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A classification of the solutions of a differential equation according to their asymptotic behaviour

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SYNOPSIS

The solutions of the differential equation $L_ny+p(x)y=0$, where $L_ny=\rho_n(\rho_{n-1}\ldots(\rho_1(\rho_0y)')'\ldots)'$ and $p(x)$ is of one sign, are classified according to their behaviour as $x\to\infty$. The solution space is decomposed into disjoint, non-empty sets S_k, $0\leqq k\leqq n$, such that $(-1)^{n-k}p(x)\leqq 0$. We study the growth properties and the density of the zeros of the solutions which belong to the different sets S_k, the structure of the sets and its connection with $(k, n-k)$-disfocality.

1. INTRODUCTION

We consider the differential equation

$$L_ny+p(x)y=0 \tag{1}$$

where $p(x)$ is continuous and of one sign on $[0,\infty)$ and L_n is the nth order differential operator

$$L_ny=\rho_n(\rho_{n-1}\ldots(\rho_1(\rho_0y)')'\ldots)'. \tag{2}$$

Here $\rho_i>0$, $\rho_i\in C^{n-i}[0,\infty)$, $i=0,\ldots,n$. We denote

$$L_0y=\rho_0y,\quad L_iy=\rho_i(L_{i-1}y)',\quad i=1,\ldots,n$$

and call $L_0y,\ldots,L_ny$ the quasi-derivatives of y.

The aim of the present work is to classify all the solutions of (1) into disjoint sets according to their behaviour as $x\to\infty$. Solutions of differential equations have been classified according to several criteria. Kiguradze [**9**] sorted the non-oscillatory solutions of

$$y^{(n)}+py=0 \tag{3}$$

so that a non-oscillatory solution y is said to belong to the set A_k (for certain integers k) if

$$y^{(i)}>0,\quad i=0,\ldots,k-1$$

$$(-1)^{j-k}y^{(j)}>0,\quad j=k,\ldots,n-1.$$

His approach is widely used even for non-linear and functional equations [**13**].

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Dolan and Klaasen [2] distinguished between the set $\mathcal{O}$ of oscillatory solutions (i.e. solutions which have infinitely many zeros) and the set $\mathcal{N}$ of non-oscillatory solutions. Kim classified the non-oscillatory solutions of (3) according to their rate of growth. In [10], S_0 is the set of bounded non-oscillatory solutions of (3), S_k, $0 \leqq k \leqq n-1$, is the set of non-oscillatory solutions such that $\lim [y(x)/x^{k-1}] > 0$ and $\lim [y(x)/x^k] = 0$ as $x \to \infty$ and S_n is the set of non-oscillatory solutions such that $\lim [y(x)/x^{n-1}] > 0$. It is proved in [10, Corollary 2] that if $\int^\infty x^{n-1} |p(x)| \, dx = \infty$ then each non-oscillatory solution of (3) belongs to one of the sets S_k, $0 \leqq k \leqq n$, such that $(-1)^{n-1} p(x) \leqq 0$.

We try to sort both oscillatory and non-oscillatory solutions of (1) according to a different criterion which generalizes the above methods and seems to fit (1) well. To state our results we introduce some notation, similar to that of [3]. Let $S(c_0, \ldots, c_n)$ denote the number of sign changes in the sequence $c_0, c_1, \ldots, c_n$, whose $n+1$ elements are non-zero. First, we assume tactically that $p > 0$ or $p < 0$. For a non-trivial solution y of (1) and a point x, $x > 0$, we denote

$$\begin{aligned} S(y, x-) &= \lim_{\xi \to x-} S(L_0 y(\xi), L_1 y(\xi), \ldots, L_n y(\xi)) \\ S(y, x+) &= \lim_{\xi \to x+} S(L_0 y(\xi), -L_1 y(\xi), \ldots, (-1)^n L_n y(\xi)). \end{aligned} \tag{4}$$

Since $p \neq 0$, every x has a punctured neighbourhood on which $L_t y \neq 0$, $t = 0, \ldots, n$, and so the limits in (4) exist.

Our classification is based on

THEOREM 1 [3, Lemma 5]. *For every non-trivial solution y of (1) there exists x_0 such that $S(y, x+)$ and $S(y, x-)$ are constants for x, $x_0 < x < \infty$. If $S(y, x+) \equiv k$ on (x_0, ∞) then $S(y, x-) \equiv n-k$ on (x_0, ∞) and*

$$(-1)^{n-k} p < 0.$$

On (x_0, ∞), $L_0 y, \ldots, L_{n-1} y$ may have only simple zeros.

For short, we denote the integer $\lim_{x \to \infty} S(y, x+)$ by $S(y)$.

EXAMPLE 1. The solutions of

$$y^{(4)} - y = 0 \tag{5}$$

satisfy $S(e^x) = S(e^x, (-1)e^x, (-1)^2 e^x, (-1)^3 e^x, (-1)^4 e^x) = 4$, $S(\sin x) = S(\sin x, (-1)\cos x, (-1)^2(-\sin x), (-1)^3(-\cos x), (-1)^4 \sin x) = 2$, $S(\cos x) = 2$, $S(e^{-x}) = 0$.

EXAMPLE 2. For the solutions of $y^{(4)} + 4y = 0$ we have $S(e^x \sin x) = S(e^x \cos x) = 3$, $S(e^{-x} \sin x) = S(e^{-x} \cos x) = 1$.

