1} y(b, \lambda) = 0$, so also $L_{j_1} (\partial/\partial\lambda) y(b, \lambda) = 0$. If $(L_t y)(x_i) = 0$ is one of the boundary conditions of (10), then $L_t y(x_i, \lambda) = 0$ holds for every λ and so

$L_i(\partial/\partial\lambda)\, y(x_i, \lambda) = 0$. Thus, if λ is a multiple zero of $\Delta(\lambda)$, the function $(\partial/\partial\lambda)\, y(x, \lambda)$ satisfies the boundary conditions of (2). For short, we denote $y(x, \lambda)$ by y and $(\partial/\partial\lambda)\, y(x, \lambda)$ by y_λ.

Differentiating equation (1) with respect to λ, we obtain

$$Ly_\lambda + \lambda p(x) y_\lambda + p(x) y = 0.$$

It is clear from this equation that y_λ is not a constant multiple of the eigenfunction y. We choose a linear combination $f = c_1 y + c_2 y_\lambda$ such that one of its quasi-derivatives, which is not given in (2), vanishes at one of the endpoints a, b. We shall prove first that

$$S(y) > S(f) = S(c_1 y + c_2 y_\lambda). \tag{33}$$

The function f satisfies $N_n(f) \geqslant n + 1$ and hence, by Lemma 2, $S(Lf) \geqslant S(f) + 1$. But

$$Lf = L(c_1 y + c_2 y_\lambda) = c_1(-\lambda p y) + c_2(-\lambda p y_\lambda - p y) = -\lambda p[(c_1 + c_2/\lambda) y + c_2 y_\lambda],$$

therefore the function $f_1 = (c_1 + c_2/\lambda) y + c_2 y_\lambda$ satisfies $S(f_1) \geqslant S(f) + 1$. f_1 is also a linear combination of y and y_λ and it satisfies the conditions of (2). Thus $N_n(f_1) \geqslant n$ and again by Lemma 2, $S(Lf_1) \geqslant S(f_1)$. Repeating this process m times, we obtain

$$S\left(\left(c_1 + \frac{mc_2}{\lambda}\right) y + c_2 y_\lambda\right) \geqslant \cdots \geqslant S(f_1) \geqslant S(f) + 1,$$

or, what is equivalent,

$$S\left(\left(\frac{c_1}{m} + \frac{c_2}{\lambda}\right) y + \frac{c_2}{m} y_\lambda\right) \geqslant S(f) + 1.$$

When $m \to \infty$,

$$f_m = \left(\frac{c_1}{m} + \frac{c_2}{\lambda}\right) y + \frac{c_2}{m} y_\lambda \to \frac{c_2}{\lambda} y.$$

By the argument which was used in the proof of (28), we obtain that $S(f_m) = S(y)$ for sufficiently large m. Now (33) follows from $S(y) = S(f_m) \geqslant S(f) + 1$.

To prove the theorem we obtain an inequality which contradicts (33). Let

$$g_m = \left(\frac{c_1}{m} - \frac{c_2}{\lambda}\right) y + \frac{c_2}{m} y_\lambda .$$

When $m \to \infty$, $g_m \to -(c_2/\lambda) y$ and as before we have $S(g_m) = S(y)$ for sufficiently large m.

g_m satisfies $N_n(g_m) \geqslant n$, hence $S(Lg_m) \geqslant S(g_m)$. Here

$$Lg_m = -\lambda p \left[\left(\frac{c_1}{m} - \frac{c_2}{\lambda} + \frac{c_2}{m\lambda} \right) y + \frac{c_2}{m} y_\lambda \right].$$

Repeating m times this process we obtain

$$S\left(\frac{c_1}{m} y + \frac{c_2}{m} y_\lambda \right)$$
$$\geqslant \cdots \geqslant S\left(\left(\frac{c_1}{m} - \frac{c_2}{\lambda} + \frac{c_2}{m\lambda} \right) y + \frac{c_2}{m} y \right) \geqslant S(g_m) = S(y).$$

or

$$S(c_1 y + c_2 y_\lambda) \geqslant S(y).$$

This inequality contradicts (33) and the proof of the theorem is completed.

