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FOCAL POINTS FOR A LINEAR DIFFERENTIAL EQUATION WHOSE COEFFICIENTS ARE OF CONSTANT SIGNS

BY

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ABSTRACT. The differential equation considered is $y^{(n)} + \Sigma p_i(x)y^{(i)} = 0$, where $\sigma_i p_i(x) \geqslant 0$, $i = 0, \ldots, n-1$, $\sigma_i = \pm 1$. The focal point $\zeta(a)$ is defined as the least value of s, $s > a$, such that there exists a nontrivial solution y which satisfies $y^{(i)}(a) = 0$, $\sigma_i \sigma_{i+1} > 0$ and $y^{(i)}(s) = 0$, $\sigma_i \sigma_{i+1} < 0$. Our method is based on a characterization of $\zeta(a)$ by solutions which satisfy $\sigma_i y^{(i)} > 0$, $i = 0, \ldots, n-1$, on $[a, b]$, $b < \zeta(a)$. We study the behavior of the function ζ and the dependence of $\zeta(a)$ on $p_0, \ldots, p_{n-1}$ when at least a certain $p_i(x)$ does not vanish identically near a or near $\zeta(a)$. As an application we prove the existence of an eigenvalue of a related boundary value problem.

1. In the study of oscillatory properties of a linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(m-1)} + \cdots + p_0(x)y = 0, \tag{1}$$

certain solutions which satisfy some particular boundary conditions have an important role. For example, the $(k, n-k)$-focal point of the equation

$$y^{(n)} + p(x)y = 0, \tag{2}$$

associated with the point a, is defined to be the infimum of the values of s, $s > a$, such that there exists a nontrivial solution of (2) which satisfies

$$y^{(i)}(a) = 0, \quad i = 0, \ldots, k-1,$$
$$y^{(i)}(s) = 0, \quad i = k, \ldots, n-1.$$

It turns out that the $(k, n-k)$-focal points and the associated solutions are useful tools for the study of the disconjugacy of (2) [**6**].

The concept of $(k, n-k)$-focal point was generalized by Keener and Travis [**5**] for the equation

$$y^{(n)} - (-1)^{n-k} \sum_{i=0}^{n-1} p_i(x)y^{(i)} = 0 \tag{3}$$

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where

$$p_i(x) \geqslant 0, \quad i = 0, \ldots, k-1,$$
$$(-1)^{i-k} p_i(x) \geqslant 0, \quad i = k, \ldots, n-1.$$

In the present study we shall define focal points for (1), when *each of the functions* $p_0(x), \ldots, p_{n-1}(x)$ *is of an arbitrary constant sign* on $(0, \infty)$. We assume that $\operatorname{sgn}[p_i(x)] = \sigma_i$, $i = 0, \ldots, n-1$, where σ_i is $+1$ or -1 (when $p_i(x) \equiv 0$ for certain i, σ_i will be determined arbitrarily). For this equation we shall be interested in the solutions which satisfy

$$\operatorname{sgn}\left[y^{(i)}(x)\right] = \sigma_i, \qquad i = 0, \ldots, n-1.$$

For such a solution, $y^{(n)} = -\sum p_i y^{(i)} \leqslant 0$, so it will be convenient to define $\sigma_n = -1$.

For equation (1) we define the *focal point of a* as the infimum of the values of s, $s > a$, such that there exists a nontrivial solution of (1) which satisfies the n boundary conditions

$$\begin{aligned} y^{(i)}(a) &= 0, \quad \sigma_i\sigma_{i+1} > 0, \\ y^{(i)}(s) &= 0, \quad \sigma_i\sigma_{i+1} < 0, \qquad i = 0, \ldots, n-1. \end{aligned} \tag{4}$$

The focal point will be denoted by $\zeta(a)$. Clearly $\zeta(a) > a$. We shall see later that if $p_0(x)$ is not eventually vanishing, then $\zeta(a) < \infty$ for every a, except perhaps for equation (3).

The number of boundary conditions of (4) at a is the number of the sign changes in the sequence $\sigma_0, -\sigma_1, \ldots, (-1)^n\sigma_n$. Denote this number by k and recall that $\sigma_n = -1$ and $\sigma_0 = \operatorname{sgn}[p_0(x)]$. If $p_0 \not\equiv 0$, σ_0 is uniquely defined, so k is even if $\operatorname{sgn}[p_0(x)] = (-1)^{n-1}$ and k is odd if $\operatorname{sgn}[p_0(x)] = (-1)^n$. We may summarize this in the single condition

$$(-1)^{n-k} p_0(x) \leqslant 0.$$

The basis of the study of (3) by Keener and Travis in **[5]** is the eigenvalue problem

$$\begin{gathered} y^{(n)} - (-1)^{n-k}\lambda \sum_{i=0}^{n-1} p_i(x) y^{(i)} = 0, \\ \begin{aligned} y^{(i)}(a) &= 0, \quad i = 0, \ldots, k-1, \\ y^{(i)}(b) &= 0, \quad i = k, \ldots, n-1. \end{aligned} \end{gathered} \tag{5}$$

(5) is replaced by an equivalent integral equation and it is proved that if

$$\begin{aligned} p_0(x) &> 0, \\ p_i(x) &\geqslant 0, \quad i = 1, \ldots, k-1 \\ (-1)^{i-k} p_i(x) &\geqslant 0, \quad i = k, \ldots, n-1, a \leqslant x \leqslant b, \end{aligned}$$

then the problem has a positive eigenvalue and this eigenvalue is a strictly monotone decreasing function of b. This is established in [5] by the theory of μ_0-positive operators with respect to a cone in a Banach space. The relation between the existence of focal points of (3) and the existence of a smallest positive eigenvalue of (5) is used to prove other properties of focal points. For related eigenvalue problems see [3], [7].

Our methods of proof are essentially different. We base our study of focal points on the following characterizations, which are proved by elementary arguments:

THEOREM 1. *$\zeta(a) > b$ if and only if* (1) *has a solution y such that*

$$\sigma_i y^{(i)}(x) > 0, \qquad i = 0, \ldots, n-1, a \leqslant x \leqslant b. \tag{6}$$

THEOREM 2. *$\zeta(a) > b$ if and only if there exists a function $f \in C^n$ such that*

$$\sigma_i f^{(i)} > 0, \quad i = 0, \ldots, n-1,$$
$$f^{(n)} + \sum p_i f_i \leqslant 0, \quad a \leqslant x \leqslant b. \tag{7}$$

We shall see that Theorems 1 and 2 have meaning even if (4) is in fact an initial value problem.

Examples (i)–(iv) will show that if we only assume that $\sigma_i p_i(x) \geqslant 0$, $i = 0, \ldots, n-1$, then ζ is not necessarily continuous nor strictly increasing. However, very slight assumptions about $p_0, \ldots, p_{n-1}$ may guarantee nice properties of $\zeta(a)$. We shall assume that either $\sigma_{n-1}\sigma_n > 0$ and $\Sigma_{i=0}^{q-1}|p_i(x)| \not\equiv 0$ near a or $\sigma_{n-1}\sigma_n < 0$ and $\Sigma_{i=0}^{q-1}|p_i(x)| \not\equiv 0$ near $\zeta(a)$, where q is the least integer such that $\sigma_q\sigma_{q+1} = \cdots = \sigma_{n-1}\sigma_n$. Then ζ is a strictly increasing, continuous function. Moreover $\zeta(a)$ depends continuously on $p_0, \ldots, p_{n-1}$ and if $p_0, \ldots, p_{n-1}$ are replaced by $\bar{p}_0, \ldots, \bar{p}_{n-1}$, $\sigma_i p_i \geqslant \sigma_i \bar{p}_i \geqslant 0$, $i = 0, \ldots, n-1$, then the focal point strictly grows (unless $p_i \equiv \bar{p}_i$, $i = 0, \ldots, n-1$).