Theorem 1 suggests to sort the solutions of (1) to the sets

$$S_k = \{ y \mid S(y, x+) \equiv k \quad \text{for sufficiently large values of } x \}$$

for integers k, $0 \leqq k \leqq n$, such that $(-1)^{n-k} p < 0$. Our main result is

THEOREM 2. *The set of non-trivial solutions of (1) is the union of the disjoint,* *non-empty sets S_k, $0 \leqq k \leqq n$, such that $(-1)^{n-k}p<0$. Each one of the above sets consists either of oscillatory or of non-oscillatory solutions only.*

Examples 1 and 2 suggest that there is a correlation between the integer $S(y)$ and the order of magnitude of y as $x \to \infty$. This property is demonstrated in Theorem 3. We also show that there are dominance relations [as defined in **2**] between the sets S_k: If $y \in \bigcup_{k \geqq q} S_k$ and u is an arbitrary solution then also $y+\varepsilon u \in \bigcup_{k \geqq q} S_k$ for sufficiently small values of ε.

The sets S_k are not linear spaces but we prove in Theorem 5 some properties of their structure. For example, each S_k, $1 \leqq k \leqq n-1$, contains a two dimensional linear space. Finally, we point out the connection between the non-oscillation of the solutions in S_k and the $(k, n-k)$-disfocality of (1).

We made the assumption that $p>0$ or $p<0$ to ensure that $L_n y=-py$ is not identically zero on any subinterval, and so no ambiguity may arise in (4). Nevertheless, our results are correct even if we only assume that $p \geqq 0$ or $p \leqq 0$ and $p \not\equiv 0$ on any ray (c, ∞). In this case we define (and use) $S(y, x+)$ only on the subintervals of $(0, \infty)$ where $p \not\equiv 0$.

(1) contains some of the equations which are most discussed in the literature. It is well known that a differential operator admits the Polya factorization (2) if and only if it is disconjugate. For example, the equation

$$y'''+qy'+ry=0 \tag{6}$$

can be written as

$$L_3 y+py=0 \tag{7}$$

if $q \leqq 0$, $r \geqq 0 (\leqq 0)$, since the operator $y''+qy$ is disconjugate if $q \leqq 0$. The same holds if $q \leqq 0$, $r-q' \geqq 0 (\leqq 0)$, since (6) can be re-written as

$$(y''+qy)'+(r-q')y=0.$$

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The problem of classifying the solutions of (1) seems at first sight to be unrelated to [**3**]. However, quite surprisingly, results which are equivalent to Theorems 1 and 2 were used in [**3**] for different purposes. To make this work as self-contained as possible, we prove here Theorem 1 [by an argument different from that of **3**, Lemma 5] and most of Theorem 2.

In [**3**] we arranged the n quasi-derivatives $L_0 y, \ldots, L_{n-1} y$ of a solution y in a cyclic order so that $L_{n-1} y$ is followed by $L_0 y$. We also enumerated their zeros in $[a, b]$ from the left to the right and denoted them by $x_1, \ldots, x_r$. Here, if several *consecutive* quasi-derivatives of y vanish at the same point, this point is denoted by a certain x_i and the number of the consecutive quasi-derivatives which vanish at x_i is denoted by $n(x_i)$. On the other hand, if several *non-consecutive* quasi-derivatives have zeros at the same point, we denote the zeros of non-consecutive quasi-derivatives by distinct subscripts. Hence, $a \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_r \leqq b$. For example, if $L_0 y(\alpha)=L_2 y(\alpha)=L_3 y(\alpha)=0$, $L_0 y(\beta)=L_{n-1}(\beta)=0$, $\alpha<\beta$, then $x_i=\alpha$, $x_{i+1}=\alpha$, $x_j=\beta$, $n(x_i)=1$, $n(x_{i+1})=2$ and $n(x_j)=2$ (since $L_0 y$ follows $L_{n-1} y$).

Following this notation, we have [**3**, Lemma 1] for every solution y of (1)

$$S(y, a)+\sum_{a<x_i<b}\langle n(x_i)\rangle+S(y, b-)\leqq n, \tag{8}$$

where $\langle q\rangle$ denotes the greatest even integer which is not greater than q.

To prove (8) first observe the simple fact that if f and f' have the same sign [opposite signs] on a right [left] neighbourhood of c, then f' changes its sign on the right [left] side of c before f vanishes. Now (8) follows if we apply this fact and Rolle's theorem to each of the n derivatives

$$(L_0y)'=L_1y/\rho_1,\quad (L_1y)'=L_2y/\rho_2,\ldots,(L_{n-1}y)'=-L_0y\cdot p/\rho_0\rho_n,$$

and trace the zeros of $L_0y, L_1y, \ldots$.

Remarks (1). If $L_ty(c)=0$ then L_ty and $L_{t+1}y$ have the same sign on a right neighbourhood of c and opposite signs on a left neighbourhood of c. Thus $S(y, c+)$ and $S(y, c-)$ are not smaller than the number of quasi-derivatives $L_0y,\ldots,L_{n-1}y$ that vanish at c.

(2) By the definition of $S(y, x-)$ and since $L_ny=-py=-p\cdot L_0y/\rho_0$, it follows that if $p<0$, $S(y, x-)$ is even and if $p>0$, $S(y, x-)$ is odd. Hence, for every x, the integers $S(y, x-)$, $S(y, x+)$ satisfy

$$(-1)^{S(y,x-)}p<0,\quad (-1)^{n-S(y,x+)}p<0. \tag{9}$$

Proof of Theorem 1. If $L_ty(b)\neq 0$, $t=0,\ldots,n$, then obviously $S(y, b+)+S(y, b-)=n$. Combining this with (8), we obtain that

$$S(y, a+)\leqq n-S(y, b-)=S(y, b+)$$

Since every point has a deleted neighbourhood in which no quasi-derivative of y vanishes, we have by the definition (4) for every $a<b$,

$$S(y, a+)\leqq S(y, b+). \tag{10}$$

Thus $S(y, x+)$ is a non-decreasing, integer valued fruction, bounded from above by n and consequently $S(y, x+)\equiv k$ for $x_0<x<\infty$, for a certain integer k, $0\leqq k\leqq n$. Similarly we prove that $S(y, x-)$ is constant for sufficiently large values of x. By comparing $S(y, x-)$ and $S(y, x+)$ at a point where no quasi-derivative vanishes, we deduce that $S(y, x-)=n-S(y, x+)\equiv n-k$. $(-1)^{n-k}p(x)<0$ follows by (9).

