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Oscillatory Solutions and Extremal Points for a Linear Differential Equation

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1. Introduction

The differential equation to be discussed is

$$L_n y + p(x) y = 0, \tag{1}$$

where $p(x)$ is a continuous, one-signed function and L_n is a n-th order linear disconjugate differential operator on $[0, \infty)$. It is well known that a disconjugate operator L_n can be written in the form

$$L_n y = \rho_n(\rho_{n-1} \ldots (\rho_1(\rho_0 y)')' \ldots)',$$

where $\rho_i > 0$, $\rho_i \in C^{n-i}$. We put

$$L_0 y = \rho_0 y, \quad L_i y = \rho_i (L_{i-1} y)', \qquad i = 1, \ldots, n,$$

where $L_0 y, \ldots, L_n y$ are called the quasi-derivatives of y.

We consider the following boundary conditions on the interval $[a, s]$:

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \ldots, i_k\}, \\ L_j y(s) &= 0; \qquad j \in \{j_1, \ldots, j_{n-k}\}, \end{aligned} \tag{2}$$

where $\{i_1, \ldots, i_k\}, \{j_1, \ldots, j_{n-k}\}$ are two arbitrary sets of indices from $\{0, \ldots, n-1\}$.

Definition. *The i-th extremal point $\theta_i(a)$ corresponding to the boundary conditions (2) is the i-th value of s in (a, ∞) for which there exists a non-trivial solution of (1), (2). The corresponding non-trivial solution is called an extremal solution.*

It is well known [7] that a necessary condition for the existence of a non- trivial solution of (1), (2) is that $n-k$ is even if $p(x) < 0$ and $n-k$ is odd if $p(x) > 0$. In the following, we assume that this condition is satisfied, thus

$$(-1)^{n-k} p(x) < 0. \tag{3}$$

Two systems of boundary conditions of type (2) play an important role in the study of (1). For the boundary conditions

$$
\begin{aligned}
&L_i y(a) = 0, \qquad i = 0, \ldots, k-1, \\
&L_j y(s) = 0, \qquad j = 0, \ldots, n-k-1,
\end{aligned}
\tag{4}
$$

which are equivalent to

$$
\begin{aligned}
&y^{(i)}(a) = 0, \qquad i = 0, \ldots, k-1, \\
&y^{(j)}(s) = 0, \qquad j = 0, \ldots, n-k-1,
\end{aligned}
\tag{5}
$$

$\theta_1(a)$ is called a *conjugate point of type* $(k, n-k)$; we shall denote it by $\eta_{k,n-k}(a)$. If $\eta_{k,n-k}(a)$ does not lie in $[a,b), a < b \le \infty$, that is, if there is no nontrivial solution which satisfies (4) for $a < s < b$, equation (1) is said to be $(k, n-k)$-disconjugate in $[a,b)$. This situation has been investigated in numerous papers (cf [1] [6] [9]).

For the boundary conditions

$$
\begin{aligned}
&L_i y(a) = 0, \qquad i = 0, \ldots, k-1, \\
&L_j y(s) = 0, \qquad j = k, \ldots, n-1,
\end{aligned}
\tag{6}
$$

$\theta_1(a)$ is called a *focal point of type* $(k, n-k)$; it will be denoted by $\zeta_{k,n-k}(a)$. If $\zeta_{k,n-k}(a)$ does not lie in $[a,b)$, (1) is said to be $(k, n-k)$-disfocal in $[a,b)$. The function $\zeta_{k,n-k}(a)$ plays an important role in the study [9]. (In denoting conjugate points by η and focal points by ζ, we have used the notation of [7] rather than that of [9].)

A solution of (1) is called *oscillatory* on $[0,\infty)$ if it has an infinite number of zeros. Equation (1) is said to be oscillatory if it has at least one oscillatory solution.

In this work we investigate the connection between the existence of extremal points and the existence of oscillatory solutions. In order to state the main result, we first introduce some notation. Let $S(c_0, \ldots, c_n)$ denote the number of sign changes in the sequence $c_0, \ldots, c_n$, whose elements are non-zero real numbers. Moreover, we write

$$
\begin{aligned}
S(y, a+) &= \lim_{x \downarrow a} S\big(L_0 y(x), -L_1 y(x), \ldots, (-1)^n L_n y(x)\big) \\
S(y, b-) &= \lim_{x \uparrow b} S\big(L_0 y(x), L_1 y(x), \ldots, L_n y(x)\big)
\end{aligned}
$$

It is easy to show that if $L_n y(a) \neq 0$, then

$$
S(y, a+) = S^{+}\big(L_0 y(a), \ldots, (-1)^n L_n y(a)\big),
$$

where S^{+} denotes the maximal number of sign changes achievable by appropriate choice of signs for the zero entries (if any). In Lemma 5 we show that

For every solution y of (1), $S(y, x+)$ is constant for sufficiently large values of x. If $S(y, x+) \equiv q$ on (c, ∞), $0 \le q \le n$, then $S(y, x-) \equiv n-q$ on (c, ∞) and $(-1)^{n-k} p(x) < 0$.

Our main result may be stated.

Theorem I. *For equation (1), the following properties are equivalent:*
(1) A conjugate point $\eta_{k,n-k}(a)$ exists for every $a \geq 0$.
(2) There exists a system of boundary conditions of type (2) and a value of a such that all the extremal points $\theta_i(a)$, $i = 1, 2, \ldots$, exist.
(3) For every system of boundary conditions of type (2) and for every value of $a \geq 0$, all the extremal points $\theta_i(a)$, $i = 1, 2, \ldots$, exist.
(4) There exists an oscillatory solution y such that $S(y, x+) \equiv k$ for sufficiently large values of x.
(5) Every solution for which $S(y, x+)$ ultimately has the constant value k is an oscillatory solution.

Theorem I is proved in Section 3. In Section 2 we prove certain preliminary results.

2. Basic properties of (1)

Let y be a solution of (1). As in [2], we arrange the n quasi-derivatives $L_0 y, \ldots, L_{n-1} y$ in a cyclic order, so that $L_{n-1} y$ is followed by $L_0 y$. Let $a \leq x_1 \leq \cdots x_r \leq b$ be the zeros of $L_0 y, \ldots, L_{n-1} y$ in a certain interval, common zeros of consecutive quasi-derivatives being considered as multiple zeros (distinct subscripts will be used for common zeros of non-consecutive derivatives). Let the number of consecutive quasi-derivatives which vanish at x_i, be denoted by $n(x_i)$. In Lemma 1 of [2] it is proved that the distribution of the zeros of $L_0 y, \ldots, L_{n-1} y$ is restricted by the inequality

$$\sum_I n(x_i) + \sum_J [n(x_j) - 1] < n, \tag{7}$$

where

$$I = \{i | x_i = a \text{ or } x_i = b \text{ or } n(x\ ) \text{ is even}\},$$
$$J = \{j | a < x_j < b \text{ and } n(x_j) \text{ is odd }\}.$$

Inequality (7) may be re-written as

$$\sum_{x_t = a,b} n(x_t) + \sum_{a < x_t < b} \langle n(x_t) \rangle \leq n, \tag{8}$$

where $< q >$ denotes the greatest even integer which is not greater than q.

Lemma 1. *Every solution y of (1) satisfies the condition*

$$N(y) \equiv S(y, a+) + S(y, b-) + \sum_{a < x_t < b} \langle n(x_t) \rangle \leq n. \tag{9}$$

Moreover $S(y, b-)$ and $n - S(y, a+)$ are both even if $p(x) < 0$ and both odd if $p(x) > 0$. If $N(y) = n$, then $L_{t+1} y$ has exactly one sign change between two consecutive zeros of $L_t y$ in $[a, b]$. In addition $L_{t+1} y$ changes sign before the first zero of $L_t y$ in $(a, b]$ if and only if $\operatorname{sgn}[L_{t+1} y(a + \varepsilon)] = \operatorname{sgn}[L_t y(a + \varepsilon)]$, and this sign change is unique. The situation is similar near the endpoint b.

Proof. First we note that (9) implies (8) and (7). Indeed if $L_t y(a) = 0$, then $\operatorname{sgn}[L_{t+1}y(a+\varepsilon)] = \operatorname{sgn}[L_t y(a+\varepsilon)]$ for sufficiently small positive ε, and if $L_t y(b) = 0$ then $\operatorname{sgn}[L_{t+1}y(b-\varepsilon)] = -\operatorname{sgn}[L_t y(b-\varepsilon)]$. Thus

$$S(y, a+) \geq \sum_{x_i = a} n(x_i), \qquad S(y, b-) \geq \sum_{x_i = b} n(x_i).$$

The rest of the proof is similar to that of Lemma 1 of [2]. Among the zeros $\{x_i\}$ of $L_0 y, \ldots, L_{n-1}y$, let $\{x_{i,t}\}$ be the zeros of $L_t y$ which are not zeros of $L_{t-1}y$. Here $n(x_{i,t})$ is the exact number of consecutive quasi-derivatives which vanish at x:

$$L_s y(x_{i,t}) = 0, \qquad t \leq s \leq t + n(x_{i,t}) - 1.$$

Let γ_t be the sum of the number of consecutive quasi-derivatives, starting with $L_t y$, which vanish in $[a, b]$. If $m_1, \ldots, m_q$ consecutive quasi-derivatives , starting with $L_{t-1}y$, vanish respectively at the points $a \leq z_1 < \cdots < z_q \leq b$, then $\gamma_{t-1} = m_1 + \cdots + m_q$. At the same points, $m_1 - 1, \ldots, m_q - 1$ consecutive quasi- derivatives respectively vanish, starting with $L_t y$. Additionally, $L_t y$ has $\sum n(x_{i,t})$ zeros in $[a, b]$ which are not zeros of $L_{t-1}y$. There

$$\gamma_t = (m_l - l) + \cdots + (m_q - 1) + \sum n(x_{i,t}) = \gamma_{t-1} + \sum n(x_{i,t}) - q.$$