As an application we show how the focal points can be used to prove that the problem

$$y^{(n)} + \lambda \sum_{i=0}^{n-1} p_i y^{(i)} = 0,$$
$$y^{(i)}(a) = 0, \quad \sigma_i\sigma_{i+1} > 0,$$
$$y^{(i)}(b) = 0, \quad \sigma_i\sigma_{i+1} < 0,$$

has a positive eigenvalue if $\Sigma_{i=0}^{q-1}|p_i(x)| \not\equiv 0$ in $[a, b]$.

2. The necessity part of Theorem 2 is a particular case of the necessity part of Theorem 1, since we may choose f as the solution of (1) which satisfies (6). And conversely, the sufficiency of Theorem 2 implies that of Theorem 1, since a solution which satisfies (6), satisfies (7) trivially. Hence, we shall show that (6) is a necessary condition and (7) is a sufficient condition for $\zeta(a) > b$.

Necessity of Theorem 1. By the definition of $\zeta(a)$, for every s, $a \leqslant s < \zeta(a)$, only the trivial solution of (1) satisfies the homogeneous boundary conditions (4). Consequently, there exists a unique solution of (1) which satisfies

$$y^{(i)}(a) = \sigma_i, \quad \sigma_i\sigma_{i+1} > 0,$$
$$y^{(i)}(s) = \sigma_i, \quad \sigma_i\sigma_{i+1} < 0. \tag{8}$$

Moreover, this solution $y_s(x)$ depends continuously on the parameter s, $a \leqslant s < \zeta(a)$. For, let $\{y_1, \ldots, y_n\}$ be an independent set of solutions of (1). Then there exists a solution $y = \sum_{j=1}^n c_j y_j$ which satisfies (8) if and only if the nonhomogeneous system of n linear equations

$$\sum_{j=1}^{n} c_j y_j^{(i)}(a) = \sigma_i, \quad \sigma_i\sigma_{i+1} > 0,$$
$$\sum_{j=1}^{n} c_j y_j^{(i)}(s) = \sigma_i, \quad \sigma_i\sigma_{i+1} < 0, \tag{9}$$

has a nonvanishing determinant, i.e., if and only if the the corresponding homogeneous system has only the trivial solution. $y_s(x) = \sum c_j(s) y_j(x)$ is a continuous function of s since $c_j(s)$, $j = 1, \ldots, n$, which are defined by (9) are continuous for s, $a \leqslant s < \zeta(a)$.

For $s = a$, $\sigma_i y_a^{(i)}(a) = 1$, $i = 0, \ldots, n-1$; hence $\sigma_i y_a^{(i)} > 0$ on a neighborhood of a. By the continuous dependence of y_s on s we have for s sufficiently close to a

$$\sigma_i y_s^{(i)}(x) > 0, \qquad i = 0, \ldots, n-1, a \leqslant x \leqslant s. \tag{10}$$

Define s_0 to be the supremum of the values s, $a \leqslant s < \zeta(a)$, such that y_s satisfies (10). Clealry, $a < s_0 \leqslant \zeta(a)$. To complete the proof, we shall show that $s_0 = \zeta(a)$ and since $b < \zeta(a)$, the solution y_b will satisfy inequalities (6).

Suppose on the contrary that $s_0 < \zeta(a)$. By the definition of s_0, y_s satisfies (10) for every $s < s_0$. But y_s depends continuously on s, for $a \leqslant s < \zeta(a)$, in particular for $s = s_0$. So as $s \to s_0-$, we have by (10)

$$\sigma_i y_{s_0}^{(i)}(x) \geqslant 0, \qquad i = 0, \ldots, n-1,$$
$$\sigma_n y_{s_0}^{(n)}(x) = \sum p_i(x) y_{s_0}(x) \geqslant 0, \qquad a \leqslant x \leqslant s_0. \tag{11}$$

However, if $\sigma_i\sigma_{i+1} > 0$ then by (8) and (11)

$$\sigma_i y_{s_0}^{(i)}(a) = 1,$$
$$\sigma_i y_{s_0}^{(i+1)}(x) = \sigma_{i+1} y_{s_0}^{(i+1)}(x) \geqslant 0, \qquad a \leqslant x \leqslant s_0,$$

and so $\sigma_i y_{s_0}^{(i)}$ is a nondecreasing, positive function on $[a, s_0]$. Similarly, if $\sigma_i\sigma_{i+1} < 0$, then

$$\sigma_i y_{s_0}^{(i)}(s_0) = 1,$$
$$\sigma_i y_{s_0}^{(i+1)}(x) = -\sigma_{i+1} y_{s_0}^{(i+1)}(x) \leqslant 0, \qquad a \leqslant x \leqslant s_0,$$

and $\sigma_i y_{s_0}^{(i)}$ is a nonincreasing, positive function on $[a, s_0]$. Thus, (10) holds also for $s = s_0$, and consequently there exists $\varepsilon > 0$ such that

$$\sigma_i y_{s_0}^{(i)}(x) > 0, \qquad i = 0, \ldots, n-1, a - \varepsilon \leqslant x \leqslant s_0 + \varepsilon. \tag{12}$$

Since we assumed that $s_0 < \zeta(a)$, y_s depends continuously on s for $s_0 \leqslant s < \zeta(a)$. It follows by (12) that for s sufficiently close to s_0 and $s > s_0$, y_s satisfies (10). This contradicts the definition of s_0 and proves that $s_0 = \zeta(a)$. Now, since $b < \zeta(a) = s_0$, the solution y_b satisfies (6).

Sufficiency of Theorem 2. Let f satisfy (7) on $[a, b]$ and suppose that $\zeta(a) \leqslant b$. Let y be a solution of (1) which satisfies the boundary conditions (4) at the points a and $\zeta(a)$, $a < \zeta(a) \leqslant b$. We shall achieve a contradiction, thus proving $\zeta(a) > b$.

Let λ_0 be the smallest value of λ, $\lambda > 0$, such that at least one of the functions $(f - \lambda y)^{(i)}$, $i = 0, \ldots, n-1$, has a zero in $[a, \zeta(a)]$. λ_0 exists since $\sigma_0 f > 0$ on $[a, \zeta(a)]$ and we may assume that $\sigma_0 y > 0$ at one point of $[a, \zeta(a)]$ (otherwise we replace y by $-y$). Furthermore, $\lambda_0 > 0$ since $f^{(i)} \neq 0$, $i = 0, \ldots, n-1$, on $[a, \zeta(a)]$. By (7) and the continuous dependence of $(f - \lambda y)^{(i)}$ on λ, we have

$$\sigma_i (f - \lambda_0 y)^{(i)} \geqslant 0, \qquad i = 0, \ldots, n-1,$$
$$\sigma_n (f - \lambda_0 y)^{(n)} \geqslant \sum p_i (f - \lambda_0 y)^{(i)} \geqslant 0, \qquad a \leqslant x \leqslant \zeta(a). \tag{13}$$

We shall show that no $(f - \lambda_0 y)^{(i)}$ can have a zero in $[a, \zeta(a)]$, thus contradicting the definition of λ_0. For $\sigma_i \sigma_{i+1} > 0$ we have by the boundary conditions (4) which y satisfies at a and at $\zeta(a)$, $\sigma_i (f - \lambda_0 y)^{(i)}(a) = \sigma_i f^{(i)}(a) > 0$, and by (13)

$$\sigma_i (f - \lambda_0 y)^{(i+1)}(x) = \sigma_{i+1} (f - \lambda_0 y)^{(i+1)}(x) \geqslant 0, \qquad a \leqslant x \leqslant \zeta(a).$$

Hence $\sigma_i (f - \lambda_0 y)^{(i)}$ is a nondecreasing positive function on $[a, \zeta(a)]$ and it does not vanish there. Similarly, for $\sigma_i \sigma_{i+1} < 0$ we have $\sigma_i (f - \lambda_0 y)^{(i)}(\zeta(a)) = \sigma_i f^{(i)}(\zeta(a)) > 0$, and by (13), $\sigma_i (f - \lambda_0 y)^{(i+1)}(x) = -\sigma_{i+1} (f - \lambda_0 y)^{(i+1)}(x) \leqslant 0$, $a \leqslant x \leqslant \zeta(a)$, i.e., $\sigma_i (f - \lambda_0 y)^{(i)}$ is a decreasing, positive function on $[a, \zeta(a)]$ and it has no zeros there. This contradicts the definition of λ_0 and the sufficiency part of Theorem 2 is proved.