For every $a, b\in(x_0,\infty)$, we have $S(y, a+)=k$, $S(y, b-)=n-k$. It follows from (8) that for every $x_i>x_0$, $\langle n(x_i)\rangle=0$, i.e. the quasi-derivatives may have only simple zeros on (x_0,∞).

Proof of Theorem 2. Theorem 1 proves that every solution of (1) belongs to one of the sets S_k which are clearly disjoint. Now we prove that they are all non-empty.

Let $1\leqq k\leqq n-1$, $(-1)^{n-k}p<0$ and let $y(x, s)$ be a non-trivial solution of (1) which satisfies the $n-1$ boundary conditions

$$\begin{aligned} L_iy(a)&=0,\quad i=0,\ldots,k-1,\\ L_jy(s)&=0,\quad j=0,\ldots,n-k-2.\end{aligned} \tag{11}$$

We prove that

$$S(y(x,s),x+)=k, \qquad a\leqq x<s$$
$$S(y(x,s),x-)=n-k, \quad a<x\leqq s. \tag{12}$$

According to Remark 1, (11) implies that

$$S(y(x,s),a+)\geqq k, \quad S(y(x,s),s-)\geqq n-k-1. \tag{13}$$

First, let x_0 be a point of (a,s) such that $L_t y(x_0,s)\neq 0$, $t=0,\ldots,n$. If $S(y(x,s),x_0+)=q$, then $S(y(x,s),x_0-)=n-q$. On $[a,x_0]$ we have by (8) and (13)

$$n\geqq S(y(x,s),a+)+S(y(x,s),x_0-)\geqq k+(n-q)$$

and on $[x_0,s]$,

$$n\geqq S(y(x,s),x_0+)+S(y(x,s),s-)\geqq q+(n-k-1).$$

By the last inequalities, $k\leqq q<k+1$. But k satisfies $(-1)^{n-k}p<0$ and so by (9), the integers $S(y(x,s),x_0-)=n-q$ and $n-k$ must have the same parity, i.e. $q=k$. This proves (12) for x_0. Since every point x, $a\leqq x<s(a<x\leqq s)$ has a right (left) neighbourhood which is included in (a,s) and on which $L_t y(x,s)\neq 0$, $t=0,\ldots,n$, (12) follows for every x by the definition (4).

(12) also implies that $y(x,s)$ is essentially unique. If there were two linearly independent solutions y_1, y_2 which satisfy (11), there would be a linear combination $y=c_1y_1+c_2y_2$ which would satisfy, in addition to (11), also $L_k y(a)=0$. This would imply that $S(y,a+)\geqq k+1$, which contradicts (12).

To obtain a solution which is in the set S_k, we normalize the solutions $\{y(x,s)\}$ so that $\sum_{i=1}^{n-1}|L_i y(a,s)|=1$ and we choose a sequence s_i, $s_i\to\infty$, and a solution y such that $L_t y(x,s_i)\to L_t y$, $t=0,\ldots,n$, as $s_i\to\infty$, uniformly on compact intervals. By Theorem 1, $S(y,x+)$ is constant on some ray (x_0,∞). To determine this constant, $S(y)$, we take $x_1\in(x_0,\infty)$ such that $L_t y(x_1)\neq 0$, $t=0,\ldots,n$. Then $\operatorname{sgn}[L_t y(x_1)]=\operatorname{sgn}[L_t y(x_1,s_i)]$, $t=0,\ldots,n$ for sufficiently large values of s_i. If also $s_i>x_1$, then by (12) $S(y,x_1+)=S(y(x,s_i),x_1+)=k$ and thus $S(y)=k$.

If $k=0$ or $k=n$ are admissible, the above proof still holds if we interprete the system (11) of $n-1$ two-point boundary value conditions as a system of $n-1$ homogeneous initial value conditions at $x=s$ or at $x=a$, respectively. If $(-1)^n p<0$ and $k=0$, $y(x,s)$, which has a zero of multiplicity $n-1$ at s, satisfies $S(y(x,s),s-)=n$ and hence by (8), it has no zeros left to s. As $s\to\infty$, the limit function y satisfies $S(y,x-)=n$, $S(y,x+)=0$ for every x, $0<x<\infty$, and

$$S_0=\{y \mid L_i y\,.\,L_{i+1}y<0 \quad \text{on} \quad [0,\infty), \quad i=0,\ldots,n-1\} \tag{14}$$

This argument is identical with that of Hartman and Wintner [**6**].

For $p<0$ and $k=n$, the solution $y(x)\equiv y(x,s)$ which is determined by $L_i y(a)=0$, $i=0,\ldots,n-2$, $L_{n-1}y(a)=1$, satisfies $L_i y>0$, $i=0,\ldots,n$ on (a,∞). Indeed, it satisfies $S(y,a+)=n$ and by (8), it has no zeros in (a,∞). Also by (10), $S(y,x+)=n$ for x, $a<x<\infty$. This property of (1) (with $p<0$) is well known and extensively used in the literature.

We have proved in [**3**] that S_k consists either of oscillatory or of non-oscillatory solutions. Here we outline the proof. Let $u,v\in S_k$ and let u be non-oscillatory.

There exists a point x_0 and k indices $i_1, \dots, i_k$ such that

$$L_i u \,.\, L_{i+1} u > 0, \quad i \in \{i_1, \dots, i_k\},$$
$$L_i u \,.\, L_{i+1} u < 0, \quad i \in \{0, \dots, n-1\} \backslash \{i_1, \dots, i_k\}, \tag{15}$$

on (x_0, ∞) and $S(v, x+) \equiv k$ on (x_0, ∞).