In each of the $q - 1$ intervals $(z_1, z_2), \ldots, (z_{q-1}, z_q)$, $L_t y$ changes its sign by Rolle's theorem. Hence $L_t y$ has at least $q - 1$ zeros $x_{i,t}$ in (z_1, z_q) for which $n(x_{i,t})$ is odd. If for every $x_{i,t} \in (z_l, z_q)$ we replace $n(x_{i,t})$ by the greatest even integer which is not greater than $n(x_{i,t})$, we obtain

$$\gamma_t \geq \gamma_{t-1} + \sum_{[a, z_l) \cup (z_q, b]} n(x_{i,t}) + \sum_{(z_1, z_q)} \langle n(x_{i,t}) \rangle - 1. \tag{10}$$

If $\operatorname{sgn}[L_t y(a+\varepsilon)] = \operatorname{sgn}[L_{t-1}y(a+\varepsilon)]$, $\epsilon > 0$, then $L_t y$ has at least one zero in (a, z_1) for which $n(x_{i,t})$ is odd. Otherwise, $L_t y$ would be one-signed in (a, z_1) and so

$$L_{t-1}y(z_1) = L_t y(a) + \int_a^{z_1} L_t y / \rho_t \, dx \neq 0.$$

In particular, this is the case when $L_{t-1}y(a) = 0$. In a similar fashion, if $\operatorname{sgn}[L_t y(b-\varepsilon)] = -\operatorname{sgn}[L_{t-1}y(b-\varepsilon)]$ (especially when $L_{t-1}y(b) = 0$), $L_t y$ has a zero in (z_q, b) with odd $n(x_{i,t})$. Hence (10) may be rewritten

$$\begin{aligned} \gamma_t \geq \gamma_{t-1} &+ S(L_{t-1}y(a+\varepsilon), -L_t y(a+\varepsilon)) \\ &+ S(L_{t-1}y(b-\varepsilon), L_t y(b-\varepsilon)) + \sum_{(a,b)} \langle n(x_{i,t}) \rangle - 1. \end{aligned} \tag{11}$$

Observe that (11) holds even if $L_{t-1}y$ has no zero in $[a, b]$. Since $\rho_n(L_{n-1}y)' = L_n y = -py$, we have similarly

$$\begin{aligned} \gamma_0 \geq \gamma_{n-1} &+ S(L_{n-1}y(a+\varepsilon), -L_n y(a+\varepsilon)) \\ &+ S(L_{n-1}y(b-\varepsilon), L_n y(b-\varepsilon)) + \sum_{(a,b)} \langle n(x_{i,0}) \rangle - 1. \end{aligned} \tag{12}$$

Adding inequalities (11), for $t = l, \ldots, n-1$, and (12) we obtain

$$\begin{aligned} n &> S(L_0 y(a+\varepsilon), -L_l y(a+\varepsilon), \ldots, (-1)^n L_n y(a+\varepsilon)) \\ &\quad + S(L_0 y(b-\varepsilon), \ldots, L_n y(b-\varepsilon)) + \sum_{a<x_i<b} \langle n(x_i) \rangle \end{aligned}$$

for sufficiently small positive ε (for which all the quasi-derivatives above are non-zero). This proves (9).

If $N(y) = n$, then none of the inequalities in (11) and (12) can be strict. This proves our assertion about the location of sign changes of the quasi-derivatives. The parities of $S(y, b-)$ and $n - S(y, a+)$ are easily determined by counting the sign changes in the sequences $L_0 y = \rho_o y, L_1 y, \ldots, L_n y = -py$ and $L_0 y, -L_1 y, \ldots, (-1)^n L_n y$.

To investigate the extremal solutions corresponding to (2), we consider solutions of (1) which satisfy only $n-1$ of the boundary conditions of (2), the condition $L_{j_{n-k}} y(s) = 0$, say, being deleted.

Lemma 2. *When only $n-1$ boundary conditions*

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \ldots, i_k\}, \\ L_j y(s) &= 0, \qquad j \in \{j_l, \ldots, j_{n-k-1}\} \end{aligned} \tag{13}$$

are considered, the following results hold:
(1) Equation (1) has an essentially unique solution $y(x,s)$ satisfying (13).
(2) At the endpoint a we have $S(y(x,s), a+) = k$ and $\operatorname{sgn}[L_{i+1} y(a+\varepsilon, s)] = \operatorname{sgn}[L_i y(a+\varepsilon, s)]$ if and only if $i \in \{i_1, \ldots, i_k\}$. In particular, no quasi-derivatives other than those specified in (13) vanish at a.
(3) At the endpoint s, we have $S(y(x,s), s-) = n-k$ and $n-k-1$ of the sign changes among consecutive quasi-derivatives are determined by $j_1, \ldots, j_{n-k-1}$. At most one quasi-derivative may vanish at s, in addition to those specified in (13).
(4) $S(y(x,s), x+) \equiv k$ for $x \in [a, s)$ and $S(y(x,s), x-) \equiv n-k$ for $x \in (a, s]$.
(5) $L_t y(x,s)$, $t = 0, \ldots, n-1$, may have only simple zeros in (a, s). $L_{t+1} y(x,s)$ has exactly one simple zero between two consecutive zeros of $L_t y$ in $[a, s]$.
(6) $L_t y(x,s)$ and its simple zeros are differentiable as functions of s.

Proof. The boundary conditions (13) imply that

$$\begin{aligned} \operatorname{sgn}[L_{i+1} y(a+\varepsilon)] &= \operatorname{sgn}[L_i y(a+\varepsilon)], \quad i \in \{i_1 \ldots, i_k\}, \\ \operatorname{sgn}[L_{j+1} y(s-\varepsilon)] &= -\operatorname{sgn}[L_j y(s-\varepsilon)], \quad j \in \{j_1, \ldots, j_{n-k-1}\} \end{aligned}$$

for sufficiently small positive ε. Hence if y satisfies (13), we have $S(y, a+) \geq k$ and $S(y, s-) \geq n-k-1$.

If there are two independent solutions y_1, y_2 which satisfy (13), there will exist a linear combination $y = c_1 y_1 + c_2 y_2$, which has an additional quasi-derivative that vanishes at a. This solution satisfies $N(y) = n$, $S(y, a+) = k+1$, and $S(y, s-) = n-k-1$. By assumption (3), $S(y, s-)$ is odd if $p(x) < 0$ and even if $p(x) > 0$, contradicting Lemma 1. This establishes the uniqueness of $y = y(x,s)$.

By the previous remark we have $S(y,a+) \geq k$ and $S(y,s-) \geq n-k-1$; according to (9), also $S(y,a+)+S(y,s-) \leq n$. Properties (2) and (3) are immediate consequences of these inequalities: otherwise Lemma 1 will be contradicted either by the parities of $S(y,a+)$ and $S(y,b-)$ or by $N(y) > n$.

To prove (4), choose first a point x_0 where no quasi-derivative of $y = y(x,s)$ vanishes. If $S(y,x_0+) = q$ then obviously $S(y,x_0-) = n-q$. On the subinterval $[a,x_0]$ we have

$$n \geq N(y) \geq S(y,a+) + S(y,x_0-) = k + (n-q)$$

and on $[x_0,s]$,

$$n \geq N(y) \geq S(y,x_0+) + S(y,s-) = q + (n-k).$$

Hence $q = k$. If $L_t y(x,s) = 0$ for a certain t, $0 \leq t \leq n-1$, choose values α, β sufficiently close to x, $\alpha < x < \beta$, where no quasi-derivative vanishes. If $x \in (a,s]$, then by definition,

$$S(y,x-) = S(y,\alpha-) = n-k,$$

and if $x \in [a,s)$,

$$S(y,x+) = S(y,\beta+) = k.$$

The function $L_t y(x,s)$, $0 \leq t \leq n-1$, can have only simple zeros in (a,s). Indeed if it has a multiple zero, then $N(y) \geq S(y,a+) + S(y,s-) + 2 > n$, contradicting (9). $L_{t+1}y$ has exactly one simple zero in (a,s) between two consecutive zeros of $L_t y$ by Lemma 1, since $N(y(x,s)) = n$.

To prove (6), let $\{u_1,\dots,u_n\}$ be an independent set of solutions of (1). We show that

$$y(x,s) = \begin{vmatrix} L_{i_1}u_1(a), & \dots, & L_{i_k}u_1(a), & L_j u_1(s), & \dots, & L_{j_{n-k-1}}u_1(s), & u_1(x) \\ \vdots & & \vdots & & & \vdots & \vdots \\ L_{i_1}u_n(a), & \dots, & L_{i_k}u_n(a), & L_j u_n(s), & \dots, & L_{j_{n-k-1}}u_n(s), & u_n(x) \end{vmatrix}.$$

Obviously, the determinant is a solution of (1) which satisfies (13). We have only to show that it is not the trivial solution. If it were identically zero, the minor consisting of the first $n-1$ columns would be of rank less than $n-1$. This would contradict the uniqueness (proved before) of the solution which satisfies (13). By the above representation and by the implicit function theorem, it follows that $L_t y(x,s)$, $t = 0,\dots,n-1$, together with its simple zeros are differentiable functions of s.

Remark. Lemma 2 depends heavily on the fact that every solution of (13) satisfies $N(y) \geq n-1$. Actually, properties (1)-(6) can be proved for solutions satisfying any boundary condition for which $N(y) \geq n-1$. Since $y(x,s)$ satisfies $n-1$ of the boundary condition (2), s is an extremal point if and only if $L_{j_{n-k}}y(x,s)$ vanishes at $x = s$. If s is an extremal point, $y(x,s)$ is the unique corresponding extremal solution.