It may be interesting to note that Theorem 1 holds even if the boundary conditions (4) are in fact initial value conditions. When $\sigma_0 = \sigma_1 = \cdots = \sigma_n = -1$, (4) turns out to be an initial value problem which, independently of s, is satisfied only by the trivial solution and so, by definition, $\zeta(a) = \infty$. In this case, the solution y of

$$y^{(n)} = \sum (-p_i)y^{(i)}, \qquad (-p_i) \geqslant 0,$$
$$y^{(i)}(a) > 0, \qquad i = 0, \ldots, n-1,$$

is easily seen to satisfy $y, y', \ldots, y^{(n-1)} > 0$ on $[a, \infty)$ an $-y$ satisfies (6) there. When $\sigma_i = (-1)^{n-i+1}$, $i = 0, \ldots, n-1$, (4) is $y^{(i)}(s) = 0$, $i = 0, \ldots, n-1$, and once again $\zeta(a) = \infty$. Now, it is known [4, p. 508] that the equation

$$y^{(n)} = \sum (-p_i)y^{(i)}, \qquad \operatorname{sgn}[-p_i] = (-1)^{n-i},$$

has a solution which satisfies $(-1)^{n-i}y^{(i)} > 0$, $i = 0, \ldots, n-1$, on $[a, \infty)$ if $p_0 \not\equiv 0$ eventually.

By Theorem 1 we may deduce that if $p_0(x)$ is not eventually zero, then $\zeta(a)$ is finite for every a, except perhaps when $\sigma_0 = \sigma_1 = \cdots = \sigma_k = -\sigma_{k+1} = \cdots = (-1)^{n-k}\sigma_n$, for certain k, $0 \leqslant k \leqslant n$, which (for $1 \leqslant k \leqslant n-1$) is the case considered in [5]. Indeed, if for every $b > a$ there exists a solution which satisfies (6) on $[a, b]$, then an appropriate subsequence converges to a nontrivial solution y such that $\sigma_i y^{(i)} \geqslant 0$, $i = 0, \ldots, n-1$, on $[a, \infty)$. Since y is monotone and $y \not\equiv 0$, we have $\sigma_0 y > 0$ on (a, ∞). But $p_0 \not\equiv 0$, therefore also $y^{(n)} = -\sum p_i y^{(i)} \not\equiv 0$, and since $y^{(i)}$, $i = 0, \ldots, n-1$, are monotone, y satisfies (6) on (c, ∞), $c \geqslant a$. If there is i, $1 \leqslant i \leqslant n-1$, such that $\sigma_i = \sigma_{i+1}$, i.e., $\sigma_i y^{(i)} > 0$ and $\sigma_i y^{(i+1)} > 0$ then $\sigma_i y^{(i)}$ is positive and increasing and so $\sigma_i y^{(i-1)} \to +\infty$. Therefore $\sigma_{i-1} = \sigma_i$. So we must have $\sigma_0 = \cdots = \sigma_k = -\sigma_{k+1} = \cdots = (-1)^{n-k}\sigma_n$ for a certain k, $0 \leqslant k \leqslant n$, which is the case mentioned above.

Our next aim is to study the solutions of (1) which are associated with $\zeta(a)$.

THEOREM 3. *There exists a solution Y of* (1), *associated with $\zeta(a)$, such that*

$$\begin{aligned} Y^{(i)}(a) &= 0, \qquad \sigma_i\sigma_{i+1} > 0, \\ Y^{(i)}(\zeta(a)) &= 0, \qquad \sigma_i\sigma_{i+1} < 0, \; i = 0, \ldots, n-1 \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\sigma_0 Y > 0, \\ &\sigma_i Y^{(i)} \geqslant 0, \qquad a < x < \zeta(a), \; i = 0, \ldots, n-1. \end{aligned} \tag{15}$$

PROOF. We have seen in the proof of Theorem 1 that for every s, $a \leqslant s < \zeta(a)$, the boundary conditions

$$\begin{aligned} y^{(i)}(a) &= \sigma_i, \quad \sigma_i\sigma_{i+1} > 0, \\ y^{(i)}(s) &= \sigma_i, \quad \sigma_i\sigma_{i+1} < 0, \end{aligned} \tag{16}$$

define a unique solution y_s of (1) and that

$$\sigma_i y_s^{(i)} > 0, \qquad i = 0, \ldots, n-1, \; a \leqslant x \leqslant s. \tag{17}$$

We normalize the solutions $\{y_s\}$ through multiplying them by a positive constant K_s so that $\Sigma_{i=0}^{n-1}|K_s y_s^{(i)}(a)| = 1$. As $s \to \zeta(a) -$, we choose a subsequence of $\{K_s y_s\}$ which converges to a nontrivial solution Y of (1). By (17), we have

$$\sigma_i Y^{(i)} \geqslant 0, \qquad i = 0, \ldots, n-1, a \leqslant x \leqslant \zeta(a), \tag{18}$$

and by (16), $|Y^{(i)}(a)| = m$ if $\sigma_i\sigma_{i+1} > 0$ and $|Y^{(i)}(\zeta(a))| = m$ if $\sigma_i\sigma_{i+1} < 0$, where $m = \lim K_s^{-1} \geqslant 0$. If $m \neq 0$, then $\sigma_i Y^{(i)}(a) > 0$, $\sigma_i\sigma_{i+1} > 0$ and $\sigma_i Y^{(i)}(\zeta(a)) > 0$, $\sigma_i\sigma_{i+1} < 0$. As in the proof of Theorem 2, these inequalities together with (18) imply that $\sigma_i Y^{(i)} > 0$, $i = 0, \ldots, n-1$, on $[a, \zeta(a)]$, which is impossible by Theorem 1. Therefore $m = 0$ (i.e., $K_s \to \infty$) and Y satisfies (14).

By (14), Y vanishes at one of the endpoints of $[a, \zeta(a)]$. If Y would have another zero in $[a, \zeta(a)]$, it would vanish identically on a whole interval, since by (18) it is monotone. This is impossible because Y is a nontrivial solution and so (15) is proved.

REMARK. In contrary to (6), inequalities of type (15) do not imply absence of focal points on any interval, since if $p_0 \equiv 0$, (1) always has the solution $y \equiv 1$.

3. The extensive literature about boundary value functions suggests to ask when is ζ a continuous, increasing function. For example, see [**1**, Theorem 2] and [**6**, Theorem 4.1]. It is also interesting whether $\zeta(a)$ depends continuously on $p_0, \ldots, p_{n-1}$ and decreases when $|p_0|, \ldots, |p_{n-1}|$ are enlarged. Indeed, it can be easily proved by using Theorems 1 and 2 alone that ζ is a nondecreasing function and $\zeta(a)$ does not grow then $|p_0|, \ldots, |p_{n-1}|$ are enlarged. However, if we only assume $\sigma_i p_i(x) \geqslant 0$, $i = 0, \ldots, n-1$, the following examples show that the focal point does not have necessarily more delicate properties.

EXAMPLES. Let $\tau(x) \equiv 0$ for $x \leqslant 0$, $\tau(x) \equiv 1$ for $x > 0$.