In the proof of Theorem 6 of [**3**] we showed that if v has at least n^2 zeros in (x_0, ∞), then by "condensing" these zeros we can obtain another solution w and two points $\alpha, \beta \in (x_0, \infty)$ such that

$$L_i w(\alpha) = 0, \quad i \in \{i_1, \dots, i_k\}$$
$$L_i w(\beta) = 0, \quad i \in \{0, \dots, n-1\} \backslash \{i_1, \dots, i_k\}. \tag{16}$$

To show that (15) and (16) are incompatible, define λ_0 as the smallest positive value of λ for which one of the quasi-derivatives $L_t u_\lambda \overset{\text{def}}{\equiv} L_t(u - \lambda w)$, $t = 0, \dots, n-1$, has a zero in $[\alpha, \beta]$. By (15), $0 < \lambda_0 < \infty$ exists (otherwise we can replace w by $-w$). By continuity argument, $L_t u_{\lambda_0}$, $t = 0, \dots, n$, do not change their signs in (α, β) and

$$L_i u_{\lambda_0} \,.\, L_{i+1} u_{\lambda_0} \geqq 0, \quad i \in \{i_1, \dots, i_k\}$$

$$L_i u_{\lambda_0} \,.\, L_{i+1} u_{\lambda_0} \leqq 0, \quad i \in \{0, \dots, n-1\} \backslash \{i_1, \dots, i_k\}, \quad \alpha \leqq x \leqq \beta.$$

Thus, for $i \in \{i_1, \dots, i_k\}$, $|L_i u_{\lambda_0}|$ increases on $[\alpha, \beta]$ and may have a zero only at α. But by (15) and (16), $L_i u_{\lambda_0}(\alpha) = L_i(u - \lambda_0 w)(\alpha) = L_i u(\alpha) \neq 0$. Similarly, $L_i u_{\lambda_0} \neq 0$ on $[\alpha, \beta]$ for $i \in \{0, \dots, n-1\} \backslash \{i_1, \dots, i_k\}$. This contradicts the definition of λ_0 and shows that a solution v which belongs to S_k cannot be oscillatory.

Example 3. In [**14**, Theorem 2], Svec proved that if the equation

$$y^{(3)} + py = 0, \quad p > 0 \tag{17}$$

has an oscillatory solution, then every solution which has a zero is oscillatory. This can be deduced also from Theorem 2. The set of non-trivial solutions of (17) is $S_0 \cup S_2$. By (14), a solution which has a zero cannot belong to S_0, thus it belongs to S_2. But if there is one oscillatory solution in S_2, every solution that belongs to S_2 is oscillatory.

A similar result was proved by Hanan [**5**, Theorem 3.4] and Lazer [**12**, Lemma 1.2″] for equation (6). More generally, we can use Theorem 2 and the further results for the oscillatory and the non-oscillatory solutions of (6) if $q \leqq 0$, $r \geqq 0 (\leqq 0)$ even without writing it explicitly in the form (7). The reason is that the solutions of (7) which belong to S_0 (if $p \geqq 0$) or to S_3 (if $p \leqq 0$) are non-oscillatory. Therefore, the oscillatory solutions of (6) must belong to S_2 (if $r \geqq 0$) or to S_1 (if $r \leqq 0$), regardless of the explicit form of their quasi-derivatives.

Kondratev [**11**] showed that if $n \geqq 5$ then for an arbitrary integer m there is an equation of type (3) and two solutions u, v such that v has at least m zeros between two consecutive zeros of u. This example discourages attempts to study the separation properties of zeros for solutions of equations of high order. However, the following corollary shows that the situation is not so bad if we compare solutions of (1) with the same $S(y, x+)$.

COROLLARY. *Let u be a solution of (1) such that $S(u, x+)=k$ and $u(x)\neq 0$ for $a<x<b$. There exists an integer N (which depends only on n) such that any solution v of (1) such that $S(v, x+)=k$ for $a<x<b$ may have at most N zeros in (a, b).*

Proof. If $u\neq 0$ in (a, b), $L_{n-1}u$ may have at most one zero in $[a, b]$ and $L_i u$, $1\leqq i\leqq n-1$, may have at most $n-i$ zeros there. It is possible, thus, to divide $[a, b)$ to $1+n(n-1)/2$ subintervals $[a_q, a_{q+1})$ such that $L_i u\neq 0$, $i=0,\ldots,n-1$ on (a_q, a_{q+1}). As in the proof of Theorem 2, we see that v has less than n^2 zeros in (a_q, a_{q+1}) and less than n^2+n zeros in $[a_q, a_{q+1})$. It follows that $N\leqq (n^2+n)(1+n(n-1)/2)\leqq n^4$ for $n>1$.

Let $N_y(x)$ denote the number of zeros of the solution y in $[0, x]$. It follows that if u, v are oscillatory solutions such that $S(u)=S(v)$, then $N_u(x)/N_v(x)$ is bounded as $x\to\infty$.

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Examples 1 and 2 suggest to study the relation between the rate of growth of a solution and the set S_k to which it belongs. For a non-oscillatory solution y, this is an easy task. Let us assume momentarily that

$$\int^{\infty}\rho_i^{-1}(s)\,ds=\infty,\quad i=1,\ldots,n-1, \tag{18}$$

which, by [**15**], is quite a natural assumption. If $L_i y>0$ and $L_{i+1}y>0$ on (a,∞) for a certain i, $1\leqq i\leqq n-1$, then

$$\begin{aligned}L_{i-1}y(x)&=L_{i-1}y(a)+\int_a^x L_i y(s)\rho_i^{-1}(s)\,ds\\&\geqq L_{i-1}y(a)+L_i y(a)\int_a^x \rho_i^{-1}(s)\,ds\to\infty\end{aligned}$$

as $x\to\infty$. Therefore, for every non-oscillatory solution y there is an integer q, $0\leqq q\leqq n$ and a number c such that

$$\begin{aligned}&L_i y>0,\quad i=0,\ldots,q-1,\\&(-1)^{j-q}L_j y>0,\quad j=q,\ldots,n,\quad c<x<\infty,\end{aligned} \tag{19}$$

