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Green's Functions for a Non-disconjugate Differential Operator

URI ELIAS

Department of Mathematics, Technion-Israel Institute of Technology, Haifa, Israel

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1. INTRODUCTION

The objective of this paper is to study the Green's function for a two-term, non-disconjugate differential operator and a family of boundary value conditions.

An nth order, linear differential operator

$$T[y] \equiv q_0 y^{(n)} + q_1 y^{(n-1)} + \cdots + q_n y$$

is said to be *disconjugate* on the interval I if no solution $y \not\equiv 0$ of $T[y] = 0$ has n zeros, counting multiplicities, on I. Green's functions associated with disconjugate operators and various boundary value conditions were studied from several points of view. For example, the sign of Green's function for a multipoint boundary value problem is well known [2]. Inequalities and monotony properties of Green's function were derived in [1, 6]. For the $(k, n-k)$-*conjugate point boundary value problem*,

$$\begin{aligned} y^{(i)}(a) &= 0, \qquad i = 0,\ldots, k-1, \\ y^{(j)}(b) &= 0, \qquad j = 0,\ldots, n-k-1,\ a < b, \end{aligned} \tag{1.1}$$

and other boundary value problems, not only is the sign of $G(x, s)$ known, but also its total positivity properties [7]. For the most simple disconjugate operator, $T[y] = y^{(n)}$, and boundary conditions (1.1) as well as others, the Green's function is explicitly known [9, 10]. On the other hand, less is known about Green's functions for non-disconjugate operators. Some results in this direction are given in [11, 12].

Peterson studied in [13] the Green's function $G(x, s)$ for the two-term operator

$$T[y] \equiv y^{(n)} + py \tag{1.2}$$

and the $(k, n-k)$-*focal point boundary conditions*

$$\begin{aligned} y^{(i)}(a) &= 0, \qquad i = 0,\ldots, k-1, \\ y^{(j)}(b) &= 0, \qquad j = k,\ldots, n-1,\ a < b, \end{aligned} \tag{1.3}$$

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where $p > 0$ or $p < 0$. It was assumed in [13] that $T[y] = y^{(n)} + py$ is *disfocal* in $[a, b]$, i.e., there is no solution $y \not\equiv 0$ of

$$y^{(n)} + py = 0 \tag{1.4}$$

such that each of the derivatives $y, y', ..., y^{(n-1)}$ has a zero in $[a, b]$. Using disfocality, which is easily seen to imply disconjugacy, it was proved in [13] that

$$(-1)^{n-k}\, G^{(i)}(x, s) > 0, \qquad i = 0,..., k-1$$

for $a < x, s < b$. Our aim is to extend and generalize this result under a weaker assumption, which does not imply disconjugacy.

Known properties of Eq. (1.4) and boundary conditions (1.1) and (1.3) suggest studying the above boundary value problems separately for $p(x) > 0$ and $p(x) < 0$. It is known [8] that if $n - k$ is even and $p > 0$ or if $n - k$ is odd and $p < 0$ then no nontrivial solution of (1.4) satisfies (1.1) or (1.3) for any $a < b$, independently of the magnitude of $|p(x)|$. In this case $G(x, s)$ trivially exists for every a, b. It can be shown that in this case $G^{(i)}(x, s)$, $i = k,..., n-1$ are not one-signed and the results of [13] are, from a certain point of view, the best possible. Here we shall consider (1.4) only for $p(x)$ of the opposite sign, namely,

$$(-1)^{n-k}\, p(x) < 0.$$

In this case the results of [13] can be generalized under weaker assumptions. In our study (1.2) will be only assumed to be $(k, n-k)$-*disfocal* in $[a, b]$, that is, no solution $y \not\equiv 0$ of (1.4) satisfies

$$\begin{aligned} y^{(i)}(\alpha) &= 0, \qquad i = 0,..., k-1, \\ y^{(j)}(\beta) &= 0, \qquad j = k,..., n-1, \end{aligned}$$

for any $\alpha, \beta \in [a, b]$, $\alpha < \beta$. It will be proved, then, that the Green's function for the $(k, n-k)$-focal point problem satisfies

$$(-1)^{n-k+(q-k)_+ + (r-n+k)_+} \frac{\partial^{q+r}}{\partial x^q\, \partial s^r}\, G(x, s) > 0, \qquad q \cdot r = 0,..., n-1, \tag{1.5}$$

for $a < x, s < b$. Here $t_+ \equiv \max\{t, 0\}$.