(i) $y'' + (1 - \tau(x) + \tau(x-1))y = 0, y(a) = y'(s) = 0.$

We have $\zeta(a) = a + \pi/2 \leqslant 0$ for $a \leqslant -\pi/2$, but $\zeta(a) > 1$ for $a > -\pi/2$.

(ii) $y'' + \alpha(1 - \tau(x) + \tau(x-1))y = 0, y(a) = y'(s) = 0.$

Here, if $\alpha = 1$ then $\zeta(-\pi/2) = 0$ but for every $\alpha < 1$, $\zeta(-\pi/2) > 1$.

(iii) Consider $y'' + p_1(x)y' + p_0(x)y = 0$, where $p_1(x) = -2$, $p_0(x) = 2\tau(x)$. The corresponding boundary conditions are $y'(a) = y(s) = 0$ and the unique solution (up to a multiplicative constant) which satisfies $y'(a) = 0$ for $a \leqslant 0$ is $y(x) \equiv 1$ for $x \leqslant 0$, $y(x) = e^x(\cos x - \sin x)$ for $x > 0$. It follows that $\zeta(a) = \pi/4$ for every $a \leqslant 0$.

(iv) Let now $\bar{p}_1(x) = -1 - \tau(x) \leqslant 0$, $\bar{p}_0(x) = 2\tau(x) \geqslant 0$. Then $p_0 \equiv \bar{p}_0 \geqslant 0$, $-p_1 \geqslant -\bar{p}_1 > 0$, $p_1 \not\equiv \bar{p}_1$ but $y'' + \bar{p}_1 y' + \bar{p}_0 y = 0$ has the same solution as in example (iii) for $a \leqslant 0$. Consequently $\bar{\zeta}(a) = \zeta(a)$.

It turns out that the irregular behaviour of focal points, which was demonstrated above, will be eliminated if the inequalities in (15) are strict. Therefore next we check when does this happen.

LEMMA 1. *Let q, $1 \leqslant q \leqslant n-1$, be the least integer such that $\sigma_q\sigma_{q+1} = \sigma_{q+1}\sigma_{q+2} = \cdots = \sigma_{n-1}\sigma_n$. An equality occurs in* (15) *if and only if $\sigma_{n-1}\sigma_n > 0$ and $p_0 = \cdots = p_{q-1} \equiv 0$ on $[a, c]$ or if $\sigma_{n-1}\sigma_n < 0$ and $p_0 \equiv \cdots \equiv p_{q-1} \equiv 0$ on $[c, \zeta(a)]$, for certain c. In this case*

$$\sigma_i Y^{(i)} > 0, \qquad i = 0, \ldots, q-1, a < x < \zeta(a),$$

and

$$\sigma_i Y^{(i)} \equiv 0, \qquad i = q, \ldots, n,$$

on $[a, c]$ or on $[c, \zeta(a)]$, respectively.

PROOF. Let an equality hold in the ith inequality of (15). Suppose, for example that $\sigma_i\sigma_{i+1} > 0$, i.e., $Y^{(i)}(a) = 0$ and let $Y^{(i)}(c) = 0$, $a < c \leqslant \zeta(a)$. $Y^{(i)}$ is monotone, so $Y^{(i)} \equiv 0$ on $[a, c]$. On the other hand, $Y^{(i)} \not\equiv 0$ on $[a, \zeta(a)]$, since integration of $Y^{(i)} \equiv 0$ and application of (14) would imply that $Y \equiv 0$. Therefore $a < c < \zeta(a)$ and $Y^{(i)}(\zeta(a)) \neq 0$.

Clearly, $Y^{(i+1)} \equiv 0$ on $[a, c]$, $Y^{(i+1)}(\zeta(a)) \neq 0$ and therefore $\sigma_i\sigma_{i+1} = \sigma_{i+1}\sigma_{i+2}$. Consider now $Y^{(i-1)}$. If $\sigma_{i-1}\sigma_i = \sigma_i\sigma_{i+1} > 0$ then $Y^{(i-1)}(a) = 0$ and $Y^{(i-1)} = \int_a^x Y^{(i)}(t)\,dt \equiv 0$ for $a \leqslant x \leqslant c$. And if $\sigma_{i-1}\sigma_i = -\sigma_i\sigma_{i+1} < 0$, then $Y^{(i-1)}(\zeta(a)) = 0$ and $Y^{(i-1)}(x) \neq 0$ for $a \leqslant x < \zeta(a)$. For if $Y^{(i-1)}(\alpha) = 0$, $a \leqslant \alpha < \zeta(a)$, then $Y^{(i-1)} \equiv 0$ on $[\alpha, \zeta(a)]$ and this would contradict $Y^{(i)}(\zeta(a)) \neq 0$.

Let q, $1 \leqslant q \leqslant n-1$, be the least integer such that $\sigma_q\sigma_{q+1} = \sigma_{q+1}\sigma_{q+2} = \cdots = \sigma_{n-1}\sigma_n$. The above discussion shows that $Y^{(i)} \equiv 0$ on $[a, c]$ for $i = q, \ldots, n$ and $Y^{(i)} \neq 0$ on $(a, c]$ for $i = 0, \ldots, q-1$. By (1), $\sum_{i=0}^{q-1} p_i Y^{(i)} = 0$ on $[a, c]$ and since $p_i Y^{(i)} \geqslant 0$, $Y^{(i)} \neq 0$, $i = 0, \ldots, q-1$, we have $p_0 \equiv \cdots \equiv p_{q-1} \equiv 0$ on $[a, c]$.

Conversely, if the above conditions are fulfilled, then $u = y^{(q)}$ satisfies, on $[a, c]$, $u^{(n-q)} + \sum_{i=q}^{n-1} p_i u^{(i-q)} = 0$, $u^{(i)}(a) = 0$, $i = 0, \ldots, n-q-1$ and therefore $y^{(q)} \equiv \cdots \equiv y^{(n)} \equiv 0$ on $[a, c]$.

In the remainder of the paper let a be a fixed point. In order to guarantee strict inequalities in (15) we assume

ASSUMPTION I. *Either $\sigma_{n-1}\sigma_n > 0$ and $\sum_{i=0}^{q-1} |p_i(x)| \not\equiv 0$ in a right neighborhood of a or $\sigma_{n-1}\sigma_n < 0$ and $\sum_{i=0}^{q-1} |p_i(x)| \not\equiv 0$ in a left neighborhood of $\zeta(a)$, where q is the least integer such that $\sigma_q\sigma_{q+1} = \sigma_{q+1}\sigma_{q+2} = \cdots = \sigma_{n-1}\sigma_n$.*

Before proceeding the study of the focal points, we prove the following lemma, which emphasizes the importance of the solutions of (1) which satisfy $\sigma_i y^{(i)} > 0$.

LEMMA 2. *Let y be a solution of* (1) *which satisfies*

$$\sigma_i y^{(i)} > 0, \qquad i = 0, \dots, n-1, a < x < s, \tag{19}$$

and let $\bar{y}$ be any solution of

$$y^{(n)} + \sum_{i=0}^{n-1} \bar{p}_i(x) y^{(i)} = 0 \tag{20}$$

where

$$\sigma_i p_i(x) \geqslant \sigma_i \bar{p}_i(x) \geqslant 0, \qquad i = 0, \dots, n-1, a \leqslant x \leqslant s. \tag{21}$$

If $\bar{y}$ satisfies at least those of the boundary conditions (4) *which are satisfied by y (perhaps none), then there exists a positive λ such that*

$$\sigma_i (y - \lambda\bar{y})^{(i)} \geqslant 0, \qquad i = 0, \dots, n, a \leqslant x \leqslant s. \tag{22}$$

Note, that no assumptions are made about the sign of $\bar{y}^{(i)}(x)$.