For an extremal solution $y = y(x,\theta_i)$ we have $\operatorname{sgn}[L_{j+1}y(\theta_i - \varepsilon)] = -\operatorname{sgn}[L_j y(\theta_i - \varepsilon)]$ if and only if $j \in \{j_1,\dots,j_{n-k}\}$, i.e. only when $L_j y(\theta_i) = 0$. Therefore by Lemma 1, $L_{t+1}y$ changes sign before (after) the first (last) zero of

$L_t y$ in $(a, \theta_i(a))$ if and only if $L_t y(a) = 0$ ($L_t y(\theta_i) = 0$). Since $L_t y(x, \theta_i)$ may have only simple zeros in $(a, \theta_i(a))$, Lemma 1 implies the following

Corollary 1. *Let y be the unique extremal solution which corresponds to $\theta_i(a)$. Then $L_{t+1}y$ has exactly one simple zero between two consecutive zeros of $L_t y$ in $[a, \theta_i(a)]$, and these are its only zeros in $(a, \theta_i(a))$.*

There exists a nontrivial solution $\sum_{i=1}^n c_i u_i$ which satisfies (2) if and only if the determinant

$$W(s) = \begin{vmatrix} L_{i_1}u_1(a), & \dots, & L_{i_k}u_1(a), & L_{j_1}u_1(s), & \dots, & L_{j_{n-k}}u_1(s) \\ \vdots & & & & & \vdots \\ L_{i_1}u_n(a), & \dots, & L_{i_k}u_n(a), & L_{j_1}u_n(s), & \dots, & L_{j_{n-k}}u_n(s) \end{vmatrix}$$

vanishes. Thus, the extremal point $\theta_i(a)$ is the i-th zero of $W(s)$ in (a, ∞). Note that $W(s) = L_{j_{n-k}}y(x, s)\big|_{x=s}$.

Theorem 1. *The extremal points of (2) are simple zeros of $W(s)$.*

Proof. To stress the dependence of $W(s)$ on the indices $\{j_1, \dots, j_{n-k}\}$, we use the notation $W(s; j_1, \dots, j_{n-k})$. Clearly

$$\begin{aligned} &\frac{d}{ds}W(s; j_1, \dots, j_{n-k}) \\ &\quad = \sum_{t=1}^{n-k} W(s; j_1, \dots, j_{t-1}, j_t + 1, j_{t+1}, \dots, j_{n-k})/\rho_{j_t}(s). \end{aligned} \tag{14}$$

Throughout the proof let s be an extremal point $\theta_i(a)$ of (2), thus

$$W(s; j_1, \dots, j_{n-k}) = 0. \tag{15}$$

All the determinants in (14) for which $j_t + 1 \neq j_{t+1}$ are non-zero. Indeed, suppose that

$$W(s; j_1, \dots, j_t + 1, \dots, j_{n-k}) = 0. \tag{16}$$

By (15), there exists a solution y_1, namely the corresponding extremal solution, which satisfies (2). By (16), there exists a solution y_2 which satisfies the conditions

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \dots, i_k\}, \\ L_j y(s) &= 0, \qquad j \in \{j_1, \dots, j_t + 1, \dots, j_{n-k}\}, \end{aligned} \tag{17}$$

If y_1, y_2 are linearly dependent, they satisfy both (2) and (17). This implies $N(y_1) \geq n + 1$, an impossibility. On the other hand, if y_1, y_2 are independent, they are two independent solutions obeying the $n - 1$ boundary conditions

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \dots, i_k\}, \\ L_j y(s) &= 0, \qquad j \in \{j_1, \dots, j_{n-k}\} \setminus \{j_t\}, \end{aligned}$$

and this is impossible by Lemma 2.

If the sum in (14) consists of a single non-zero term, as is the case for the boundary conditions (4), the proof is complete. To prove that $W'(s) \neq 0$ in the general case, we show that every two non-zero terms of (14), for example, the q-th term and the r-th term, have the same sign. Let $j_q < j_r$ and let

$$y_1(x,s) = \begin{vmatrix} L_{i_1}u_1(a), & \dots, & L_{j_1}u_1(s), & \dots & & \dots, & L_{j_{n-k}}u_1(s), u_1(x) \\ \vdots & & & & \overset{j_q}{\text{missing}} & & \\ L_{i_1}u_n(a), & \dots, & L_{j_1}u_n(s), & \dots & & \dots, & L_{j_{n-k}}u_n(s), u_n(x) \end{vmatrix}.$$

As in Lemma 2, $y_1(x,s)$ is the essentially unique solution of (1) which satisfies

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \dots, i_k\}, \\ L_j y(s) &= 0, \qquad j \in \{j_1, \dots, j_{n-k}\} \setminus \{j_q\}. \end{aligned} \tag{18}$$

Since the extremal solution corresponding to $s = \theta_i$ satisfies (18), it follows by uniqueness that $y_1(x,s)$ is a constant multiple of the extremal solution and satisfies (2). By differentiation we have

$$(-1)^{n-k-q+1} L_{j_q+1} y_1(x,s)\Big|_{x=s} = W(s; j_1, \dots, j_q + l, \dots, j_{n-k}) \equiv W_q. \tag{19}$$

Similarly, if

$$y_2(x,s) = \begin{vmatrix} L_{i_1}u_1(a), & \dots, & L_{j_1}u_1(s), & \dots & & \dots, & L_{j_{n-k}}u_1(s), u_1(x) \\ \vdots & & & & \overset{j_r}{\text{missing}} & & \\ L_{i_1}u_n(a), & \dots, & L_{j_1}u_n(s), & \dots & & \dots, & L_{j_{n-k}}u_n(s), u_n(x) \end{vmatrix},$$

then

$$(-1)^{n-k-r+1} L_{j_r+1} y_2(x,s)\Big|_{x=s} = W(s; j_1, \dots, j_r + l, \dots, j_{n-k}) \equiv W_r. \tag{20}$$

Using (19) and (20) we shall now show that $W_q W_r > 0$.

It was remarked above that $y_1(x,s)$ satisfies (2). Therefore

$$\text{sgn }[L_{j+1} y_1(x,s)\Big|_{s-\varepsilon}] = -\text{sgn }[L_j y_1(x,s)\Big|_{s-\varepsilon}]$$

if and only if $j \in \{j_1, \dots, j_q, \dots, j_{n-k}\}$. In the sequence $L_0 y_1(x,s)\Big|_{s-\varepsilon}, \dots,$ $L_{j_q+1}\Big|_{s-\varepsilon}$ there are exactly q changes of sign. Thus

$$\begin{aligned} \text{sgn }[W_q] &= (-1)^{n-k-q+1} \text{sgn }[L_{j_q+1} y_1(x,s)\Big|_{s-\varepsilon}] \\ &= (-1)^{n-k+1} \text{sgn }[L_1 y_1(x,s)\Big|_{s-\varepsilon}]. \end{aligned}$$

A similar relation holds between W_r and $y_2(x,s)$. Since $y_1(x,s), y_2(x,s)$ are both constant multiples of the extremal solution, we get

$$\text{sgn }[W_q W_r] = \text{sgn }[y_1(x,s)\Big|_{s-\varepsilon} y_2(x,s)\Big|_{s-\varepsilon}] = \text{sgn }[y_1(x,s) y_2(x,s)]$$

and it suffices to show that the constant ratio of $y_1(x,s)$ and $y_2(x,s)$ is positive. Choose an index i_{k+1}, different from $i_1, \dots, i_k$; say $i_1 < \dots < i_k < i_{k+1} \leq n-1$. To prove the last assertion it is sufficient to show that

$$\text{sgn }[L_{i_{k+1}} y_1(a,s)] = \text{sgn }[L_{i_{k+1}} y_2(a,s)]. \tag{21}$$

The choice of i_{k+1} ensures that the quasi-derivatives in (21) are non zero, else we would have

$$N(y_1(x,s)) \geq S(y-1,a+) + S(y,s-) \geq (k+1) + (n-k) > n.$$

We apply $L_{i_{k+1}}$ to the determinant $y_1(x,s)$ and substitute $x = a$. In this determinant there exists a column which corresponds to j_r but the column corresponding to j_q is missing. We bring the last column to the $(k+1)$-th place and the column which corresponds to j_r , to the last place. Now

$$L_{i_{k+1}} y_1(a,s) = (-1)^{n-k-1}(-1)^{n-k-r-1} \times \begin{vmatrix} L_{i_1}u_1(a), \ldots, & L_{i_k}u_1(a), L_{i_{k+1}}u_1(a), L_{j_1}u_1(s), \ \ldots & \ldots, \ L_{j_{n-k}}u_1(s), L_{j_r}u_1(s) \\ \vdots & \underset{missing}{j_q, j_r} & \\ L_{i_1}u_1(a), \ldots & & \ldots, \quad L_{j_r}u_n(s) \end{vmatrix}$$

(The indices j_q, j_r are missing among the columns which correspond to $j_1, \ldots, j_{n-k}$.) Similarly,

$$L_{i_{k+1}} y_2(a,s) = (-1)^{n-k-1}(-1)^{n-k-q-2} \times \begin{vmatrix} L_{i_1}u_1(a), \ldots, & L_{i_k}u_1(a), L_{i_{k+1}}u_1(a), L_{j_1}u_1(s), \ \ldots & \ldots, \ L_{j_{n-k}}u_1(s), L_{j_q}u_1(s) \\ \vdots & \underset{missing}{j_q, j_r} & \\ L_{i_1}u_1(a), \ldots & & \ldots, \quad L_{j_q}u_n(s) \end{vmatrix}$$

Observe that the cofactors in the last two relations differ by (-1). This is explained as follows: In the determinant $y_2(x,s)$ the column corresponding to j_r is missing. Since $j_q < j_r \leq j_{n-k}$, one less exchange of columns is necessary to bring the column which corresponds to j_q to the last entry of $L_{i_{k+1}} y_2(a,s)$ than for the corresponding exchange in $L_{i_{k+1}} y_1(a,s)$.