Our assumption is weaker than that of [13] since it is clear that $(k, n-k)$-disfocality (for a fixed k) does not imply, in general, disfocality. Moreover, $(k, n-k)$-disfocality doesn't imply disconjugacy (see Example 4 of [4]), and hence none of our results can be deduced from the intensively studied properties of disconjugate operators.

If (1.2) is $(k, n-k)$-disfocal on $[0, \infty)$, we consider the Green's function corresponding to (1.2), (1.3) as a function of the endpoint b. We shall prove that as $b \to \infty$,

$$G(x, s, b) \to G_\infty(x, s).$$

Here, the convergence is monotone and the limit function G_∞, too, satisfies (1.5) for $a < x, s < \infty$. G_∞ may be used to link the solutions of (1.4) with the solutions of the forced equation

$$y^{(n)} + py = f. \tag{1.6}$$

For example, it will be proved that if (1.3) has a solution satisfying

$$\begin{aligned} y^{(i)} &> 0, \qquad i = 0,\ldots, k-1, \\ (-1)^{j-k} y^{(j)} &> 0, \qquad j = k,\ldots, n-1, \qquad a \leqslant x < \infty \end{aligned} \tag{1.7}$$

and if

$$(-1)^{n-k} f \geqslant 0,$$

$$\int^\infty x^{n-k} \, | f | < \infty,$$

then the forced equation (1.6), too, has a solution satisfying (1.7).

In Sections 3 and 4 we present a large family of boundary value problems with different boundary conditions at $x = b$, for each of which the Green's function exists (provided (1.3) is $(k, n-k)$-disfocal) and satisfies

$$(-1)^{n-k} G(x, s) > 0, \qquad a < x, s < b.$$

More exactly, for each of these Green's functions

$$0 < (-1)^{n-k} G_{\mathrm{conj}}(x, s) \leqslant (-1)^{n-k} G(x, s) \leqslant (-1)^{n-k} G_{\mathrm{foc}}(x, s),$$

where G_{foc} and G_{conj} are Green's functions for the $(k, n-k)$-focal point and the $(k, n-k)$-conjugate point problems, respectively. Moreover we prove quite surprisingly that for each of these Green's functions

$$G(x, s, b) \to G_\infty(x, s) \qquad \text{as} \quad b \to \infty,$$

where G_∞ is the same function which was defined above. Note that the common limit G_∞ satisfies (1.5) for $a < x, s < \infty$, while the only Green's function $G(x, s, b)$ which satisfies these inequalities for $a < x$, $s < b$, is $G_{\mathrm{foc}}(x, s, b)$. This property is another reason why the $(k, n-k)$-focal point problem arises naturally in the study of oscillation properties of (1.4).

2. Green's Function of the Focal Point Problem

For simplicity we introduced our results for Eq. (1.4). In fact, we can study without any additional difficulty, the more general equation

$$L^{(n)}y + py = 0, \tag{2.1}$$

where

$$L^{(n)}y = q_0 y^{(n)}(+ q_1 y^{(n-1)} + \cdots + q_n y, \qquad q_0 > 0,$$

is disconjugate and $p(x)$ is of constant sign. According to the known result of Polya [2], the disconjugate operator $L^{(n)}y$ can be written on a compact interval as

$$L^{(n)}y = \rho_n(\rho_{n-1} \cdots (\rho_1(\rho_0 y)')' \cdots)',$$

where $\rho_i > 0$, $\rho_i \in C^{(n-i)}$, $i = 0,..., n$. We put

$$L^{(0)}y = \rho_0 y, \qquad L^{(i)}y = \rho_i(L^{(i-1)}y)', \qquad i = 1,..., n,$$

where $L^{(0)}y,..., L^{(n)}y$ are called the quasi-derivatives of y. If $q_i \in C^{(n-i)}$, the formal adjoint of $L^{(n)}$ is $(-1)^n M^{(n)}$, where

$$M^{(n)}v = \rho_0(\rho_1 \cdots (\rho_{n-1}(\rho_n v)')' \cdots)'.$$

Here we put

$$M^{(0)}v = \rho_n v, \quad M^{(i)}v = \rho_{n-i}(M^{(i-1)}v)', \qquad i = 1,..., n.$$

The Green's function for the operator

$$T[y] \equiv L^{(n)}y + py \tag{2.2}$$

and n linear, homogeneous boundary value conditions is defined as follows:

(i) For each fixed s, $G(x, s)$ as a function of x satisfies (2.1) on $[a, s) \cup (s, b]$.

(ii) For each fixed s, $G(x, s)$ as a function of x satisfies the given boundary conditions.