PROOF. Since $\sigma_i y^{(i)} \geqslant 0$, (22) is equivalent to $|y^{(i)}| \geqslant \lambda|\bar{y}^{(i)}|$. Roughly saying, this means that whenever $y^{(i)}$ vanishes, $\bar{y}^{(i)}$ tends to zero at least as quickly as $y^{(i)}$ does. For the first derivatives this is almost trivial. However, $p_0, \dots, p_{n-1}$ may all vanish at the same point or vanish identically on the same interval and by $y^{(n)} = -\sum p_i y^{(i)}$, the behaviour of $y^{(n-1)}$ near its zeros may be problematical.

The only zeros which $y, \dots, y^{(n-1)}$ may have in $[a, s]$ are the zeros which are specified in (4) or part of them. For, if $\sigma_i \sigma_{i+1} > 0$ then $\sigma_i y^{(i)} > 0$ and $\sigma_i y^{(i+1)} = \sigma_{i+1} y^{(i+1)} > 0$ on (a, s), i.e., $\sigma_i y^{(i)}$ may have a zero only at a. But at this point also the corresponding derivative of $\bar{y}$ vanishes according to our assumption. Assume for example that $\sigma_{n-1}\sigma_n > 0$ and let q be defined as in the statement of Assumption I. Then $\sigma_{q-1}\sigma_q < 0$ and $y^{(q-1)}(a) \neq 0$. Consequently the zeros of $y, \dots, y^{(q-1)}$ at a are at least of the same multiplicities as those of $\bar{y}, \dots, \bar{y}^{(q-1)}$. Since $\sigma_{n-1}\sigma_n > 0$ and $y^{(n-1)}(s) \neq 0$, the same is true also at s. It follows that there exists $\lambda > 0$ such that $|y^{(i)}| \geqslant \lambda|\bar{y}^{(i)}|$, $i = 0, \dots, q-1$, $a \leqslant x \leqslant s$. Since $\sigma_i y^{(i)} > 0$, we have

$$\sigma_i y^{(i)} \geqslant \lambda \sigma_i \bar{y}^{(i)}, \qquad i = 0, \dots, q-1, a \leqslant x \leqslant s. \tag{23}$$

At the points a and s, (22) holds trivially also for $i = q, \dots, n-1$. For if $y^{(i)}$ vanishes there, then as we have already remarked, also $\bar{y}^{(i)}$ vanishes. And if $y^{(i)}(a) \neq 0$, i.e., $\sigma_i y^{(i)}(a) > 0$, then (22) holds for that i and $x = a$ if λ is sufficiently small.

In order to prove (22) on $[a, s]$ for $i = q, \dots, n$ we denote $w = \sigma_q(y - \lambda\bar{y})^{(q)}$. Recall that $\sigma_q\sigma_{q+1} = \cdots = \sigma_{n-1}\sigma_n > 0$ i.e., $\sigma_q = \sigma_{q+1} = \cdots = \sigma_n = -1$. Therefore by (21) and (23),

$$w^{(n-q)} + \sum_{i=0}^{n-q-1} p_{i+q} w^{(i)} = -\left[y^{(n)} + \sum_{i=q}^{n-1} p_i y^{(i)} \right] + \lambda \left[\bar{y}^{(n)} + \sum_{i=q}^{n-1} \bar{p}_i \bar{y}^{(i)} \right]$$
$$= \sum_{i=0}^{q-1} p_i y^{(i)} - \lambda \sum_{i=0}^{q-1} \bar{p}_i \bar{y}^{(i)} \geqslant 0$$

and $w^{(i)}(a) = \sigma_{i+q}(y - \lambda\bar{y})^{i+q}(a) \geqslant 0$, $i = 0, \ldots, n - q - 1$, since (22) has already been proved for $x = a$. But

$$w^{(n-q)} \geqslant \sum_{i=0}^{n-q-1} (-p_{i+q}) w^{(i)}, \qquad (-p_{i+q}) \geqslant 0,$$
$$w^{(i)}(a) \geqslant 0, \qquad i = 0, \ldots, n - q - 1,$$

imply that $w^{(i)} = \sigma_{i+q}(y - \lambda\bar{y})^{(i+q)} \geqslant 0$, $i = 0, \ldots, n - q$, $a \leqslant x \leqslant s$, and the lemma is proved.

The first application of Assumption I will be the following strengthening of Theorem 3.

THEOREM 3′. *There exists an essentially unique solution of* (1) *associated with* $\zeta(a)$, *which satisfies the boundary conditions* (14). *This solution satisfies*

$$\begin{aligned} &\sigma_i Y^{(i)} > 0, \qquad \sigma_i\sigma_{i+1} > 0, \quad a < x \leqslant \zeta(a), \\ &\sigma_i Y^{(i)} > 0, \qquad \sigma_i\sigma_{i+1} < 0, \quad a \leqslant x < \zeta(a). \end{aligned} \tag{24}$$

PROOF. By Assumption I, an equality in (15) is excluded and the solution Y which is defined in the proof of Theorem 3 satisfies (24). To prove the uniqueness, suppose that there exists another solution $\bar{Y}$ which satisfies the boundary conditions (14). Note that we do not assume anything about the signs of $\bar{Y}(x), \ldots, \bar{Y}^{(n-1)}(x)$.

Let λ_0 be the maximal value of λ such that $\sigma_i(Y - \lambda\bar{Y})^{(i)} \geqslant 0$, $i = 0, \ldots, n - 1$, $a \leqslant x \leqslant \zeta(a)$. Since Y and $\bar{Y}$ satisfy the conditions of Lemma 2 on $[a, \zeta(a)]$, we have $0 < \lambda_0 < \infty$. Clearly

$$\sigma_i\left(Y - \lambda_0\bar{Y}\right)^{(i)} \geqslant 0, \qquad i = 0, \ldots, n - 1, a \leqslant x \leqslant \zeta(a),$$

and $Y - \lambda_0\bar{Y}$ satisfies the boundary conditions (14). Therefore, if $Y - \lambda_0\bar{Y} \not\equiv 0$, then by Assumption I we have

$$\sigma_i\left(Y - \lambda_0\bar{Y}\right)^{(i)} > 0, \qquad i = 0, \ldots, n - 1, a < x < \zeta(a).$$

But then we may apply Lemma 2 for the solutions $Y - \lambda_0\bar{Y}$ and $\bar{Y}$ on $[a, \zeta(a)]$ and obtain $\lambda_1 > 0$ such that

$$\sigma_i\left(Y - (\lambda_0 + \lambda_1)\bar{Y}\right)^{(i)} > 0, \qquad i = 0, \ldots, n - 1, a < x < \zeta(a),$$

in contrary of the maximality of λ_0. This proves that $Y \equiv \lambda_0\bar{Y}$.

The following theorems are routine applications of Lemma 2.

THEOREM 4. *Let $\bar{\zeta}(a)$ be the focal point which corresponds to the equation*

$$y^{(n)} + \sum_{i=0}^{n-1} \bar{p}_i(x) y^{(i)} = 0, \qquad \sigma_i \bar{p}_i \geqslant 0. \tag{25}$$

If $\sigma_i p_i(x) \geqslant \sigma_i \bar{p}_i(x)$, $i = 0, \ldots, n-1$, $a \leqslant x \leqslant \zeta(a)$, then $\bar{\zeta}(a) \geqslant \zeta(a)$. If Assumption I *holds for* (1), *then $\bar{\zeta}(a) = \zeta(a)$ if and only if* (1) *and* (25) *are identical on* $[a, \zeta(a)]$.

PROOF. Let y be a solution of (1) which, according to Theorem 1, satisfies (6) on $[a, b]$, $b < \zeta(a)$. Then $y^{(n)} + \sum \bar{p}_i y^{(i)} \leqslant y^{(n)} + \sum p_i y^{(i)} = 0$ on $[a, b]$ and, by Theorem 2, $\bar{\zeta}(a) > b$. Since this holds for every b, $b < \zeta(a)$, it follows that $\bar{\zeta}(a) \geqslant \zeta(a)$.