COROLLARY 5. *The eigenvalue λ_i is a differentiable function of the boundary points and it depends continuously on the coefficient $p(x)$.*

Proof. λ_i is a simple zero of $\Delta(\lambda)$ and $\Delta(\lambda)$ is a differentiable function of x_t. The differentiability of $\lambda_i(x_t)$ follows by the implicit functions theorem. Since $\Delta(\lambda)$ depends continuously on $p(x)$, the same holds for λ_i by the intermediate value property.

5. Two-Point Boundary Value Problems

Now we consider the boundary value problem (3), where all the boundary conditions are given at the two endpoints a and b. For these boundary conditions, the formulation of our results is simpler. The eigenfunctions and their quasi-derivatives may have only simple zeros in (a, b). In particular, the i-th eigenfunction, u_i, has exactly $i - 1$ simple zeros in (a, b). For $i > l$, $N(u_i) = n$ and by Lemma 1, $L_t u_i$ has one simple zero between two consecutive zeros of $L_{t-1} u_i$ in $[a, b]$ and these are the only zeros of $L_t u_i$ in (a, b). Thus, in the proof of Lemma 2, inequality (8) becomes an equality, and so

$$S(L_h u_i) = S(u_i) + N_h(u_i) - h$$

or

$$S(L_h u_i) = (i - 1) + N_h(u_i) - h. \tag{34}$$

Here, $S(L_h u_i)$ is equal to the number of zeros of $L_h u_i$ in (a, b). Since no boundary conditions are given at points of $(a\ \ b)$, $N_h(u_i)$ is the number of the boundary conditions imposed in (3) on $L_0 y, \ldots, L_{h-1} y$ and it does not depend on u_i.

Our next result concerns the effect of a change in the boundary conditions of (3) on the eigenvalues.

THEOREM 6. *Given the boundary value problem* (3) *with the eigenvalues* λ_i, $i = 1, 2, \ldots$. *Consider the boundary values problem*

$$\begin{aligned} &Ly + \lambda p(x) y = 0, \\ &\quad (L_i y)(a) = 0, \qquad i \in \{\tilde{i}_1, \ldots, \tilde{i}_k\}, \\ &\quad (L_j y)(b) = 0, \qquad j \in \{\tilde{j}_1, \ldots, \tilde{j}_{n-k}\}, \end{aligned} \tag{35}$$

where $\tilde{i}_1 \geqslant i_1, \ldots, \tilde{j}_{n-k} \geqslant j_{n-k}$ *and at least one inequality is strict. Denote its eigenvalues by* $\tilde{\lambda}_i$, $i = 1, 2, \ldots$. *Then*

$$|\tilde{\lambda}_i| \leqslant |\lambda_i|, \qquad i = 1, 2, \ldots$$

and

$$|\tilde{\lambda}_i| < |\lambda_i|$$

if $\lambda_i \neq 0$.

Proof. Without loss of generality we may assume that the indices $i_1, \ldots, i_k$ and $j_1, \ldots, j_{n-k}$ are arranged in increasing order. Clearly it is enough to prove the theorem when (35) is obtained from (3) by transferring one boundary condition from a certain quasi-derivative in (3) to its consecutive quasi-derivative. So assume that the boundary condition $(L_{j-1}y)(b) = 0$, $0 \leqslant j - 1 \leqslant n - 2$, of (3) is replaced by $(L_j y)(b) = 0$ (which has not appeared in (3)) to obtain (35).

First it will be proved that if l independent eigenfunctions correspond to the eigenvalue $\lambda = 0$ in (3) then at least l eigenfunctions correspond to $\lambda = 0$ in (35). That is, if $0 = \lambda_1 = \cdots = \lambda_l \neq \lambda_{l+1}$ then $0 = \tilde{\lambda}_1 = \cdots = \tilde{\lambda}_l$.