(iii) $L_x^{(i)}G(x, s)$ is continuous on $a \leqslant x$, $s \leqslant b$ for $i = 0,..., n-2$.

(iv) $L_x^{(n-1)}G(s+0, s) - L_x^{(n-1)}G(s-0, s) = 1/\rho_n(s)$.

Here $L_x^{(i)}$ denotes quasi-differentiation with respect to x. The jump discontinuity (iv) corresponds to the jump of the ordinary $(n-1)$-th derivative by $1/q_0(s)$, where $L^{(n)}y = q_0 y^{(n)} + \cdots$. If $G(x, s)$ exists, the unique solution of

$$L^{(n)}y + py = f$$

satisfying the given boundary value conditions is given by

$$y(x) = \int_a^b G(x, s) f(s)\, ds.$$

Consider the $(k, n-k)$-focal point problem consisting of the operator (2.2) and the boundary value conditions

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0,\dots, k-1, \\ L^{(j)}y(b) &= 0, \qquad j = k,\dots, n-1. \end{aligned} \tag{2.3}$$

The Green's function for this problem exists if and only if no nontrivial solution of (2.1) satisfies (2.3). Therefore, it will be useful to define:

Definition. Equation (2.1) is said to be $(k, n-k)$-*disfocal* on an interval I if no solution $y \not\equiv 0$ of (2.1) satisfies

$$\begin{aligned} L^{(i)}y(\alpha) &= 0, \qquad i = 0,\dots, k-1, \\ L^{(j)}y(\beta) &= 0, \qquad j = k,\dots, n-1, \end{aligned}$$

for any $\alpha, \beta \in I$, $\alpha < \beta$.

As we mentioned in the Introduction, it is known that if $n-k$ is even and $p(x) > 0$ or if $n-k$ is odd and $p(x) < 0$, i.e., $(-1)^{n-k} p(x) > 0$, then (2.1) is $(k, n-k)$-disfocal on any interval [8]. We shall not discuss this case; all along in this paper it will be assumed, even without further remark, that

$$(-1)^{n-k} p(x) < 0. \tag{2.4}$$

In this case, $(k, n-k)$-disfocality is characterized by the following lemma:

Lemma 2.1. *Equation* (2.1), *where* $(-1)^{n-k} p < 0$, *is* $(k, n-k)$-*disfocal on the interval* $[a, b]$ *if and only if* (2.1) *has a solution* u *such that*

$$\begin{aligned} L^{(t)}u &> 0, \qquad t = 0,\dots, k-1, \\ (-1)^{t-k} L^{(t)}u &> 0, \qquad t = k,\dots, n-1, \end{aligned} \tag{2.5}$$

on $[a, b]$. *Equations* (2.5) *may be rewritten for convenience as*

$$(-1)^{(t-k)_+} L^{(t)}u > 0, \qquad t = 0,\dots, n-1. \tag{2.6}$$

The lemma was proved in [3] for $I = (a, b)$. The assertion follows immediately for $I = [a, b]$, since if (2.1) is $(k, n-k)$-disfocal on $[a, b]$, it is also $(k, n-k)$-disfocal on $(a-\epsilon, b+\epsilon)$ for sufficiently smalll, positive ϵ.

Out first result for the focal point problem is

THEOREM 2.2. *If* (2.1) *is* $(k, n-k)$*-disfocal on* $[a, b]$, *then the Green's function* $G(x, s)$ *for the* $(k, n-k)$*-focal point problem* (2.2), (2.3) *satisfies*

$$(-1)^{n-k+(q-k)_+ +(r-n+k)_+} L_x^{(q)} M_s^{(r)} G(x, s) > 0, \qquad q, r = 0,..., n-1, \quad (2.7)$$

for $a < x, s < b$. *In particular, for* (1.2), (1.3) *we have*

$$(-1)^{n-k+(q-k)_+ +(r-n+k)_+} \frac{\partial^{q+r}}{dx^q \, \partial s^r} G(x, s) > 0.$$

Proof. The Green's function can be written as $G(x, s) = -\sum_{t=1}^{n-k} u_t(x)\, v_t(s)$ for $x \in [a, s)$, $= \sum_{t=n-k+1}^{n} u_t(x)\, v_t(s)$ for $x \in (s, b]$, where $u_1 ,..., u_n$ are linearly independent solutions of (2.1), such that $u_1 ,..., u_{n-k}$ satisfy the k boundary value conditions of (2.3) at $x = a$ and $u_{n-k+1} ,..., u_n$ satisfy the $n-k$ conditions of (2.3) at $x = b$. It is known that $v_1 ,..., v_n$ are solutions of the adjoint equation.
We have

$$L_x^{(q)} G(s+0, s) - L_x^{(q)} G(s-0, s) = \delta_{q,n-1}/\rho_n(s), \qquad q = 0,..., n. \quad (2.8)$$