It is sufficient to prove the second part of the theorem only when (25) too satisfies Assumption I. For choose $\tilde{p}_i$, $\sigma_i \bar{p}_i \leqslant \sigma_i \tilde{p}_i \leqslant \sigma_i p_i$, such that the intermediate equation is not identical with (1) but satisfies Assumption I. By the above argument, $\bar{\zeta}(a) \geqslant \tilde{\zeta}(a)$, and if we are able to prove that $\tilde{\zeta}(a) > \zeta(a)$, then $\bar{\zeta}(a) > \zeta(a)$ follows.

The remainder of the proof is similar to that of Theorem 3′. Suppose $\bar{\zeta}(a) = \zeta(a)$. Let Y be the solution of (1) which satisfies the boundary conditions (14) at a and at $\zeta(a) = \bar{\zeta}(a)$ and which satisfies inequalities (24) on $[a, \zeta(a)] = [a, \bar{\zeta}(a)]$, and let $\bar{Y}$ be the solution of (25) which satisfies (14) at the same points. Let λ_0 be the largest value of λ such that $\sigma_i(Y - \lambda\bar{Y})^{(i)} \geqslant 0$, $i = 0, \ldots, n-1$, $a \leqslant x \leqslant \zeta(a) = \bar{\zeta}(a)$. Clearly,

$$\sigma_i\left(Y - \lambda_0 \bar{Y}\right)^{(i)} \geqslant 0, \qquad i = 0, \ldots, n-1, a \leqslant x \leqslant \zeta(a), \tag{26}$$

$$\sigma_n\left(Y - \lambda_0 \bar{Y}\right)^{(n)} = \sum p_i Y^{(i)} - \lambda_0 \sum \bar{p}_i \bar{Y}^{(i)} \geqslant \sum \bar{p}_i\left(Y - \lambda_0 \bar{Y}\right)^{(i)}. \tag{27}$$

$Y - \lambda_0 \bar{Y}$ satisfies the boundary conditions (14) at a and at $\zeta(a)$. As in the proof of Lemma 1 we obtain (by using the differential inequality (27) rather than a differential equation) that if $Y - \lambda_0 \bar{Y} \not\equiv 0$ and Assumption I holds for (25), then $\sigma_i(Y - \lambda_0 \bar{Y})^{(i)} > 0$, $i = 0, \ldots, n-1$, $a < x < \zeta(a)$. But this leads to a contradiction as in the proof of the previous theorem. Therefore $Y \equiv \lambda_0 \bar{Y}$ and equations (1) and (27) are identical.

COROLLARY. *Let $\bar{\zeta}(a)$ be the focal point for the $(n - r)$th order equation $(2 \leqslant n - r < n)$,*

$$y^{(n-r)} + p_{n-1} y^{(n-r-1)} + \cdots + p_r y = 0. \tag{28}$$

Then $\bar{\zeta}(a) \geqslant \zeta(a)$ and the inequality is strict unless $p_0 \equiv \cdots \equiv p_{r-1} \equiv 0$.

The boundary conditions which correspond to (28) are

$$y^{(i)}(a) = 0, \qquad \sigma_{i+r}\sigma_{i+r+1} > 0,$$
$$y^{(i)}(s) = 0, \qquad \sigma_{i+r}\sigma_{i+r+1} < 0, \qquad i = 0, \dots, n-r-1.$$

If $\sigma_r\sigma_{r+1} = \cdots = \sigma_{n-1}\sigma_n$, the corollary is trivial since we have in fact initial value conditions and $\zeta(a) = \infty$. Otherwise, $\zeta(a)$ is identical with the focal point of $y^{(n)} + \sum_{i=r}^{n-1} p_i(x)y^{(i)} = 0$ and the corollary follows by Theorem 4.

THEOREM 5. *ζ is strictly increasing on a right neighborhood of a.*

PROOF. First we show that ζ is always nondecreasing. By Theorem 1, for every b, $a_1 < b < \zeta(a_1)$, there is a solution y of (1) such that $\sigma_i y^{(i)} > 0$, $i = 0, \dots, n-1$, $a_1 \leqslant x \leqslant b$. For every a_2, $a_1 < a_2 < \zeta(a_1)$, y satisfies the same inequalities on the smaller interval $[a_2, b]$; thus by Theorem 1, $\zeta(a_2) > b$. Since this holds for every $b < \zeta(a_1)$, $\zeta(a_2) \geqslant \zeta(a_1)$ and ζ is nondecreasing.

Recall now that Assumption I holds for $[a, \zeta(a)]$ and suppose $\zeta(a) = \zeta(\bar{a})$ for certain $\bar{a}$, $a < \bar{a} < \zeta(a)$. Let Y be the solution associated with $\zeta(a)$ and which satisfies (24) on $[a, \zeta(a)]$ and let $\bar{Y}$ be a solution associated with $\zeta(\bar{a})$. Consider Y and $\bar{Y}$ on $[\bar{a}, \zeta(\bar{a})]$. Since $a < \bar{a} < \zeta(a) = \zeta(\bar{a})$, we have

$$\sigma_i Y^{(i)} > 0, \qquad i = 0, \dots, n-1, \bar{a} \leqslant x < \zeta(\bar{a}) = \zeta(a) \tag{29}$$

while

$$\bar{Y}^{(i)}(\bar{a}) = 0, \qquad \sigma_i\sigma_{i+1} > 0,$$
$$\bar{Y}^{(i)}(\zeta(\bar{a})) = 0, \qquad \sigma_i\sigma_{i+1} < 0. \tag{30}$$

We shall show (29) and (30) are incompatible. Let λ_0 be the largest value of λ such that $\sigma_i(Y - \lambda\bar{Y})^{(i)} \geqslant 0$, $i = 0, \dots, n$, $\bar{a} \leqslant x \leqslant \zeta(\bar{a}) = \zeta(a)$. Since Y and $\bar{Y}$ satisfy the conditions of Lemma 2 on $[\bar{a}, \zeta(\bar{a})]$, we have $\lambda_0 > 0$ and

$$\sigma_i\left(Y - \lambda_0\bar{Y}\right)^{(i)} \geqslant 0, \qquad i = 0, \dots, n, \bar{a} \leqslant x \leqslant \zeta(\bar{a}) = \zeta(a). \tag{31}$$

If we show that

$$\sigma_i\left(Y - \lambda_0\bar{Y}\right)^{(i)} > 0, \qquad i = 0, \dots, n-1, \bar{a} < x < \zeta(\bar{a}) = \zeta(a), \tag{32}$$

we shall obtain a contradiction as in the previous proofs, thus confirming that $\zeta(a) > \zeta(\bar{a})$.

If $\sigma_{n-1}\sigma_n > 0$ then $\sigma_{n-1}(Y - \lambda_0\bar{Y})^{(n-1)}(\bar{a}) = \sigma_{n-1}Y^{(n-1)}(\bar{a}) > 0$ by (29) and (30) and $\sigma_{n-1}(Y - \lambda_0\bar{Y})^{(n)} = \sigma_n(Y - \lambda_0\bar{Y})^{(n)} \geqslant 0$ by (31). Consequently $\sigma_{n-1}(Y - \lambda_0\bar{Y})^{(n-1)}$ is a nondecreasing, positive function on $[\bar{a}, \zeta(\bar{a})]$ and (32) follows from (31).