[Compare with **9**]. Obviously, $q=S(y)$. Since $L_{q-1}y$ is positive and increasing and $L_q y$ is positive and decreasing, there are positive constants A, B such that

$$\begin{aligned}B\rho_0^{-1}(x)\int_c^x \rho_1^{-1}(s_1)\,ds_1\int_c^{s_1}\cdots\int_c^{s_{q-2}}\rho_{q-1}^{-1}(s_{q-1})\,ds_{q-1}\leqq y(x)\\ \leqq A\,.\,\rho_0^{-1}(x)\int_c^x\rho_1^{-1}(s_1)\,dx_1\int_c^{s_1}\cdots\int_c^{s_{q-1}}\rho_q^{-1}(s_q)\,ds_q\end{aligned}$$

for large values of x. It follows that if y_1, y_2 are non-oscillatory, (18) holds and $S(y_1)>S(y_2)$, then

$$|L_t y_1/L_t y_2|\to\infty,\quad t=0,\ldots,n-1 \tag{20}$$

as $x\to\infty$.

It is more complicated to compare oscillatory solutions. We prove (without any assumptions on the $\rho_i - s$) the following:

THEOREM 3. *For every pair of solutions y_1, y_2 such that $S(y_1) > S(y_2)$, there exists a positive constant c so that (in the extended real numbers)*

$$\max_{0 \leqq t \leqq n-1} |L_t y_1(x)/L_t y_2(x)| \geqq c > 0, \quad 0 \leqq x < \infty. \tag{21}$$

If y_2 is non-oscillatory then

$$\limsup_{x \to \infty} |L_r y_1(x)/L_r y_2(x)| > 0, \quad r = 0, \dots, n-1. \tag{22}$$

First we prove a lemma.

LEMMA 1. *If $S(y_i) = k$, $i = 1, 2, \dots$ and $y_i \to y$ as $i \to \infty$, then $S(y) \leqq k$.*

Proof. By Theorem 1, there is x_0 such that $S(y, x+) \equiv S(y)$ on (x_0, ∞). Therefore, it is sufficient to determine $S(y, x+)$ for one point of (x_0, ∞). Let $a \in (x_0, \infty)$ be such that $L_t y(a) \neq 0$, $t = 0, \dots, n$. Since $y_i \to y$, and $L_i y(a) \neq 0$,

$$\operatorname{sgn}[L_t y_i(a)] = \operatorname{sgn}[L_t y(a)], \quad t = 0, \dots, n, \quad i \geqq i_0$$

and so

$$S(y) = S(y, a+) = S(y_i, a+), \quad i \geqq i_0. \tag{23}$$

Since $S(y_i) = k$, there is x_i such that $S(y_i, x+) \equiv k$ on (x_i, ∞). If we take $a_i > \max\{x_i, a\}$, then $S(y_i, a_i) = k$ and by (23) and (10),

$$S(y) = S(y_i, a+) \leqq S(y_i, a_i+) = k.$$

Proof of Theorem 3. To prove (21) suppose on the contrary that for every $\varepsilon > 0$ there is a sequence $x_i \to \infty$ such that

$$\max_{0 \leqq t \leqq n-1} |L_t y_1(x_i)/L_t y_2(x_i)| < \varepsilon, \quad i = 1, 2, \dots$$

i.e.

$$\varepsilon |L_t y_2(x_i)| > |L_t y_1(x_i)|, \quad t = 0, \dots, n-1, \quad i = 1, 2, \dots$$

Then

$$\operatorname{sgn}[\varepsilon L_t y_2(x_i)] = \operatorname{sgn}[L_t(\varepsilon y_2 + y_1)(x_i)], \quad t = 0, \dots, n-1, \quad i = 1, 2, \dots$$

and consequently

$$S(y_2) = S(\varepsilon y_2 + y_1).$$

As $\varepsilon \to 0$, $\varepsilon y_2 + y_1 \to y_1$ (pointwisely and uniformly on compact sets) and by Lemma 1,

$$S(y_1) \leqq S(\varepsilon y_2 + y_1) = S(y_2).$$

This contradicts our assumption and proves (21), on a certain interval $[a, \infty)$. Since the only points where $|L_t y_1/L_t y_2|$ is not continuous are those where the quotient is equal to $+\infty$, (21) is true also on $[0, a]$.

Suppose now that (22) is not true for a certain r, $0 \leqq r \leqq n-1$, i.e. for every $\varepsilon > 0$, $|L_r y_1(x)/L_r y_2(x)| < \varepsilon$ on (x_ε, ∞). Then for every $|\lambda| \geqq \varepsilon$, $L_r y_1(x)/L_r y_2(x) - \lambda \neq 0$ on (x_ε, ∞) and since $L_r y_2 \neq 0$, $L_r(y_1 - \lambda y_2) \neq 0$. But then $y_1 - \lambda y_2$ is a non-oscillatory solution and $L_t(y_1 - \lambda y_2) \neq 0$, $t = 0, \ldots, n-1$ on $(\tilde{x}_\varepsilon, \infty)$. This means that $L_t y_1 / L_t y_2 \neq \lambda$, $t = 0, \ldots, n-1$, for $|\lambda| \geqq \varepsilon$, i.e. $|L_t y_1 / L_t y_2| < \varepsilon$ on $(\tilde{x}_\varepsilon, \infty)$. Therefore $L_t y_1 / L_t y_2 \to 0$ as $x \to \infty$ for every t, $t = 0, \ldots, n-1$, contradicting (21).