Now define a third function

$$y_3(x,s) = \begin{vmatrix} L_{i_1}u_1(a), \ldots, & L_{i_{k+1}}u_1(a), L_{j_1}u_1(s), \ \ldots & \ldots, \ L_{j_{n-k}}u_1(s), u_1(x) \\ \vdots & \underset{missing}{j_q, j_r} & \\ L_{i_1}u_n(a), \ldots, & L_{i_{k+1}}u_1(a), L_{j_1}u_n(s), \ \ldots & \ldots, \ L_{j_{n-k}}u_n(s), u_n(x) \end{vmatrix},$$

which is the unique solution satisfying the boundary conditions

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \ldots, i_k, i_{k+l}\}, \\ L_j y(s) &= 0, \qquad j \in \{j_1, \ldots, j_{n-k}\} \setminus \{j_q, j_r\}. \end{aligned} \tag{22}$$

Using $y_3(x,s)$, we have

$$\begin{aligned} L_{i_{k+1}} y_1(a,s) &= (-1)^r L_{j_r} y_3(x,s)\Big|_{x=s}, \\ L_{i_{k+1}} y_2(a,s) &= (-1)^{q-1} L_{j_r} y_3(x,s)\Big|_{x=s}. \end{aligned} \tag{23}$$

To prove (21), we now determine the signs of $L_0 y_3(x,s), \ldots, L_{n-1} y_3(x,s)$ on a left neighborhood of s. By (22) we have $S(y_3, a+) \geq k+1$, $S(y_3, s-) \geq n-k-2$, and $N(y_3) \geq n-1$. But $S(y_3, a+) = k+1$, $S(y_3, s-) = n-k-1$ is impossible by

Lemma 1 and assumption (3), and therefore the sign change

$$\text{sgn}\ [L_{j+1}y_3(x,s)\Big|_{s-\varepsilon}] = -\text{sgn}\ [L_jy_3(x,s)\Big|_{s-\varepsilon}] \tag{24}$$

must hold precisely for the $n-k-2$ indices $\{j_1,\dots,j_{n-k}\}\setminus\{j_q,j_r\}$. Among the functions $L_{j_q}y_3,\dots,L_{j_r}y_3$, (24) holds exactly for the $r-q-1$ indices $\{j_{q+1},\dots,j_{r-1}$ and so

$$\text{sgn}\ [L_{j_r}y_3(x,s)\Big|_s] = (-1)^{r-q-1}\text{sgn}\ [L_{j_q}y_3(x,s)\Big|_s] \tag{25}$$

Relations (25) and (23) imply (21), which is equivalent to $W_qW_r > 0$. This completes the proof of Theorem 1.

Theorem 2. *θ_i is a differentiable strictly increasing function, whose domain is of the form $[0,b)$, $0\le b\le\infty$.*

Proof. If $\theta_i(a)$ exists, it is a simple zero of $W(a,s)$. By the implicit function theorem, θ_i is differentiable in a neighborhood of a and

$$\frac{d\theta_i}{da} = -\frac{\partial W}{\partial a}\Big/\frac{\partial W}{\partial}\Big|_{s=\theta_i(a)}.$$

Moreover, θ_i may be continued as long as it is bounded. As in Theorem 1, one shows that

$$\frac{\partial W}{\partial a}\Big|_{s=\theta_i(a)} = \sum_{t=1}^{k} W(a,\theta_i(a);i_1,\dots,i_t+1,\dots,i_k,j_1,\dots,j_{n-k}) \ne 0.$$

Thus $\theta_i(a)\ne 0$ and θ_i is monotonic. Furthermore it is increasing, for otherwise it could be continued until the inequality $a<\theta_i(a)$ fails.

If $\theta_i(a)$ exists, then θ_i is defined in some open interval A containing a. Let $a'=\inf A$. For every $b\in(a',a]$, $\theta_i(b)$ exists, that is $W(b,\theta_i(b))=0$. Therefore by continuity, $\theta_i(a')$ also exists. If $a'>0$ (((1) is defined on $[0,\infty)$), then θ_i is defined on a neighbourhood of a', contradicting the definition of a'. Thus, $a'=0$. This completes the proof of the theorem.

The next result describes the zeros of $y(x,s)$ as functions of s.

Theorem 3. *The number of simple zeros of $L_ry(x,s)$, $0\le r\le n-1$, in (a,s) can vary, as s increases, only when a simple zero enters (a,s) through the variable endpoint s.*

Proof. By Lemma 2, two simple zeros of $L_ry(x,s)$ from (a,s) cannot meet in $[a,s]$, nor can a simple zero meet the endpoint a, as s varies. Since the simple zeros are continuous functions of s, their number in (a,s) can vary only when one simple zero enters (a,s) or leaves it through the endpoint s. We shall prove that as s increases, simple zeros of $L_ry(x,s)$ can only enter (a,s).

It is sufficient to prove the theorem only for quasi-derivatives on which no boundary condition at s is imposed by (13). Indeed, suppose that the boundary conditions

$$L_jy(s)=0,\qquad j=r,r+1,\dots,r+q-1 \tag{26}$$

are included in (13) for a certain q, $q \geq 1$, but $r+q \notin \{j_1, \ldots, j_{n-k-1}\}$. If $L_r y(x,s)$ has a zero in a given left neighbourhood of s, then by (26) the function $L_{r+q} y(x,s)$ also has a zero in the same neighbourhood. Conversely, assume that, for certain values of s, $L_r y(x,s)$ has no zero in an appropriate left neighbourhood of s. Then $\operatorname{sgn}[L_r y(x,s)\big|_{s-\varepsilon}]$, is fixed for $0 < \varepsilon < \varepsilon_0$, ε_0 independent of s. By (26),

$$\operatorname{sgn}\ [L_{r+q} y(x,s)\Big|_{s-\varepsilon}] = (-1)^q \operatorname{sgn}\ [L_r y(x,s)\Big|_{s-\varepsilon}].$$

Therefore $\operatorname{sgn}[L_{r+q} y(x,s)|_{s-\varepsilon}]$ also remains fixed for the same values of s, and $L_{r+q} y(x,s)$ has no zero in the appropriate left neighbourhoods of s. Hence without loss of generality we may assume that $r \notin \{j_1, \ldots, j_{n-k-1}\}$.

First we prove the theorem under the additional restriction $r+1 \notin \{j_1, \ldots, j_{n-k-1}\}$. In this case, if $L_r y(x, s_0)$ vanishes at $x = s_0$ then $L_{r+1} y(x, s_0)\big|_{s_0} \neq 0$ and s_0 is a simple zero of $L_r y(x, s_0)$. By the implicit function theorem, there exists a simple zero $x(s)$ of $L_r y(x,s)$ such that $x(s_0) = s_0$ and

$$x'(s_0) = -\frac{\partial}{\partial s} L_r y(x,s) \Big/ \frac{\partial}{\partial x} L_r y(x,s)\Big|_{s=s_0, x=x(s_0)=s_0}$$

Differentiating the determinant $L_r y(x,s)$ with respect to x and s and substituting $s = s_0$, $x = x(s_0) = s_0$, we obtain

$$\frac{\partial}{\partial x} L_r y(x,s)\Big|_{(s_0,s_0)} = \rho_{r+1}^{-1}(s_0) W(s_0; j_1, \ldots, j_{n-k-1}, r+1) \tag{27}$$

$$\frac{\partial}{\partial s} L_r y(x,s)\Big|_{(s_0,s_0)} = \sum_{t=1}^{n-k-1} \rho_{j_t+1}^{-1}(s_0) W(s_0; j_1, \ldots, j_t + 1, \ldots j_{n-k-1}, r) \tag{28}$$

But s_0 is an extremal point corresponding to the boundary conditions

$$\begin{aligned} L_i y(a) &= 0, & i &\in \{i_1, \ldots, i_k\}, \\ L_j y(s) &= 0, & j &\in \{j_1, \ldots, j_{n-k-1}, r\} \end{aligned} \tag{29}$$

and it satisfies $W(s_0; j_1, \ldots, j_{n-k-1}, r) = 0$. Therefore by the proof of Theorem 1 all the determinants in (27)-(28) have the same sign; hence $x'(s_0) \leq 0$. The distance between the simple zero $x(s)$ and the variable endpoint s, $d(s) = x(s) - s$, satisfies $d(s_0) = 0$, $d'(s_0) \leq -1$. Therefore as s increases and passes through s_0, $x(s)$ enters the interval (a, s).

On the other hand, if $r+1 \in \{j_1, \ldots, j_{n-k-1}\}$, the zero of $L_r y(x, s_0)$ at s_0 is not simple and the previous argument fails. In this case let s_1 be the extremal point of (29) which follows s_0. (Regarding the existence of s_1, see the remark at the end of the proof.) To prove that a simple zero is added to (a,s) as s passes through s_0, it is sufficient to show that $L_r y(x, s_1)$ has more zeros in (a, s_1) than $L_r y(x, s_0)$ has in (a, s_0).

In the set $\{j_1, \ldots, j_{n-k-1}, r\}$ there exists an index, say j_q, such that j_{q+1} does not belong to the set. Let $y_1(x,s)$ be the unique solution satisfying the conditions

$$\begin{aligned} L_i y(a) &= 0, & i &\in \{i_1, \ldots, i_k, \}, \\ L_j y(s) &= 0, & j &\in \{j_1, \ldots, j_{n-k-1}, r\} \setminus \{j_q\}. \end{aligned} \tag{30}$$

For the extremal point s_0 of (29), the extremal solution $y(x,s_0)$ satisfies (30), and by uniqueness we may assume that $y_1(x,s_0) \equiv y(x,s_0)$. Similarly $y_1(x,s_1) \equiv y(x,s_1)$. By the choice of j_q, we may apply the first part of the proof to $y_1(x,s)$ and deduce that $L_{j_q}y_1(x,s_1) \equiv L_{j_q}y(x,s_1)$ has one more zero in (a,s_1) than $L_{j_q}y_1(x,s_0) \equiv L_{j_q}y(x,s_0)$ has in (a,s_0). By Corollary 1, $L_{t+1}y(x,s_0)$ has exactly one simple zero between two consecutive simple zeros of $L_ty(x,s_0)$ in $[a,s_0]$, and those are its only zeros in (a,s_0). The same property holds for $L_{t+1}y(x,s_1)$ in (a,s_1). In view of this relation among the zeros of consecutive quasi-derivatives of extremal solutions and the above description of the zeros of $L_{j_q}y(x,s_0)$ and $L_{j_q}y(x,s_1)$, it follows that for every t, $0 \le t \le n-1$, $L_ty(x,s_1)$ has exactly one more simple zero in (a,s_1) than $L_ty(x,s_0)$ has in (a,s_0). In particular this holds for $L_ry(x,s_1)$ and $L_ry(x,s_0)$, completing the proof of the theorem.