For $q = 0,..., n-1$, (2.8) follows by conditions (iii)–(iv) of the definition of Green's function. For $q = n$, (2.8) follows by the equation $L_x^{(n)} G(x, s) = -p(x)\, G(x, s)$, which is satisfied for $x \in [a, s) \cup (s, b]$. By using the above representation of $G(x, s)$, (2.8) is written as

$$\sum_{t=1}^{n} L^{(q)} u_t(s)\, v_t(s) = \delta_{q,n-1}/\rho_n(s), \qquad q = 0,..., n,$$

or in terms of the notation $M^{(0)} v = \rho_n v$,

$$\sum_{t=1}^{n} L^{(q)} u_t(s)\, M^{(0)} v_t(s) = \delta_{q,n-1}, \qquad q = 0,..., n. \quad (2.9)$$

Differentiating (2.9) and substituting

$$(L^{(q)} u)' = L^{(q+1)} u/\rho_{q+1}, \qquad (M^{(0)} v)' = M^{(1)} v/\rho_{n-1}$$

we have for $q = 0,..., n-1$

$$\begin{aligned}\sum_{t=1}^{n} L^{(q)} u_t \cdot M^{(1)} v_t &= -(\rho_{n-1}/\rho_{q+1}) \sum_{t=1}^{n} L^{(q+1)} u_t M^{(0)} v_t \\ &= -(\rho_{n-1}/\rho_{q+1})\, \delta_{q+1,n-1} = -\delta_{q,n-2}.\end{aligned}$$

This identity holds for $q = n$, too, since $L^{(n)} u_t = -p u_t$. Similarly, by more differentiations we get

$$\sum_{t=1}^{n} L^{(q)} u_t(s)\, M^{(r)} v_t(s) = (-1)^r \delta_{q,n-r-1}, \qquad q, r = 0,..., n. \quad (2.10)$$

We now take fixed r and s and consider the function $H(x, s) \equiv M_s^{(r)}G(x, s)$. For each fixed s, $H(x, s)$ is a solution of (2.1) on $[a, s) \cup (s, b]$ satisfying boundary conditions (2.3). Moreover, $H(x, s)$ satisfies by (2.10)

$$L_x^{(q)}H(s + 0, s) - L_x^{(q)}H(s - 0, s) = (-1)^r\delta_{q,n-r-1}, \qquad q = 0,..., n. \quad (2.11)$$

It may be easily proved that the above conditions determine $H(x, s)$ uniquely.

In order to establish (2.7), it suffices to show that

$$(-1)^{n-k+(q-k)_+ +(r-n+k)_+} L_x^{(q)}H(x, s) > 0, \qquad q = 0,..., n - 1 \quad (2.12)$$

for $a < x, s < b$. Suppose (2.12) is not true. Then we first show that there are $x_0 \in (a, b)$ and $0 \leqslant q_0 \leqslant n - 1$ such that

$$(-1)^{n-k+(q_0-k)_+ +(r-n+k)_+} L_x^{(q_0)}H(x_0, s) < 0. \quad (2.13)$$

Indeed, if $L_x^{(q)}H(x_1, s) = 0$ and $H(x, s) \not\equiv 0$ near x_1 then either $L_x^{(q)}H(x, s)$ or $L_x^{(q+1)}H(x, s)$ changes its sign at $x = x_1$ and one of them satisfies (2.13) for a suitably chosen x_0 near x_1. If $H(x, s) \equiv 0$ on some interval, then being a piecewise solution of a linear equation, $H(x, s) \equiv 0$ on $[a, s)$ or $(s, b]$. For example, let $H(x, s) \equiv 0$ on $[a, s)$. Then $w(x) = H(x, s)$ is a solution of (2.1) on $(s, b]$, by (2.11) we have

$$\begin{aligned} L^{(t)}w(s) &= 0, \qquad t = 0,..., n - 1, t \neq n - r - 1, \\ L^{(n-r-1)}w(s + 0) &= (-1)^r \end{aligned} \quad (2.14)$$

and by (2.3)

$$L^{(t)}w(b) = 0, \qquad t = k,..., n - 1.$$

If $n - k > 1$, this is impossible since $(n - 1) + (n - k) > n$ quasi-derivatives of a solution w of (2.1) cannot vanish at two points $x = s, b$ [8]. If $n - k = 1$, then for every $0 \leqslant q_0 \leqslant n - 1$, the required inequality (2.13) is