Let $\mathscr{A}, \mathscr{B}$ be two sets of solutions of a linear, homogeneous differential equation. According to the definition of Dolan and Klaasen [**2**], $\mathscr{A}$ is said to *dominate* $\mathscr{B}$ [dominate at 0] if for every $y_1 \in \mathscr{A}$ and $y_2 \in \mathscr{B}$, $y_1 + \varepsilon y_2 \in \mathscr{A}$ for sufficiently small values of ε. Here $\mathscr{B}$ may be empty. Dominance is denoted by $\mathscr{A} > \mathscr{B}$. In [**2**], the set $\mathscr{O}$ of oscillatory solutions and the set $\mathscr{N}$ of non-oscillatory solutions are compared with respect to dominance. It is proved, for example, that if the equation

$$L_3 y + py = 0, \quad p > 0 \tag{24}$$

has oscillatory solutions, then $\mathscr{O} > \mathscr{N}$, while for

$$L_3 y + py = 0, \quad p < 0, \tag{25}$$

we have $\mathscr{N} > \mathscr{O}$. On the other side, it is observed that for the equation

$$y^{(4)} - y = 0 \tag{5}$$

there is no dominance relation between $\mathscr{N}$ and $\mathscr{O}$, since $\mathscr{N}$ includes both e^x and e^{-x}.

Both the examples (24) and (25) and the counter-example (5) of [**2**] show that for equation (1), it may be interesting to study dominance relations between the sets S_k rather than between $\mathscr{O}$ and $\mathscr{N}$. For (24), we shall prove, for example, that $S_2 > S_0$. If $\mathscr{O} \neq \varnothing$ then $S_2 = \mathscr{O}$, $S_0 = \mathscr{N}$ and our result is equivalent to that of [**2**]. But S_2, S_0 are non-empty and $S_2 > S_0$ has meaning even of $\mathscr{O} = \varnothing$. Similarly, for (25) $\mathscr{N} > \mathscr{O}$ can be replaced by $S_3 > S_1$. For (6) we have $S_4 > S_2 > S_0$. More generally,

THEOREM 4. *For every q, $\bigcup_{k \geqq q} S_k$ dominates the whole solution space of* (1), *i.e. if $y_1 \in \bigcup_{k \geqq q} S_k$ and y_2 is any solution then $y_1 + \varepsilon y_2 \in \bigcup_{k \geqq q} S_k$ for sufficiently small ε.*

Theorem 4 is an immediate consequence of Lemma 1. Since $y_1 + \varepsilon y_2 \to y_1$ as $\varepsilon \to 0$ and $S(y_1) \geqq q$, $S(y_1 + \varepsilon y_2)$ cannot be less than q for sufficiently small values of ε.

We cannot improve Theorems 3 and 4 for the general equation of type (1). Indeed, the equation

$$(x^2 y')' - x^{-2} y = 0, \quad 1 \leqq x \leqq \infty, \tag{26}$$

(where $L_0 y = y$, $L_1 y = x^2 y'$, $L_2 y = (x^2 y')'$ and (18) is not satisfied) has the solutions $y_1 = \exp(-1/x) \in S_2$ and $y_2 = \exp(1/x) \in S_0$ for which $L_t y_1 / L_t y_2 \to 1$ as $x \to \infty$, $t = 0, 1, 2$. Also, $y_1 + \varepsilon y_2 \in S_2$ if and only if $|\varepsilon| < 1$. However, it will be interesting to obtain conditions on ρ_i, $i = 0, \ldots, n$ and p which ensure that solutions in different sets have strictly different order of magnitude, i.e. that (21) can be replaced by

$$\max_{0 \leqq t \leqq n-1} |L_t y_1(x)/L_t y_2(x)| \to \infty \quad \text{as} \quad x \to \infty.$$

A related problem is to find conditions such that for every $y_1 \in S_k$, $y_2 \in S_l$, $k > l$, we shall have $y_1 + cy_2 \in S_k$ for every c. Probably, (18) is a part of such conditions.

4

Let us add the trivial solution to each S_k. The sets S_k are not linear spaces even so. For the solutions $y_1 = \sin x$, $y_2 = \sin x + e^{-x}$ of (5) we have, for example, $S(y_1) = S(y_2) = 2$ but $S(y_2 - y_1) = 0$. The next theorem yields some information about the structure of S_k and the decomposition of the solution space to subspaces which are related to the $S_k - s$.

THEOREM 5. *Let a be a fixed point. There exists a basis $\{u_0, \ldots, u_{n-1}\}$ of the solution space of* (1) *such that*

(i) *u_i, $i = 0, \ldots, n-1$, has a zero of multiplicity i at a.*

(ii) *For $1 \leqq k \leqq n-1$, $(-1)^{n-k} p < 0$, S_k contains span $\{u_{k-1}, u_k\}$, the two-dimensional subspace spanned by u_{k-1} and u_k. If $(-1)^n p < 0$ then $u_0 \in S_0$ and if $p < 0$ then $u_{n-1} \in S_n$.*

(iii) *If the set S_k, $1 \leqq k \leqq n-1$, consists of non-oscillatory solutions then span $\{u_{k-1}, u_k\}$ contains two solutions v, w such that $L_t v / L_t w \to \infty$ monotonously as $x \to \infty$, $t = 0, \ldots, n-1$. If S_k consists of oscillatory solutions then the zeros of every two independent solutions in span $\{u_{k-1}, u_k\}$ interlace in (a, ∞).*

Proof. For $1 \leqq k \leqq n-1$, $(-1)^{n-k} p < 0$ let $y_1(x, s)$ be the solution of (11) which was used in the proof of Theorem 2 and let $y_2(x, s)$ be the solution of the $n-1$ boundary conditions

$$\begin{aligned} L_i y(a) &= 0, \quad i = 0, \ldots, k-2, k, \\ L_j y(s) &= 0, \quad j = 0, \ldots, n-k-2 \end{aligned} \tag{27}$$

normalized so that $\sum_{t=0}^{n-1} |L_t y(a, s)| = 1$. We define $u_k(x) = \lim y_1(x, s)$, $u_{k-1}(x) = \lim y_2(x, s)$ as s tends to infinity through a proper sequence.