Assume that (3) satisfies the assumptions of Lemma 5 for the l indices $r_1, \ldots, r_l$. We shall prove that (35) satisfies the same assumptions at least for l indices.

Let $r_i \leqslant j - 1 < r_{i+1}$ for some $0 \leqslant i \leqslant l$, where $i = 0$ means $0 \leqslant j - 1 < r_1$ and $i = l$ means $r_l \leqslant j - 1 < n$. Since no boundary condition is imposed on $L_{r_{i+1}-1}y$, and since $j \leqslant n - 2$, we have $r_i \leqslant j - 1 < r_{i+1} - 1$. By Lemma 5 at least $j - i$ boundary conditions are posed in (3) on $L_0 y, \ldots, L_{j-1} y$. Two cases will be considered separately.

(a) If more than $j - i$ boundary conditions are posed in (3) on $L_0 y, \ldots, L_{j-1} y$, then, when one condition is transferred from $L_{j-1} y$ to $L_j y$, still at least $j - i$ conditions are posed on $L_0 y, \ldots, L_{j-1} y$ in (35). So by Lemma 5, (35) has eigenfunctions which correspond to $\lambda = 0$ for the indices $r_1, \ldots, r_l$ and for no other indices. Thus the same number of eigenfunctions belong to $\lambda = 0$ in (3) and in (35).

(b) Now we assume that exactly $j - i$ boundary conditions are imposed in (3) on $L_0 y, \ldots, L_{j-1} y$. When we replace the condition $(L_{j-1} y)(b) = 0$ by $(L_j y)(b) = 0$, only $j - i - 1$ conditions are posed on $L_0 y, \ldots, L_{j-1} y$. Therefore the index j satisfies the assumptions of Lemma 5 and (35) has an eigenfunction of the form $\sum_{t=0}^{j-1} a_t y_t$, $a_{j-1} \neq 0$. On the other hand, when $i < l$, the index r_{i+1} does not satisfy the assumptions of Lemma 5 for (35). Because exactly $r_{i+1} - (i + 1)$ conditions are posed in (3) on $L_0 y, \ldots, L_{r_{i+1}-1} y$ and so $L_j y, \ldots, L_{r_{i+1}-1} y$ satisfy exactly $(r_{i+1} - i - 1) - (j - i) = r_{i+1} - j - 1$ of these conditions. The same $r_{i+1} - j$ quasi-derivatives satisfy $r_{i+1} - j$ conditions of (35), so r_{i+1} does not satisfy the assumptions of Lemma 5. Now it is easily checked that r_{i+2} satisfies the assumptions of Lemma 5 and that the remaining indices are not affected at all by the change. So when $j < r_l$, $\lambda = 0$ has the same number of eigenfunctions for (3) and for (35), only the index r_{i+1} is replaced by j. If $i = l$, i.e. $r_l \leqslant j - 1 < n - 1$, then, while (3) has l eigenfunctions corresponding to $\lambda = 0$, (35) has an $(l + 1)$th eigenfunction $\sum_{t=0}^{j-1} a_{t,l+1} y_t$ for the same eigenvalue. Thus, we have proved that (35) has in all cases at least as many eigenfunctions which belong to $\lambda = 0$ as (3) has.

If $\tilde{\lambda}_i \neq 0$ is an eigenvalue of (35), then also the corresponding eigenvalue of (3), λ_i, is different from zero. Now, the nonzero eigenvalues of (3) were characterized by the simple zeros of $L_{j_1} y(x, \lambda)$ which enter (a, b) through b. Of course, the same may be done also by the zeros of $L_{j-1} \hat{y}(x, \lambda)$, where $\hat{y}(x, \lambda)$ is the unique solution of (10), when now the condition $(L_{j-1} y)(b) = 0$ is deleted instead of $(L_{j_1} y)(b) = 0$. Then the nonzero eigenvalues of (3) are obtained when simple zeros of $L_{j-1} \hat{y}(x, \lambda)$ enter (a, b) through b, while the nonzero eigenvalues of (35) are obtained when simple zeros of $L_j \hat{y}(x, \lambda)$ enter (a, b).