Note that in the last case the zero which is added to (a,s) does not necessarily cross the endpoint s from its right side to the left side. It may appear equally well, for example, when a multiple zero at s splits into simple zeros.

Remark. We may assume without loss of generality that the extremal point s_1, $s_1 > s_0$, exists. Choose c , $c > s_0$, and on $[c,\infty)$ define $p(x) \equiv p(c)$, $\rho_i(x) \equiv \rho_i(c)$. This definition does not alter $y(x,s)$ for $s \in [a,c]$, but on $[c,\infty)$, on the other hand, (1) becomes an equation with constant coefficients. For this equation $\theta_i(c)$ exists for every i. Indeed, $\theta_i(c)$ is an extremal point for $y^{(n)}n + Ay = 0$ and (29) if and only if $\lambda = [\theta_i(c) - c]^n$ is an eigenvalue of $y^{(n)} + \lambda Ay = 0$ with the corresponding boundary conditions at $a = 0, s = 1$. Now there exists an infinite sequence of such eigenvalues [3], so $\theta_i(c)$ exists for every i. Since $a < c$, we have thus shown that $\theta_i(c)$ exists for every i by Theorem 2.

In the course of the last proof we have seen that, for two consecutive extremal points $s_0 < s_1$, $L_ty(x,s_1)$, has one zero more in (a,s_1) than $L_ty(x,s_0)$ has in (a,s_0). This yields

Corollary 2. The quasi-derivative $L_ty(x,\theta_i(a)$ of the extremal solution $y(x,\theta_i(a))$ has $i+\ell_t$ simple zeros in $(a,\theta_i(a))$, $i = 1,2,\ldots$. For the boundary conditions (4), $y(x,\theta_i(a))$ has exactly $i-1$ simple zeros in $(a,\theta_i(a))$.

Since $L_ty(x,\theta_i(a))$ is a continuous function of a, and the number of its simple zeros in $(a,\theta_i(a))$ cannot vary with a, the constants $\ell_0,\ldots,\ell_{n-1}$ depend only on the indices $i_1,\ldots,j_{n-k}$. By Corollary 1, any one of the constants ℓ_t determines all the others; they can be found easily by considering $y(x,s)$ for s sufficiently close to a. For the boundary conditions (4), for example, it follows that $\ell_0 = -1$ by the known fact that $y(x,\theta_1(a)) \ne 0$ on $(a,\theta_1(a))$.

Corollary 3. Let $y_j(x)$ satisfy (13) at the points $a_j, s_j,\ j = 1,2$. If $(a_2,s_2) \supset (a_1,s_1)$ then L_ry_2 has at least as many zeros in (a_2,s_2) as L_ry_1 has in (a_1,s_1).

Indeed, the number of zeros of $L_ry(x,s)$ in (a,s) varies only when s passes through an extremal point θ_i of (29). Let $\theta_{i-1}(a_1) < s_1 \le \theta_i(a_1)$. By Corollary 2 and Theorem 3, L_ry_1 has exactly $i+\ell_r$ zeros in (a_1,s_1). Since $a_2 \le a_1$, we have

$$s_2 \ge s_1 > \theta_{i-1}(a_1) \ge \theta_{i-1}(a_2).$$

Hence $L_r y_2$ has at least $i + \ell_r$ zeros in (a_2, s_2) and the corollary follows immediately.

Corollary 4. *Let $\lambda_i(a,s) \neq 0$ be the i-th eigenvalue of the equation $Ly + \lambda p(x) y = 0$ with boundary conditions (2). Then $|\lambda_i(a,s)|$ is a decreasing function of the interval (a,s) (with respect to inclusion).*

Let $y(x,s,\lambda)$ be the solution of $Ly + \lambda p(x) y = 0$ which satisfies (13). The result follows by observing from Corollary 3 and [3] that the number of zeros of $L_{j_{n-k}} y(x,s,\lambda)$ in (a,s) increases as (a,s) and $|\lambda|$ increase. The differentiability of $\lambda_i(a,s)$ with respect to s was proved in [3].

The boundary conditions (6) play an important role in [9], since the Green's function which corresponds to the operator L_n and (6) can be calculated explicitly and is independent of s. The focal point $\zeta_{k,n-k}(a)$ appears quite naturally in the forthcoming proofs, due to the following characterization.

Lemma 3. *Equation (1) is $(k, n-k)$-disfocal in (a, b) if and only if there exists a solution y which satisfies*

$$\begin{aligned} L_i y &> 0, \qquad i = 0, \dots, k, \\ (-1)^{j-k} L_j y &> 0, \qquad j = k, \dots, n, \end{aligned} \tag{31}$$

on (a,b).

Proof. *Sufficiency.* Let y satisfy (31) and assume for contradiction that (1) is not disfocal in (a,b). That is, assume there exists a solution u which satisfies (6) at the points $a', b' = \zeta_{k,n-k}(a')$, $a < a' < b' < b$. We consider the solution $w_\lambda = y - \lambda u$ on $[a', b']$. For $\lambda = 0$ we have $L_t w_0 = L_t y \neq 0$ on $[a', b']$ by (31). Let λ_0 be the smallest positive value of λ such that $L_t w_\lambda$ vanishes in $[a', b']$ for some t, $0 \le t \le n-1$. No quasi-derivative of $L_t w_{\lambda_0}$ changes sign in (a', b'). Otherwise, by continuity, $L_t w_\lambda$ would change sign for λ sufficiently close to λ_0, contradicting the definition of λ_0. Therefore

$$\begin{aligned} L_i w_{\lambda_0} &\ge 0, \qquad i = 0, \dots, k, \\ (-1)^{j-k} L_j w_{\lambda_0} &\ge 0, \qquad j = k, \dots, n, \qquad a' \le x \le b'. \end{aligned} \tag{32}$$

At the endpoints we have by (6) and (31),

$$\begin{aligned} L_i w_{\lambda_0}(a') = L_i y(a') &> 0, \qquad i = 0, \dots, k-1, \\ (-1)^{j-k} L_j w_{\lambda_0}(b') = (-1)^{j-k} L_j y(b') &> 0, \qquad j = k, \dots, n-1. \end{aligned} \tag{33}$$

Relations (32)-(33) imply that $L_t w_{\lambda_0} \neq 0$ on $[a', b']$, contradicting the definition of λ_0. Indeed, for $0 \le t \le k-1$, we have $L_t w_{\lambda_0}(a') > 0$ and $L_{t+1} w_{\lambda_0} \ge 0$ on $[a', b']$. Hence $L_t w_{\lambda_0}$ increases and does not vanish in $[a', b']$. For $k \le t \le n-1$ we have also $(-1)^{t-k} L_t w_{\lambda_0}(b') > 0$ and $(-1)^{t-k+1} L_{t+1} w_{\lambda_0} \ge 0$ on $[a'b']$. Thus $(-1)^{t-k} L_t w_{\lambda_0}$ decreases and does not vanish in $[a'b']$. This contradiction proves that (1) is $(k, n-k)$-disfocal on (a,b).

Necessity. Let $a < s < b$, and let $y(x,s)$ be the solution which satisfies

$$\begin{aligned} L_i y(a) &> 0, \qquad i = 0, \dots, k-1, \\ L_j y(s) &> 0, \qquad j = k, \dots, n-2, \end{aligned} \tag{34}$$

If s is sufficiently close to a, then $L_{n-1}y(x,s) \neq 0$ on $[a,s]$, since it is impossible that every quasi-derivative of a nontrivial solution will have a zero in an arbitrary small interval. A zero of $L_{n-1}y(x,s)$ appears in (a,s) only when s passes through $\zeta_{k,n-k}(a)$. If (1) is $(k,n-k)$-disfocal on (a,b), by the continuity of $\zeta_{k,n-k}$ it is also $(k,n-k)$-disfocal on $[a,b)$ and $L_{n-1}y(x,b) \neq 0$ on $[a,b)$. If $L_t y(x,b)$, $0 \leq t \leq n-2$, has a zero in (a,b), we find by (34) and Rolle's theorem that $L_{n-1}y(x,b)$ vanishes in (a,b), which is impossible. Therefore $L_t y(x,b) \neq 0$ on (a,b). By (34) and (3) it is easily seen that the sign of $L_t y(x,b)$ in (a,b) is given by (31).

Lemma 4. *If $\eta_{k,n-k}(a)$ exists, then $\zeta_{k,n-k}(a) < \eta_{k,n-k}(a)$.*

Proof. Let $\theta_i(a)$ be the first extremal point for the conditions

$$\begin{aligned} &L_i y(a) = 0, \qquad i = 0,\dots,k-1,\\ &L_j y(s) = 0, \qquad j \in \{j_1,\dots,j_q,\dots,j_{n-k}\}. \end{aligned} \tag{35}$$

To prove the lemma, it suffices to show that for the conditions

$$\begin{aligned} &L_i y(a) = 0, \qquad i = 0,\dots,k-1,\\ &L_j y(s) = 0, \qquad j \in \{j_1,\dots,j_q+1,\dots,j_{n-k}\}, \end{aligned} \tag{36}$$

where $j_q + 1 < j_{q+1}$, there is an extremal point in $(a, \theta_1(a))$.

Let $y(x,s)$ be the solution satisfying

$$\begin{aligned} &L_i y(a) = 0, \qquad i = 0,\dots,k-1,\\ &L_j y(s) = 0, \qquad j \in \{j_1,\dots,j_{n-k}\} \setminus \{j_q\}. \end{aligned} \tag{37}$$

It will be shown below that as s increases from a to $\theta_1(a)$, a zero is added to $L_{j_q+1}y(x,s)$ in (a,s). This zero must enter (a,s) from the right endpoint for a certain value of s, and thus (36) has an extremal point in $(a,\theta_1(a))$.