We have seen in the proof of Theorem 2 that $S(u_k) = k$. The same argument shows here that $S(u_{k-1}) = k$. Indeed, (27) implies that $y_2(x, s)$ also satisfies (13) and the rest of the proof follows. Consequently, u_i $(i = k-1, k)$ has at $x = a$ a zero exactly of multiplicity i. For if $L_{k-1} u_{k-1}(a) = 0$ or $L_k u_k(a) = 0$ then by (27) or by (11), respectively, u_{k-1} or u_k would satisfy $L_t u(a) = 0$, $t = 0, \ldots, k$, i.e., $S(u, a+) \geqq k+1$. But since $S(u, x+)$ is non-decreasing, this would contradict $S(u_k) = k$ or $S(u_{k-1}) = k$.

To prove (ii), we consider a linear combination $y(x, s) = c_1 y_1(x, s) + c_2 y_2(x, s)$. By (11) and (27), $y(x, s)$ satisfies the $n-2$ boundary conditions

$$\begin{aligned} L_i y(a) &= 0, \quad i = 0, \ldots, k-2 \\ L_j y(s) &= 0, \quad j = 0, \ldots, n-k-2. \end{aligned}$$

These boundary conditions imply that $S(y(x, s), a+) \geqq k-1$, $S(y(x, s), s-) \geqq n-k-1$. By (9) and the assumption $(-1)^{n-k} p < 0$ it follows that $S(y(x, s), a+) \geqq k$ and $S(y(x, s), s-) \geqq n-k$. By (8) we conclude that in fact $S(y(x, s), a+) = k$ and

$S(y(x, s), s-) = n-k$. Now we prove, as for $y_1(x, s)$, that the limit function $\lim_{s\to\infty} y(x, s) = c_1u_k + c_2u_{k-1}$ belongs to S_k. Moreover, since $c_1u_k + c_2u_{k-1}$ has a zero of multiplicity $k-1$ at a and since $S(y, x+)$ is non-decreasing, we have $S(c_1u_k + c_2u_{k-1}, x+) = k$ for all x in $[a, \infty)$.

If $p<0$, $k=n$ is admissible. We take (as in the proof of Theorem 2) u_{n-1} as the essentially unique solution with a zero of multiplicity $n-1$ at a. If $(-1)^n p<0$, $k=0$ is admissible and we choose (according to (14)) u_0 as a solution which satisfies $(-1)^t L_t y>0$, $t=0, \dots, n-1$, on $[0, \infty)$.

If the set S_k, $1 \leqq k \leqq n$, consists of non-oscillatory solutions then $L_t u_k \neq 0$, $t=0, \dots, n-1$, on (c, ∞). For every r, $0 \leqq r \leqq n-1$, $L_r u_{k-1}/L_r u_k$ is a monotone function on (c, ∞). Otherwise, $(L_r u_{k-1}/L_r u_k)'$ would vanish at a certain point x_0, $x_0>c$, i.e. $(L_{r+1}u_{k-1}L_r u_k - L_r u_{k-1}L_{r+1}u_k)(x_0) = 0$, and a certain linear combination $y = c_1u_k + c_2u_{k-1}$ would satisfy $L_r y(x_0) = L_{r+1}y(x_0) = 0$. By (8),

$$S(y, a+) + \langle n(x_0)\rangle + S(y, b-) \leqq n$$

for every b, $b>x_0$. Now $n(x_0) \geqq 2$ and $S(y, a+) = k$, therefore $S(y, b-) \leqq n-k-2$ and if $L_t y(b) \neq 0$, $t=0, \dots, n$, then $S(y, b+) \geqq k+2$. This is impossible since $y \in \operatorname{span}\{u_{k-1}, u_k\} \subset S_k$. Therefore we have proved that $L_r u_{k-1}/L_r u_k$ tends monotonously to a limit, finite or infinite. If this limit is ∞, it follows that $L_t u_{k-1}/L_t u_k \to \infty$ for all t, $t=0, \dots, n-1$. If the limit is λ, $-\infty<\lambda<\infty$, we choose $v = u_k$, $w = u_{k-1} - \lambda u_k$.

Suppose now that S_k consists of oscillatory solutions and that there are solutions v, $w \in \operatorname{span}\{u_{k-1}, u_k\}$ whose zeros do not interlace. If $L_r v(\alpha) = L_r v(\beta) = 0$, $a<\alpha<\beta<\infty$, but $L_r w \neq 0$ on $[\alpha, \beta]$, then there is a point x_0, $\alpha<x_0<\beta$, such that $(L_r v/L_r w)'(x_0) = 0$. By the above argument, there is a linear combination $y = c_1v + c_2w = d_1u_{k-1} + d_2u_k$ such that $L_r y(x_0) = L_{r+1}y(x_0) = 0$. If $L_r v$ and $L_r w$ have a common zero, a similar linear combination with a multiple zero exists obviously. In either case $S(y, a+) = k$ and $n(x_0) \geqq 2$ and a contradiction is obtained as above. This completes the proof of Theorem 5. For $n=3$ [see **1**, Theorem 2 and **7**, Theorem 5].

Closely related to the questions which arise about the dominance of the sets S_k, is the problem of obtaining conditions under which $\bigcup_{k \leqq q} S_k$ is a linear space. For example, this is the case if $\int^\infty \rho_i^{-1}\,dt = \infty$, $i=1, \dots, n-1$, and all the solutions of (1) are non-oscillatory. In this case we have n solutions $v_0, \dots, v_{n-1}$ such that $L_t v_i/L_t v_j \to \infty$ for $t=0, \dots, n-1$ when $i>j$. This follows by Theorem 5, (iii) and by (20). Now $\bigcup_{k \leqq q} S_k$ is the subspace spanned by $v_0, \dots, v_q$. Another example is the equation

$$y^{(4)} + py = 0, \quad p<0$$

where $p(x)$ is a smooth, monotone (decreasing or increasing) function. Here S_0 and $S_0 \cup S_2$ are one and three dimensional linear spaces, respectively [**8**]. In particular it is interesting (if $(-1)^n p<0$) when S_0 is a one-dimensional linear space, i.e. when the solution of (1) which satisfies

$$(-1)^t L_t y>0, \quad t=0, \dots, n$$

is essentially unique. For (26) this is not the case since both $\sinh(1/x)$ and $\cosh(1/x)$ belong to S_0. For this problem, see [**7**, Theorem 4 and **4**].