In order to prove that $| \tilde{\lambda}_i | < | \lambda_i |$, it is enough to show that $L_j \hat{y}(x, \lambda_i)$ has more simple zeros in (a, b) than $L_j \hat{y}(x, \tilde{\lambda}_i)$ has. Indeed, by (34), the eigenfunction $\hat{y}(x, \lambda_i)$ of (3) and the eigenfunction $\hat{y}(x, \tilde{\lambda}_i)$ of (35) satisfy

$$S(L_j \hat{y}(x, \lambda_i)) = (i - 1) + N_j(\hat{y}(x, \lambda_i)) - j$$

and

$$S(L_j \hat{y}(x, \tilde{\lambda}_i)) = (i - 1) + N_j(\hat{y}(x, \tilde{\lambda}_i)) - j.$$

But

$$N_j(\hat{y}(x, \lambda_i)) = N_j(\hat{y}(x, \tilde{\lambda}_i)) + 1,$$

since $L_{j-1} \hat{y}(b, \lambda_i) = 0$ adds 1 to the sum $N_j(\hat{y}(x, \lambda_i))$ while $L_j \hat{y}(b, \tilde{\lambda}_i) = 0$ does not affect $N_j(\hat{y}(x, \tilde{\lambda}_i))$. Therefore $S(L_j \hat{y}(x, \lambda_i)) > S(L_j \hat{y}(x, \tilde{\lambda}_i))$ and the inequality $| \tilde{\lambda}_i | < | \lambda_i |$ is proved.

Remark. It is easily seen that for every $\lambda \neq 0$, between two consecutive zeros of $L_{j-1} \hat{y}(x, \lambda)$ there is exactly one simple zero of $L_j \hat{y}(x, \lambda)$. Therefore between two simple zeros of $L_{j-1} \hat{y}(x, \lambda)$ which enter (a, b), exactly one simple zero of $L_j \hat{y}(x, \lambda)$ enters (a, b). So when (3) and (35) differ only by one boundary condition, then between two nonzero eigenvalues of (3) there is exactly one eigenvalue of (35). Repeating this argument we obtain that there exist a constant m, which depends only on the indices $i_1, \ldots, j_{n-k}$ and $\tilde{i}_1, \ldots, \tilde{j}_{n-k}$, such that the nonzero eigenvalues of (3) and (35) satisfy

$$| \lambda_{i-m} | < | \tilde{\lambda}_i | < | \lambda_i |$$

EXAMPLE. The eigenvalues of $y'' + \lambda y = 0$ which belong to different boundary conditions are the following: The boundary conditions $y(0) = y(\pi) = 0$ have the eigenvalues $\lambda_i = i^2$, $y(0) = y'(\pi) = 0$ have the eigenvalues $\tilde{\lambda}_i = (i - 1/2)^2$ and $y'(0) = y'(\pi) = 0$ have the eigenvalues $\tilde{\tilde{\lambda}}_i = (i - 1)^2$ $i = 1, 2, \ldots$.

COROLLARY 6. *The ith eigenvalue of* (3) *is minimal for the boundary conditions.*

$$\begin{aligned} (L_i y)(a) &= 0, \qquad n - k \leqslant i \leqslant n - 1, \\ (L_j y)(b) &= 0, \qquad k \leqslant j \leqslant n - 1, \end{aligned} \tag{36}$$

and it is maximal for the boundary conditions of (4).

For the boundary conditions (36), $\min(k, n - k)$ independent eigenfunctions belong to $\lambda = 0$. Hence, for (3), at most $[n/2]$ independent eigenfunctions may belong to the eigenvalue $\lambda = 0$.

ACKNOWLEDGMENT

The author expresses his gratitude to Professor David London for his advice and suggestions during the preparation of this paper.

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