If $\sigma_{n-1}\sigma_n < 0$ then $\sigma_{q-1}\sigma_q > 0$ and by a similar reasoning $\sigma_{q-1}(Y - \lambda_0\bar{Y})^{(q-1)} > 0$ on $[\bar{a}, \zeta(\bar{a})]$, which proves (32) for $i = 0, \dots, q-1$. But according to Assumption I, $\Sigma_{i=0}^{q-1}|p_i(x)| \not\equiv 0$ near $\zeta(a) = \zeta(\bar{a})$ (since $\sigma_{n-1}\sigma_n < 0$) and therefore $(Y - \lambda_0\bar{Y})^{(n)} = -\Sigma\, p_i(Y - \lambda_0\bar{Y})^{(i)} \not\equiv 0$ near $\zeta(a)$. Since $Y^{(n-1)}(\zeta(a)) = \bar{Y}^{(n-1)}(\zeta(a)) = 0$,

$$\sigma_{n-1}\left(Y - \lambda_0\bar{Y}\right)^{(n-1)}(x) = -\int_x^{\zeta(a)} \sigma_{n-1}\left(Y - \lambda_0\bar{Y}\right)^{(n)}\, dt > 0$$

for $\bar{a} \leqslant x < \zeta(a) = \zeta(\bar{a})$ and (32) follows. As we have already remarked, this completes the proof.

Example (iii) shows that ζ is not necessarily strictly increasing in a left neighborhood of a.

Examples (i) and (ii) show that even if Assumption I holds for an interval $[a, \zeta(a)]$, ζ is not necessarily continuous at a and $\zeta(a)$ does not depend continuously on $p_0, \ldots, p_{n-1}$. This is not surprising, since the continuity of ζ depends on its behaviour in a whole neighborhood of a. This is the reason that in the next two theorems we modify our assumptions.

THEOREM 6. *ζ is always left-continuous. If $\sigma_{n-1}\sigma_n > 0$ and $\Sigma_{i=0}^{q-1}|p_i| \not\equiv 0$ in a right neighborhood of a or $\sigma_{n-1}\sigma_n < 0$ and $\Sigma_{i=0}^{q-1}|p_i| \not\equiv 0$ in a right neighborhood of $\zeta(a)$, then ζ is continuous at a.*

PROOF. ζ is a nondecreasing function; therefore the one sided limits $\zeta(a-)$, $\zeta(a+)$ exist. First we show that $\zeta(a-) = \zeta(a)$. For every $\alpha < a$, $\zeta(\alpha) < \infty$ and there is a solution Y_α which satisfies the boundary conditions (14) at α and at $\zeta(\alpha)$. As $\alpha \to a-$, we normalize $\{Y_\alpha\}$ and by taking an appropriate subsequence, we obtain a nontrivial solution of (1) which satisfies (4) for $s = \zeta(a-)$. Thus $\zeta(a) \leqslant \zeta(a-)$. But since ζ does not decrease, $\zeta(a) = \zeta(a-)$.

Suppose that $\zeta(a-) < s < \zeta(a+) \leqslant \infty$. For $\alpha > a$, we cannot consider a solution Y_α as above since $\zeta(\alpha) = \infty$ is possible. However, for every $\alpha > a$, $\zeta(\alpha) \geqslant \zeta(a+) > s$; therefore we can proceed as in the proof of Theorem 3. The boundary conditions $y^{(i)}(\alpha) = \sigma_i$, $\sigma_i\sigma_{i+1} > 0$ and $y^{(i)}(s) = \sigma_i$, $\sigma_i\sigma_{i+1} < 0$ define a solution y_α such that $\sigma_i y_\alpha^{(i)} > 0$, $i = 0, \ldots, n-1$, $\alpha \leqslant x \leqslant s$. As $\alpha \to a+$, we normalize $\{y_\alpha\}$ so that $\Sigma|K_\alpha y_\alpha^{(i)}(s)| = 1$ and choose a subsequence which converges to a nontrivial solution y. y satisfies $\sigma_i y^{(i)} \geqslant 0$, $i = 0, \ldots, n-1$ on $[a, s]$, $|y^{(i)}(a)| = m$, $\sigma_i\sigma_{i+1} > 0$ and $|y^{(i)}(s)| = m$, $\sigma_i\sigma_{i+1} < 0$ for certain $m \geqslant 0$. Since $s > \zeta(a-) = \zeta(a)$, we have, as in the proof of Theorem 3, $m = 0$, i.e., y satisfies boundary conditions (4). But we assumed that either $\sigma_{n-1}\sigma_n > 0$ and $\Sigma_{i=0}^{q-1}|p_i| \not\equiv 0$ on $(a, a+\varepsilon)$ or $\sigma_{n-1}\sigma_n < 0$ and $\Sigma_{i=0}^{q-1}|p_i| \not\equiv 0$ on $(\zeta(a), \zeta(a)+\varepsilon)$. Therefore as in the proof of Lemma 1,

$$\sigma_i y^{(i)} > 0, \qquad i = 0, \ldots, n-1, a < x < \min\{\zeta(a) + \varepsilon, s\}.$$

On the other hand, there is a solution $\bar{y}$ which satisfies (14) at a and at $\zeta(a)$. Now we prove that y and $\bar{y}$ are incompatible as we have done this for (29) and (30). This contradiction proves that $\zeta(a-)=\zeta(a+)$.

THEOREM 7. *Under the assumption of Theorem* 6, $\zeta(a)$ *depends continuously on* $p_0, \ldots, p_{n-1}$.

PROOF. Let $\zeta(a)$ be the focal point for (1) and let $\bar{\zeta}(a)$ be the focal point for

$$y^{(n)}+\sum_{i=0}^{n-1}\bar{p}_i(x)y=0, \qquad \sigma_i\bar{p}_i \geqslant 0. \tag{33}$$

Given $\varepsilon>0$, we shall prove that $\zeta(a)+\varepsilon>\bar{\zeta}(a)>\zeta(a)-\varepsilon$, provided that $|p_i(x)-\bar{p}_i(x)|<\delta$ on $[\zeta(a), \zeta(a)+1]$, $i=0, \ldots, n-1$, $\delta=\delta(\varepsilon)$.

According to Theorem 1, there exists a solution y of (1) such that $\sigma_i y^{(i)}>0$, $i=0, \ldots, n-1$, $a \leqslant x \leqslant \zeta(a)-\varepsilon$. By the continuous dependence of a solution of an initial value problem on the coefficients of the equation, also (33) has a solution which satisfies similar inequalities on $[a, \zeta(a)-\varepsilon]$, provided that $|p_i-\bar{p}_i|<\delta$ on $[a, \zeta(a)]$. This proves that $\bar{\zeta}(a)>\zeta(a)-\varepsilon$.

In order to prove that $\zeta(a)+\varepsilon>\bar{\zeta}(a)$, it is not sufficient to exchange the roles of (1) and (33) since the above δ depends not only on $p_0, \ldots, p_{n-1}$ but, what is worse, on y. Suppose on the contrary that there exists $\varepsilon_0>0$ and a sequence of equations $y^{(n)}+\sum_{i=0}^{n-1}p_i^j y^{(i)}=0$, $j=1, 2, \ldots,$ such that for $i=0, \ldots, n-1$, $p_i^j \rightarrow p_i$ uniformly on $[a, \zeta(a)+1]$ as $j \rightarrow \infty$ and none of these equations has a focal point on $[a, \zeta(a)+\varepsilon_0]$. Now we continue the proof as that of Theorem 3. Denote $s=\zeta(a)+\varepsilon_0$ and let y_j be the solution of the jth equation which is defined as in (16). Since the jth equation has no focal point on $[a, s]$ we have $\sigma_i y_j^{(i)}>0$, $i=0, \ldots, n-1$, $a \leqslant x \leqslant s$. We normalize $\{y_j\}$ so that $\sum_{i=0}^{n-1}|K_j y_j^{(i)}(a)|=1$ and choose a subsequence $\{K_{j_l} y_{j_l}\}$ such that the sequence of vectors $(K_{j_l} y_{j_l}(a), \ldots, K_{j_l} y_{j_l}^{(n-1)}(a))$ converges. By a standard theorem [2, p. 17], $K_{j_l} y_{j_l}$ converge uniformly on $[a, s]$ to a nontrivial solution of (1). Now the proof is completed as that of Theorem 6.