5

We recall that if

$$\int^{\infty} \rho_i^{-1}(s)\, ds = \infty, \quad i = 1, \ldots, n-1, \tag{18}$$

then every non-oscillatory solution of (1) satisfies for a certain integer q, $0 \leqq q \leqq n$, $(-1)^{n-q}p < 0$,

$$\begin{aligned} L_i y > 0, &\quad i = 0, \ldots, q-1 \\ (-1)^{j-q} L_j y > 0, &\quad j = q, \ldots, n-1 \end{aligned} \tag{19}$$

on some ray (c, ∞). Here, of course, $q = S(y)$.

Equation (1) is called $(k, n-k)$-disconjugate on an interval I if and only if for every $\alpha, \beta \in I$, $\alpha < \beta$, no non-trivial solution of (1) satisfies the n boundary conditions

$$L_i y(\alpha) = 0, \quad i = 0, \ldots, k-1,$$
$$L_j y(\beta) = 0, \quad j = 0, \ldots, n-k-1.$$

It is called $(k, n-k)$-disfocal if no non-trivial solution satisfies

$$L_i y(\alpha) = 0, \quad i = 0, \ldots, k-1,$$
$$L_j y(\beta) = 0, \quad j = k, \ldots, n-1.$$

(1) is called eventually $(k, n-k)$-disconjugate [-disfocal] if it is $(k, n-k)$-disconjugate [-disfocal] on (c, ∞) for some c.

We have proved in [**3**] that if $(-1)^{n-k}p < 0$, then (1) is $(k, n-k)$-disfocal on $[a, b]$ if and only if it has a solution y such that

$$L_i y > 0, \quad i = 0, \ldots, k-1$$
$$(-)^{j-k} L_j y > 0, \quad j = k, \ldots, n-1$$

on $[a, b]$. We also proved that if (18) holds then (1) is eventually $(k, n-k)$-disfocal if and only if it is eventually $(k, n-k)$-disconjugate. Combining this fact with (19), we conclude:

THEOREM 6. *Let* $\int^{\infty} \rho_i^{-1}(s)\, ds = \infty$, $i = 1, \ldots, n-1$. S_k *consists of non-oscillatory solutions if and only if* (1) *is eventually* $(k, n-k)$*-disfocal* (*and eventually* $(k, n-k)$*-disconjugate*).

We may now use known results about disconjugacy of (1) for the study of the sets S_k. For example, let S_k consist of non-oscillatory solutions and let $(-1)^{n-k}p \leqq (-1)^{n-k}\tilde{p} < 0$. Then the set $\tilde{S}_k$, which corresponds to the equation

$$L_n y + \tilde{p} y = 0,$$

consists of non-oscillatory solutions, too. Similarly, the solutions in S_k are of the same type as the solutions in S^*_{n-k}, which corresponds to the adjoint equation

$$L_n^* y + (-1)^n py = 0.$$

For example, the non-trivial solutions of the self-adjoint equation

$$y^{(4)} + py = 0, \quad p > 0,$$

are either all oscillatory or all non-oscillatory since they belong to S_1 and $S_3 = S_{4-1}$ [see **14**, Theorem 1].

In our last example we demonstrate Theorem 5 and the dominance relations between different sets.

Example 4. The Euler equation

$$y^{(6)} + 4x^{-6}y = 0, \quad 0 < x < \infty \tag{28}$$

has solutions of the form x^r if

$$\varphi(r) \overset{\text{def}}{\equiv} r(r-1)\ldots(r-5)+4=0. \tag{29}$$

$\varphi(r) > 0$ on $(-\infty, 0] \cup [1, 2] \cup [3, 4] \cup [5, \infty)$, therefore φ may have real zeros only in $(0, 1) \cup (2, 3) \cup (4, 5)$. Since $\varphi(r) = \varphi(5-r)$, $\min_{2 \leq r \leq 3} \varphi(r) = \varphi(5/2) > 0$ while $\varphi(1/2) = \varphi(9/2) < 0$. Therefore (29) has 4 real solutions $0 < r_0 < r_1 < 1$, $4 < r_4 < r_5 < 5$, $r_{5-i} = 5 - r_i$, and two complex solutions $r_{2,3} = 5/2 \pm i\mu$. x^{r_0}, x^{r_1} satisfy

$$y^{(i)} > 0, \quad i = 0,$$

$$(-1)^{j-1} y^{(j)} > 0, \quad j = 1, \ldots, 5, \quad x > 0,$$

thus $x^{r_0}, x^{r_1} \in S_1$. Similarly, $x^{r_4}, x^{r_5} \in S_5$. Consequently, the oscillatory solutions $x^{5/2} \cos(\ln x)$, $x^{5/2} \sin(\ln x)$ belong to S_3. In other terms, equation (28) is $(1, 5)$-disfocal and $(5, 1)$-disfocal on $(0, \infty)$ but it is not eventually $(3, 3)$-disfocal [-disconjugate]. The phenomena which are described in Theorems 3–5 are easily observed for equation (28). This example also shows that for $k \neq l$, there is not, in general, a simple relationship between the $(k, n-k)$- and $(l, n-l)$-disfocality of (1) and the above mentioned [**14**, Theorem 1] cannot be generalized for $n > 4$.

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