First we show that each $L_t y(x,s)$, $0 \leq t \leq j_q$, has a zero in $[a,s)$ for every s. If $0 \leq t \leq k-1$, then $L_t y(a,s) = 0$ by (37). Assume that $k \leq j_q \leq n-1$ and let $k \leq t \leq j_q$. Since (37) imposes no boundary condition at s on either $L_{j_q}y$ or $L_{j_q+1}y$, at most $n-t-2$ conditions are placed on the $n-t$ quasi-derivatives $L_t y,\dots,L_{n-1}y$. Thus at least $(n-k-1)-(n-t-2) = t-k+1$ of the functions $L_0 y(x,s),\dots,L_{t-1}y(x,s)$ vanish at s. Counting the zeros of $L_0 y(x,s),\dots,L_{n-1}y(x,s)$ at a and s, we find by applying Rolle's theorem t times that $L_t y(x,s)$ has at least $k+(t-k+1)-t = 1$ zeros in (a,s). In particular, $L_{j_q}y(x,s)$ has a zero in $[a,s)$. Let us denote the last zero of $L_{j_q}y(x,s)$ in $[a,s)$ by $x(s)$. For $a < s < \theta_1(a)$, $x(s)$ is a continuous function of s, because the number of zeros of $L_{j_q}y(x,s)$ in (a,s) increases only when s passes through the extremal point $\theta_1(a)$ of (35).

We claim that, when s is sufficiently close to a, the function $L_{j_q+1}y(x,s)$ has no zero in $(x(s);s]$. Suppose the contrary. Then, due to the additional zero of $L_{j_q+1}y(x,s)$ in $(x(s);s]$, we find by applying the preceding argument that each of the following quasi-derivatives also has a zero in $[a,s]$. But it is impossible that each quasi-derivative of a nontrivial solution will have a zero in an arbitrary small interval.

On the other hand, for $s = \theta_1(a)$, $L_{j_q}y(x,\theta_1(a))$ vanishes at $\theta_1(a)$ as well as at $x(\theta_1(a))$. Therefore $L_{j_q+1}y(x,\theta_1(a))$ has a zero in $\big(x(\theta_1(a)),\theta_1(a)\big)$. This zero of

$L_{j_q+1}y(x,s)$ is added to (a,s) for some s, $a < s < \theta_1(a)$, and so (36) has an extremal point in $(a, \theta_1(a))$.

An alternative proof appears in [4], though $\zeta_{k,n-k}$ is not mentioned explicitly.

3. Extremal points and oscillatory solutions

The following sequence of theorems will prove Theorem I.

Theorem 4. *If all the extremal points $\theta_i(a)$, $i = 1, 2, \ldots$, exist for one system of boundary conditions of type (2), all the extremal points exist for every system of boundary conditions of type (2).*

Proof. It is sufficient to prove that all the extremal points $\theta_i(a)$, $i = 1, 2, \ldots$, of (2) exist if and only if all the extremal points $\tilde{\theta}_i(a)$, $i = 1, 2, \ldots$, corresponding to the conditions

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \ldots, i_{n-k}\} \\ L_j y(s) &= 0, \qquad j \in \{j_1, \ldots, j_q + 1, \ldots, j_{n-k}\}, \end{aligned} \tag{38}$$

where $j_q + 1 \notin \{j_1, \ldots, j_q, \ldots, j_{n-k}\}$, exist. We shall prove that the series of points $\{\theta_i(a)\}$ and $\{\tilde{\theta}_i(a)\}$ separate each other.

Let $y(x,s)$ be the solution satisfying

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \ldots, i_{n-k}\} \\ L_j y(s) &= 0, \qquad j \in \{j_1, \ldots, j_{n-k}\} \setminus \{j_q\}. \end{aligned} \tag{38}$$

The extremal solutions of (2) and (38) which correspond to $\theta_i(a)$ and $\tilde{\theta}_i(a)$ are $y(x, \theta_i(a))$ and $y(x, \tilde{\theta}_i(a))$ respectively.

By Corollary 2, $L_t y(x, \theta_{i+1}(a))$ has one more simple zero in $(a, \theta_{i+1}(a))$ than $L_t y(x, \theta_i(a))$ has in $(a, \theta_i(a))$. In particular this holds for $t = j_q + 1$. By Theorem 3, the additional zero of $L_{j_q+1}y(x,s)$ enters (a,s) through the right endpoint for a certain $\tilde{s}$, $\theta_i(a) < \tilde{s} < \theta_{i+1}(a)$. Thus (38) has an extremal point in $(\theta_i(a), \theta_{i+1}(a))$. By the same argument, a simple zero is added to $L_{j_q}y(x,s)$ in (a,s) as s increases from $\tilde{\theta}_i(a)$ to $\tilde{\theta}_{i+1}(a)$, and (2) has an extremal point in $(\tilde{\theta}_i(a), \tilde{\theta}_{i+1}(a))$.

When a system of boundary conditions differs from (2) by one boundary condition at the endpoint a, we define $y(x,s)$ by deleting a boundary condition at a instead of s.

Theorem 5. *If the conjugate point $\eta_{k,n-k}(a)$ exists for every $a \geq 0$ then*
(1) Equation (1) has an oscillatory solution y on $[0, \infty)$ which satisfies $S(y, x+) \equiv k$ for sufficiently large values of x.
(2) For every system of type (2) and for every $a \geq 0$, all the extremal points $\theta_i(a)$, $i = 1, 2, \ldots$, exist.

Proof (cf. [4]). Let an arbitrary value a be given. We obtain a solution with infinitely many zeros in $[a, \infty)$ and prove that all the extremal points corresponding to (4) exist. By Theorem 4, it follows that the extremal points of every system of type (2) exist.

The definition of the quasi-derivatives $L_0 y$, $\ldots$, $L_{n-1}y$, and in general, the extremal points of (2), also depend on the factorization of the disconjugate operator L_n into the form

$$L_n y = \rho_n(\rho_{n-1} \dots (\rho_1(\rho_0 y)')' \dots)'.$$

It is well known that such a factorization of L_n is not unique. However, the extremal points of (4) themselves are independent of the factorization because

$$L_0 y(c) = \dots = L_{i-1} y(c) = 0, \quad L_i y(c) \neq 0$$

if and only if

$$y(c) = \dots = y^{(i-1)}(c) = 0, \quad y^{(i)}(c) \neq 0,$$

for any choice of $\rho_0, \dots, \rho_n$. Therefore in studying the extremal points of (4) we may choose any convenient factorization of L_n. It is known [10] that $\rho_1, \dots, \rho_{n-1}$ can be chosen so that

$$\int^{\infty} \rho_i^{-1}(x)\, dx = \infty, \qquad i = 1, \dots, n-1. \tag{40}$$

In the following part of the proof, we assume that L_n has a representation (39), (40).

Let $y(x, s)$ be the solution which satisfies

$$\begin{aligned} y^{(i)}(a) &= 0, \qquad i = 0, \dots, k-1, \\ y^{(j)}(s) &= 0, \qquad j = 0, \dots, n-k-2. \end{aligned} \tag{41}$$

For any factorization $\overline{\rho}_n(\overline{\rho}_{n-1} \dots (\overline{\rho}_1(\overline{\rho}_0 y)')' \dots)'$ of $L_n y$, and for the corresponding quasi-derivatives $\overline{L}_0 y = \overline{\rho}_0 y, \dots, \overline{L}_{n-1} y, \overline{L}_n y = L_n y$, the solution $y(x, s)$ satisfies

$$\begin{aligned} \overline{L}_i y(a) &= 0, \qquad i = 0, \dots, k-1, \\ \overline{L}_j y(s) &= 0, \qquad j = 0, \dots, n-k-2. \end{aligned}$$

Hence by Lemma 2,

$$S\big(\overline{L}_0 y(x+\varepsilon, s), \dots, (-1)^n \overline{L}_n y(x+\varepsilon, s)\big) \equiv k \qquad \text{for } x \in [a, s).$$

As in [4], we shall prove that the number of zeros of $y(x, s)$ in (a, s) tends to infinity as $s \to \infty$, and each of those zeros is a bounded continuous function of s. Since the zeros enter (a, s) through the endpoint s, we obtain an infinite sequence of extremal points. Since each zero is bounded, it follows that $\lim y(x, s)$ has in infinitely many zeros in $[a, \infty)$.

Assume on the contrary that as $s \to \infty$ the functions $L_0 y(x, s), \dots, L_{n-1} y(x, s)$ have at most a bounded number of zeros in (a, s) which are bounded from above as functions of s. By Lemma 2, the zeros are continuous functions of s. Let $M - 1$ be their commun upper bound. The other zeros (if any) are unbounded. We choose a sequence $\{s_i\}$ so that the unbounded zeros of $L_0 y(x, s_i), \dots, L_{n-1} y(x, s_i)$ tend to infinity as $s_i \to \infty$. We normalize the solutions $y(x, s_i)$ by

$$\sum_{t=0}^{n-1} [L_t y(M, s_i)]^2 = 1.$$

The normalized family $\{y(x, s_i)\}$ is a compact set of solutions, and therefore it is possible to choose a subsequence $\{s_{i_j}\}$, $s_{i_j} \to \infty$, and a nontrivial solution y so that $L_t y(x, s_{i_j}) \to L_t y$, $t = 0, \ldots, n$, uniformly on compact intervals.

The zeros of $L_0 y(x, s_{i_j}), \ldots, L_{n-1} y(x, s_{i_j})$ in $[M, s_{i_j})$, if any, tend to infinity as $s_{i_j} \to \infty$ and their other zeros are in $(a, M-1]$. Hence $L_0 y(x, s_{i_j}), \ldots, L_{n-1} y(x, s_{i_j})$ do not vanish on an arbitrarily large right neighborhood of the point $x = M - 1/2$, provided s_{i_j} is sufficiently large. Therefore the quasi-derivatives of $y = \lim_{s\to\infty} y(x, s)$ do not change their signs on $(M-1/2, \infty)$ and in fact, they do not vanish on $[M, \infty)$.