4. The study of focal points in [5] is based on the existence of a minimal positive eigenvalue of (5). As an application we show how the above properties of the focal points can be used for a corresponding eigenvalue problem.

THEOREM 8. *If* $\sum_{i=0}^{q-1}|p_i(x)| \not\equiv 0$ *in* $[a, b]$, *then the problem*

$$y^{(n)}+\lambda \sum_{i=0}^{n-1} p_i y^{(i)}=0, \tag{34}$$

$$\begin{aligned} y^{(i)}(a)&=0, \quad \sigma_i\sigma_{i+1}>0, \\ y^{(i)}(b)&=0, \quad \sigma_i\sigma_{i+1}<0, \end{aligned} \tag{35}$$

has a least positive eigenvalue.

PROOF. For every positive λ, we denote the focal point for equation (34) by $\zeta(a, \lambda)$, $\zeta(a, \lambda) \leqslant \infty$. We shall show that $\zeta(a, \lambda) < b$ as $\lambda \to \infty$ and $b < \zeta(a, \lambda) \leqslant \infty$ as $\lambda \to 0$. According to example (ii) $\zeta(a, \lambda)$ is not necessarily a continuous function of λ and we do not have the intermediate value property. Nevertheless, we shall show that for a certain $\lambda = \lambda_0$, a solution of (34) satisfies (35) even if $\zeta(a, \lambda_0) < b$.

For $\lambda = 0$ we define a solution y of $y^{(n)} = 0$ by setting $y^{(n-1)} \equiv \sigma_{n-1}$, $y^{(i)}(a) = \sigma_i$ for $\sigma_i\sigma_{i+1} > 0$ and $y^{(i)}(b) = \sigma_i$ for $\sigma_i\sigma_{i+1} < 0$, $i = 0, \ldots, n-2$. y is easily obtained by repeated integrations between the appropriate endpoints and $\sigma_i y^{(i)} > 0$ on $[a, b]$, $i = 0, \ldots, n-1$. By the continuous dependence of a solution of an initial value problem on the coefficients of the equation, we have for sufficiently small values of λ a solution of (34) which satisfies similar inequalities. Consequently, $\zeta(a, \lambda) > b$.

To prove that $\zeta(a, \lambda) < b$ as $\lambda \to \infty$, suppose on the contrary that $b \leqslant \zeta(a, \lambda) \leqslant \infty$ for arbitrarily large values of λ. By our assumption, there is r, $0 \leqslant r \leqslant q-1$, such that $\sigma_r p_r(x) \geqslant m > 0$ on $[\alpha, \beta] \subset [a, b]$. By Theorems 4 and 5, the focal point $\bar{\zeta}$ for the equation

$$y^{(n)} + \lambda\sigma_r m y^{(r)} = 0 \tag{36}$$

which corresponds to the boundary conditions (4), satisfies $\bar{\zeta}(\alpha, \lambda) \geqslant \beta$ and (36) has for every λ a solution y such that $\sigma_i y^{(i)} > 0$ on (α, β). But this is impossible. For, $u_\lambda(x) = y^{(r)}(\alpha + (\lambda m)^{1/(n-r)}(x - \alpha))$ is a solution of $u^{(n-r)} + \sigma_r u = 0$ such that $\sigma_{i+r} u_\lambda^{(i)} > 0$, $i = 0, \ldots, n-r$, on $(\alpha, \alpha + (\lambda m)^{1/(n-r)}(\beta - \alpha))$ and as $\lambda \to \infty$ we obtain a nontrivial solution u such that $\sigma_{i+r} u^{(i)} > 0$ on (α, ∞). But this equation may have solutions which do vanish on a half line only of the form e^{-x} or $e^{x}(1 + O(1))$, according to the sign of σ_r and the parity of $n - r$. Either of these solutions corresponds to the case $\sigma_r\sigma_{r+1} = \cdots = \sigma_{n-1}\sigma_n$, in contrary to the definition of r.

Let λ_0 be the infimum of the values of λ such that $\zeta(a, \lambda) \leqslant b$. By the above considerations, $0 < \lambda_0 < \infty$. For $\lambda > \lambda_0$, $\zeta(a, \lambda) \leqslant b$ and there is a solution y_λ of (34) which satisfies boundary conditions (4) at a and at $\zeta(a, \lambda)$. When $\lambda \to \lambda_0 +$, we obtain a nontrivial solution of

$$y^{(n)} + \lambda_0 \sum_{i=0}^{n-1} p_i y^{(i)} = 0 \tag{37}$$

which satisfies the same boundary conditions at a and at $\zeta(a, \lambda_0 +)$; thus $\zeta(a, \lambda_0) \leqslant \zeta(a, \lambda_0 +)$. Since $\zeta(a, \lambda)$ is a nonincreasing function of λ, we have $\zeta(a, \lambda_0) = \zeta(a, \lambda_0 +)$. If $\zeta(a, \lambda_0) = b$, λ_0 is the required eigenvalue.

Suppose that $\zeta(a, \lambda_0) < b$ and recall that $b < \zeta(a, \lambda) \leqslant \infty$ for $\lambda < \lambda_0$. Now we complete the proof as the proof of Theorem 3. The solution y_λ which is defined by $y^{(i)}(a) = \sigma_i$, $\sigma_i\sigma_{i+1} > 0$ and $y^{(i)}(b) = \sigma_i$, $\sigma_i\sigma_{i+1} < 0$, satisfies $\sigma_i y_\lambda^{(i)} > 0$, $i = 0, \ldots, n-1$ on $[a, b]$. After suitable normalization, we choose a subsequence which converges as $\lambda \to \lambda_0 -$ to a nontrivial solution y of (37) such that $\sigma_i y^{(i)} \geq 0$, $i = 0, \ldots, n-1$, $a \leq x \leq b$, $|y^{(i)}(a)| = m$, $\sigma_i \sigma_{i+1} > 0$ and $|y^{(i)}(b)| = m$, $\sigma_i \sigma_{i+1} < 0$, for certain $m \geq 0$. It is seen that $m \neq 0$ is impossible since $\zeta(a, \lambda_0) < b$, therefore $m = 0$. Thus y satisfies (35) and λ_0 is the least positive eigenvalue.

Note that in [5], an assumption similar to our Assumption I is used to prove the μ_0-positivity of the integral operator which corresponds to (5), though it is not explicitly stated.

Finally, note that the condition $\sum_{i=0}^{q-1} |p_i(x)| \not\equiv 0$ is necessary. If $p_0 \equiv \cdots \equiv p_{q-1} \equiv 0$ then $u = y^{(q)}$ is a solution of $u^{(n-q)} + \lambda \sum_q^{n-1} p_i u^{(i-q)} = 0$ which satisfies the initial value conditions $u = \cdots = u^{(n-q-1)} = 0$ at a or at b. Thus $y^{(q)} \equiv 0$ and it follows that $y \equiv 0$.

If Assumption I holds for the interval $[a, b]$, the eigenfunction y satisfies $\sigma_i y^{(i)} > 0$ on (a, b) and it is easy to see that $\zeta(a, \lambda_0) = b$. If Assumption I holds also for every interval sufficiently close to $[a, b]$, then the eigenvalue is a strictly decreasing function of the basic interval. Indeed, $b = \zeta(a, \lambda)$ is a strictly decreasing, continuous function of λ and the inverse function $\lambda(a, b)$ has similar properties as a function of b. The roles of a and b can be exchanged by the transformation $\bar{x} = -x$, $\bar{y}(\bar{x}) = y(x)$, $\bar{p}_i(\bar{x}) = (-1)^i p_i(x)$.

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