Let q, $i \le q \le n$, be the greatest index such that the two consecutive quasi-derivatives $L_{q-1}y, L_q y$ are of the same sign, say positive, on $[M, \infty)$, if such an index exists. Since $L_q y > 0$, it is clear that $L_{q-1} y$ increases, and by (40)

$$
\begin{aligned}
L_{q-2} y(x) &= L_{q-2}(M) + \int_M^x L_{q-1} y / \rho_{q-l} \, dx \\
&\ge L_{q-2}(M) + L_{q-1} y(M) \int_M^x \rho_{q-l}^{-1} \, dx \to \infty
\end{aligned}
$$

as $x \to \infty$. Since $L_{q-2} \neq 0$ on $[M, \infty)$, it is necessarily positive there. Similarly we have

$$
L_t y > 0, \qquad t = 0, \ldots, q, \quad x \in [M, \infty). \tag{42}
$$

Since q is the greatest index with the above property, it follows that

$$
(-1)^{t-q} L_t y > 0, \qquad t = q, \ldots, n, \quad x \in [M, \infty). \tag{43}
$$

Therefore, for sufficiently large s_{i_j},

$$
\begin{aligned}
L_t y(M, s_{i_j}) &> 0, \qquad t = 0, \ldots, q, \\
(-1)^{t-q} L_t y(M, s_{i_j}) &> 0, \qquad t = q, \ldots, n.
\end{aligned}
$$

Hence $S(y(x, s_{i_j}), M+) = q$ and by Lemma 2 also $q = k$. Substituting $q = k$ into (42), (43), we conclude by Lemma 3 that (1) is $(k, n-k)$-disfocal in (M, ∞). By Lemma 4, $\eta_{k,n-k}$ does not exist on (M, ∞), in contradiction to the hypothesis of the theorem.

The last contradiction shows that, contrary to our assumption, the number of zeros of $y(x, s)$ in (a, s) which are bounded functions of s cannot remain bounded as $s \to \infty$. In other words, as $s \to \infty$, the number of zeros of $y(x, s)$ in (a, s) increases indefinitely and all of them are bounded from above. Since all these zeros are also bounded from below by a, the solution $y = \lim_{s\to\infty} y(x, s)$ has infinitely many zeros in $[a, \infty)$. Since $S(y(x, s), x+) \equiv k$ in $[a, s)$, and since by (9) only a finite number of zeros of $y(x, s)$ may coincide and form multiple zeros of y, we have $S(y, x+) \equiv k$ for sufficiently large values of x.

Also, the number of zeros of $L_{n-k-1} y(x, s)$ in (a, s) increases indefinitely as $s \to \infty$. By Theorem 3, such a zero can be added to (a, s) only when s passes through an extremal point of (4). Therefore all the extremal points $\theta_i(a)$, $i = 1, 2, \dots,$ of (4) exist. Theorem 5 is proved.

Observe that in fact we have proved that when (40) holds the existence of $\zeta_{k,n-k}(a)$ for every a implies properties (1) and (2) of Theorem 5.

Theorem 5 verifies the implication (1) $\to$ (3) in Theorem I. Since (3) $\to$ (1) is trivial, we have established the equivalence (1) $\leftrightarrow$ (3). We may also deduce from Theorem 5 that non-oscillation of (1) implies its eventual disconjugacy (see [4]).

Lemma 5. *For every solution y of (1), there exists a half infinite interval (c, ∞) on which the functions $L_t y$, $t = 0, \dots, n-1$, may have only simple zeros, on which $L_{t+1}y$ has exactly one simple zero between two consecutive zeros of $L_t y$, and on which $S(y, x+)$ is constant. If $S(y, x+) \equiv q$, then $S(y, x-) \equiv n - q$ and $(-1)^{n-q}p(x) < 0$.*

Proof. For a non-oscillatory solution the assertion is trivial, because none of its quasi-derivatives vanish on $[c, \infty)$ if $c > 0$ is appropriately chosen.

Let y be an oscillatory solution. Then $L_0 y, \dots, L_{n-1}y$ cannot have more than $[n/2]$ multiple zeros, since otherwise we would have $N(y) \geq 2([n/2] + 1) > n$.

Let α, β be two consecutive zeros of $L_t y$. Since

$$\operatorname{sgn}[L_{t+1}y(\alpha + \varepsilon)] = -\operatorname{sgn}[L_{t+1}y(\beta - \varepsilon)],$$

it follows that $L_{t+1}y$ has an odd number of zeros (counting multiplicities) in (α, β). Consequently if $L_{t+1}y$ has more than one zero in (α, β), it has at least three zeros, i.e., two more than the minimum that can be deduced by Rolle's theorem. Moreover there can be at most $[n/2]$ pairs of consecutive zeros of $L_0 y, \dots, L_{n-1}y$ such that between a pair of zeros of $L_t y$ there is more than one zero of $L_{t+1}y$. For assume that in $[a, b]$ there are more than $[n/2]$ such pairs and denote by ℓ the number of zeros of y in $[a, b]$. By applying Rolle's theorem n times it follows that $L_n y$ has at least $\ell + 2([n/2] + 1) - n > \ell$ zeros, contradicting $L_n y = -py$.

Since the above cases may occur only a finite number of times, there exists a value c such that $L_t y$, $t = 0, \dots, n-1$, has only simple zeros on (c, ∞), and $L_{t+1}y$ has exactly one simple zero between two zeros of $L_t y$.

To prove that $S(y, x+)$ is constant on (c, ∞), it is sufficient to show that $S(y, x+)$ does not vary as x passes through the simple zeros of the quasi-derivatives. Let $a > c$ be a simple zero of $L_t y$ for which $L_{t+1}y(a) > 0$. We claim that $L_{t-1}y(a) < 0$. Suppose to the contrary that $L_{t-1}y(a) > 0$ and denote by α and β the nearest zeros of $L_{t-1}y$ on the left and right of the point a. Then $L_{t-1}y > 0$ in (α, β) and so

$$L_t y(\alpha + \varepsilon) > 0, \quad L_t y(\beta - \varepsilon) < 0, \qquad \varepsilon > 0.$$

On the other hand, in view of $L_t y(a) = 0$ and $L_{t+1}y(a) > 0$, we have $L\, y(a - \varepsilon) < 0$ and $L_t y(a + \varepsilon) > 0$. Consequently $L_t y$ changes its sign at least three times in (α, β): in (α, a), (a, β) and, of course, at a. This is impossible in (c, ∞), whence $L_{t-1}y(a) < 0$.

Since $L_{t-1}y(a) < 0$, $L_t y(a) = 0$, and $L_{t+1}y(a) > 0$, it follows that

$$L_{t-1}y(a - \varepsilon) < 0, \quad L_{t-\varepsilon}y(a) < 0, \quad L_{t+1}y(a - \varepsilon) > 0,$$

and that

$$L_{t-1}y(a+\varepsilon) < 0, \quad L_{t+\varepsilon}y(a) > 0, \quad L_{t+1}y(a+\varepsilon) > O,$$

Hence the functions $S(y, x+)$ and $S(y, x-)$ do not vary as x passes through a; consequently they are constants on (c, ∞).

At a point $x_0 > c$ where no quasi-derivative of y vanishes, we have obviously $S(y, x_0+) + S(y, x_0-) = n$. Hence if $S(y, x+) \equiv q$ on (c, ∞), then $S(y, x-) \equiv n-q$. If $c < x_1 < x_2$, we have

$$S(y, x_1+) = q, \qquad S(y, x_2-) = n-q$$

and so $(-1)^{n-q}p(x) < 0$ according to Lemma 1.

Theorem 6. *Let $y(x)$ be an oscillatory solution of (1) which satisfies $S(y, x+) \equiv k$ for sufficient large values of x. Then for every system of type (2) and for every $a \geq 0$, all the extremal points $\theta_i(a)$, $i = 1, 2, \ldots$, exist.*

Proof. By Lemma 5, $S(y, x+)$ is constant on some half line (c, ∞), say $S(y, x+) \equiv k$. Let $a > c$ be arbitrary. For an arbitrarily given integer m, we choose two points t_1 and s_1 in (a, ∞) such that each quasi-derivative of y has at least m simple zeros in (t_1, s_1) and no quasi-derivative vanishes at either t_1 or s_1. Since $S(y, t_1+) = k$, we have $S(y, s_1-) = n-k$. The inequality

$$L_{i+1}y(t_1)/L_iy(t_1) = c_i > 0$$

holds for exactly k values of i , say $\{i_1, \ldots, i_k\}$. Let

$$L_{j+1}y(s_1)/L_jy(s_1) = -d_j < 0$$

holds for $j \in \{j_1, \ldots, j_{n-k}\}$. For $c < t < s$, let $u(x, t, s)$ be the solution which satisfies the boundary conditions

$$\begin{aligned} L_{i+1}u(t) - c_iL_iu(t) &= 0, \qquad i \in \{i_1, \ldots, i_k\}, \\ L_{j+1}u(s) + d_jL_ju(s) &= 0, \qquad j \in \{j_1, \ldots, j_{n-k-1}\}. \end{aligned} \tag{44}$$

Since every solution u of (44) satisfies

$$N(u) \geq S(u, t+) + S(u, s-) \geq k + (n-k-1) = n-1,$$

it follows that $u(x, t, s)$ is unique and has the properties specified in Lemma 2 (see the remark after Lemma 2). The oscillatory solution y satisfies (44) for $t = t_1$, $s = s_1$; hence by the uniqueness $u(x, t_1, s_1) \equiv y(x)$. Thus $L_ru(x, t_1, s_1)$, $0 \leq r \leq n-1$, has at least m simple zeros in (t_1, s_1).

By Lemma 2 simple zeros of $L_ru(x, t_1, s)$ in (t_1, s) cannot coincide, and moreover cannot meet t_1 as s varies. Furthermore, when s is sufficiently close to t_1, it is impossible for every quasi-derivative of $u(x, t_1, s)$ to have zeros in (t_1, s). Therefore, as s decreases from s_1 towards t_1, zeros of the quasi-derivatives must leave (t_1, s) through the endpoint s. Suppose that the first zero of $L_{j_{n-k}}u(x, t_1, s)$ leaves (t_1, s) for $s = s_2$, $t_1 < s_2 < s_1$, that is

$$L_{j_{n-k}}u(x, t_1, s_2)\Big|_{x=s_2} = 0.$$

While the number of zeros of $L_{j_{n-k}}u(x,t_1,s)$ in (t_1,s) decreases by one, the number of zeros of $u(x,t_1,s)$ may decrease at most by $j_{n-k} < n$. Hence $u(x,t_1,s_2)$ has at least $m-n$ simple zeros in (t_1,s_2).

Let $v(x,t,s)$ be the solution which satisfies

$$\begin{aligned} L_{i+1}v(t) - c_i L_i v(t) &= 0, \qquad i \in \{i_1,\dots,i_k\},\\ L_{j+1}v(s) + d_j L_j v(s) &= 0, \qquad j \in \{j_1,\dots,i_{n-k-2},\},\\ L_{j_{n-k}}v(s) &= 0. \end{aligned}$$

Since $v(x,t,s)$ is unique it enjoys the properties (1)-(6) of Lemma 2, and $v(x,t_1,s_2) \equiv u(x,t_1,s_2)$. Now we let s decrease until a zero of $L_{j_{n-k-1}}v(x,t_1,s)$ leaves (t_1,s) through the endpoint s, for $s=s_3$, $t_1 < s_3 < s_2 < s_1$. Repeating the process of decreasing s and increasing t, we finally obtain an interval $[\overline{t},\overline{s}] \subset (t_1,s_1)$ and a solution w which satisfies

$$\begin{aligned} L_i w(\overline{t}) &= 0, \qquad i \in \{i_1,\dots,i_k\},\\ L_j w(\overline{s}) &= 0, \qquad j \in \{j_1,\dots,j_{n-k}\}. \end{aligned} \tag{45}$$

Moreover, w has at least $m-n^2$ simple zeros in $(\overline{t},\overline{s})$.

Now $\overline{s}$ is an extremal point, say $\theta_i(\overline{t})$, of (45). By Corollary 2 we have $i+\ell_0 \geq m-n^2$. But m was chosen arbitrarily, so that $\theta_i(\overline{t})$ exists for every i. Since $a < t_1 < \overline{t}$, it follows from Theorem 2 that $\theta_i(a)$ also exists for every i. Thus by Theorem 4 all the extremal points of every system of type (2) exist.

The next theorem is the converse of Theorem 6.

Theorem 7. *If for one system of boundary conditions of type (2) and for one value of a, all the extremal points $\theta_i(a)$, $i=1,2,\dots$, exist, then (1) has an oscillatory solution y which satisfies $S(y,x+) \equiv k$ for sufficiently large values of x.*

Proof. By Theorem 4 we may assume without loss of generality that $\{\theta_i(a)\}$ are the extremal points of (4). Let $y(x,s)$ be the solution satisfying (41). Suppose for contradiction that as $s \to \infty$ there are only a finite number of zeros (in (a,s)) of $y(x,s)$ and its quasi-derivatives which are bounded functions of s. Then as shown in the proof of Theorem 5, (1) is $(k,n-k)$-disfocal on (M,∞). We now use the existence of $\theta_i(a)$, $i=1,2,\dots$, to get a contradiction.

Since $\theta_i(a)$, exists for every i, the number of zeros of the extremal solutions $y(x,\theta_i(a))$, in $(a,\theta_i(a))$ increases indefinitely as $i \to \infty$. Only a bounded number of them are in (a,M). Therefore $y(x,s_0)$ has an arbitrarily given number of zeros in (M,s_0) for sufficiently large s_0. Moreover $S(y(x,s_)),x+) \equiv k$ in $[a,s_0)$. As in the proof of Theorem 6, by contracting the interval (M,s_0) we see that the focal point $\zeta_{k,n-k}(c)$, as well as other extremal points, exists for some c, $M < c < s_0$. This contradicts the previous conclusion that (1) is $(k,n-k)$-disfocal in (M,∞). This proves that as $s \to \infty$ the number of zeros of $y(x,s)$ in (a,s) increases indefinitely, and all of them are bounded functions of s. Therefore $\lim_{s\to\infty} y(x,s)$ is the required oscillatory solution.

Theorems 6 and 7 prove the implications (2) $\to$ (4), (4) $\to$ (3) of Theorem I. Since (3) $\to$ (2) is trivial, we have established the equivalence (2) $\leftrightarrow$ (3) $\leftrightarrow$ (4). To complete the proof of Theorem I, we have only to verify (4) $\to$ (5), because (5) $\to$ (4) also is trivial.

Theorem 8. *The solutions of (1) which have the same final constant value of $S(y, x+)$ are either all oscillatory or all non-oscillatory.*

Proof. Let y_1 be a non-oscillatory solution of (1), and let $S(y_1, x+) \equiv k$ for sufficiently large values of x. The quasi-derivatives of y_1 do not vanish on $[M, \infty)$ for an appropriately chosen M; hence there are k indices $i_1, \ldots, i_k$ such that

$$\text{(46)} \qquad \begin{aligned} \operatorname{sgn}\,[L_{i+1}y] &= \operatorname{sgn}\,L_i y], \qquad i \in \{i_1, \ldots, i_k\}, \\ \operatorname{sgn}\,[L_{j+1}y] &= \operatorname{sgn}\,L_j y], \qquad j \in \{0, \ldots, n-1\} \setminus \{i_1, \ldots, i_k\}, \end{aligned}$$

on $[M, \infty)$. (If (40) holds, then $\{i_1, \ldots, i_k\} = \{0, \ldots, k-1\}$ as in (42), (43)).

If, on the other hand, there exists an oscillatory solution y_2 such that $S(y_2, x+) \equiv k$, then by Theorem 6, $\theta_i(a)$ exists for every a, every i, and every system of boundary conditions of type (2). In particular, $\theta_1(M)$ exists for the conditions

$$\text{(47)} \qquad \begin{aligned} L_i y(M) &= 0, \qquad i \in \{i_1, \ldots, i_k\}, \\ L_j y(s) &= 0, \qquad j \in \{0, \ldots, n-1\} \setminus \{i_1, \ldots, i_k\}, \end{aligned}$$

But, exactly as in the proof of sufficiency in Lemma 3, the conditions (46) and the existence of $\theta_1(M)$ for (47) are incompatible. This completes the proof of Theorem I.

For the sake of completeness, we conclude by proving the following analogue of Theorem 4.3 of [9].

Lemma 6. *Let a be a fixed point and assume that*

$$\text{(40)} \qquad \int^{\infty} \rho_i^{-1}(x)\, dx = \infty, \qquad i = 1, \ldots, n-1.$$

Then $\eta_{k,n-k}(a)$ exists in (a, ∞) if and only if $\zeta_{k,n-k}(a)$ exists in (a, ∞).

Proof. In Lemma 4 we have shown, without any assumption about (1), that if $\eta_{k,n-k}(a)$ exists then also $\zeta_{k,n-k}(a)$ exists. It remains to prove the converse implication.

Let $y(x, s)$ be the solution of (41). If $\eta_{k,n-k}(a)$ does not exist, then

$$L_n y(x, s) = -p(x) y(x, s) \neq 0 \qquad \text{on } (a, s).$$

Let $x_t(s)$ be the first zero of $L_t y(x, s)$, $1 \leq t \leq n-2$, in (a, s). This exists in view of the fact that $y(x, s)$ has $k + (n-k-1) = n-1$ zeros at a and s. If also $L_{n-1}y(x, s)$ has a zero in (a, s), it is unique (since $L_n y(x, s) \neq 0$); we denote it by $x_{n-1}(s)$. It is easily verified that

$$\begin{aligned} a < x_k(s) < x_{k-1}(s) < \ldots < x_1(s) < s, \\ a < x_k(s) < x_{k+1}(s) < \ldots < x_{n-2}(s) < x_{n-1}(s). \end{aligned}$$

By Lemma 2, $y(x, s)$ satisfies (31) on $(a, x_k(s))$; hence (1) is $(k, n-k)$-disfocal on that interval and, by Theorem 2, also on $[a, x_k(s))$. To complete the proof it suffices to show that $x_k(s) \to \infty$ as $s \to \infty$.

Assume to the contrary that $x_k(s)$ is bounded as $s \to \infty$. Let q, $k+1 \leq q \leq n$, be the first index such that $x_q(s)$ is unbounded or $L_q y(x, s)$ has no zero in (a, s). By

Lemma 2,

$$\text{sgn } [L_q y(a,s)] = \text{ sgn } L_{q-1} y(a,s)]$$

and also $L_{q-1}y(x,s)$ changes sign at $x_{q-1}(s)$. Hence

$$\text{sgn } [L_q y(x,s)] = \text{ sgn } L_{q-1} y(x,s)] \qquad \text{on } (x_{q-1}(s), x_q(s)).$$

Here $x_{q-1}(s)$ is bounded and $x_q(s) \to \infty$ as $s \to \infty$. With the help of (40), we find as in the proof of Theorem 5 that there exists a value M such that $L_i y(M,s) > 0$, $i = 0, \dots, q$, for sufficiently large s. Hence $S(y, M+) \geq q \geq k+1$, which in turn implies that

$$N(y) \geq S(y, M+) + S(y, s-) \geq (k+1) + (n-k) > n.$$

This contradiction shows that $x_k(s) \to \infty$, completing the proof. (It is interesting to compare the last result to that of [5].)

Remark. All the results of this work remain valid if (1) is defined on $[a,b)$ instead of on $[0,\infty)$, provided that

$$\int^b \rho_i^{-1}(x)\,dx = \infty, \qquad i = 1, \dots, n-1.$$

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