$$(-1)^{1+(r-1)_+} L^{(q_0)}w(x_0) < 0. \quad (2.15)$$

According to the initial values (2.14) at $x = s$, we have for some $\epsilon > 0$

$$(-1)^r L^{(t)}w > 0, \qquad t = 0,..., n - r - 1, \quad (2.16)$$

for $s < x < s + \epsilon$. But $(-1)^{n-k} p = -p < 0$, therefore $(-1)^r L^{(n)} w = -(-1)^r pw < 0$ and consequently

$$(-1)^r L^{(t)}w < 0, \qquad t = n - r,..., n, \quad (2.17)$$

for $s < x < s + \epsilon$. If $r = 0$ then (2.16) implies (2.15) with $s < x_0 < s + \epsilon$ and if $r \geqslant 1$ then (2.17) implies (2.15) with $q_0 \geqslant n - r$. In either case (2.13) holds for some q_0 and x_0.

Since it is given that (2.1) is $(k, n - k)$-disfocal in $[a, b]$, there exists, according to Lemma 2.1 a solution u such that

$$(-1)^{n-k+(q-k)_+ + (r-n+k)_+} L^{(q)} u > 0, \qquad q = 0,..., n - 1, \; a \leqslant x \leqslant b. \quad (2.18)$$

Note that $(-1)^{n-k+(r-n+k)_+}$ is a fixed number since r is fixed. We shall derive a contradiction from (2.13) and (2.18), thus proving (2.12).

Let λ_0 be the maximal value of λ such that

$$(-1)^{n-k+(q-k)_+ + (r-n+k)_+} L^{(q)}(u + \lambda H) \geqslant 0, \qquad q = 0,..., n - 1,$$

on $[a, b]$. By (2.18) it follows that $\lambda_0 > 0$ and by (2.13), $\lambda_0 < \infty$; for $\lambda = \lambda_0$ we have

$$(-1)^{n-k+(q-k)_+ + (r-n+k)_+} L^{(q)}(u + \lambda_0 H) \geqslant 0, \qquad q = 0,..., n, \qquad (2.19)$$

and equality holds at least for one q and one $x \in [a, b]$. The inequality for $q = n$ follows by the inequality for $q = 0$, Eq. (2.1) and (2.4).

For $q \neq n - r - 1$, $L^{(q)}(u + \lambda_0 H)$ is monotone on $[a, b]$ since it is continuous and according to (2.19) its derivative is of a constant sign. Furthermore, $L^{(n-r-1)}(u + \lambda_0 H)$, too, is monotone on $[a, b]$ in spite of the discontinuity of $L^{(n-r-1)} H$ at $x = s$. Indeed, we have by (2.19) (for $q = n - r$)

$$(-1)^r L^{(n-r)}(u + \lambda_0 H) \geqslant 0,$$

hence $(-1)^r L^{(n-r-1)}(u + \lambda_0 H)$ increases on $[a, s)$ and on $(s, b]$. Moreover, according to (2.11), this function increases by $\lambda_0 > 0$ as we pass from $x = s - 0$ to $x = s + 0$. Therefore $L^{(n-r-1)}(u + \lambda_0 H)$ too is monotone on $[a, b]$.

Examining the signs in (2.19) we see that for $q = 0,..., k - 1$, the functions $(-1)^{n-k+(q-k)_+ + (r-n+k)_+} L^{(q)}(u + \lambda_0 H)$ are increasing on $[a, b]$; by (2.18) and (2.3) we have for $x = a$

$$(-1)^{n-k+(q-k)_+ + (r-n+k)_+} L^{(q)}(u + \lambda_0 H)(a) = (-1)^{n-k+(q-k)_+ + (r-n+k)_+} L^{(q)} u(a) > 0.$$

Consequently these k functions are strictly positive on $[a, b]$. In a similar fashion we see that for $q = k,..., n - 1$ the functions $(-1)^{n-k+(q-k)_+ + (r-n+k)_+} L^{(q)} \times (u + \lambda_0 H)$ are decreasing on $[a, b]$. Moreover, by (2.18) and (2.3) they are positive at $x = b$, hence on all $[a, b]$. But this contradicts the definition of λ_0 and thus proves (2.12). Hence the theorem is proved.

Remark 2.3. The symmetry of (2.7) is a result of the relations between adjoint boundary value problems. The adjoint of (2.1) is

$$(-1)^n M^{(n)}v + pv = 0 \tag{2.20}$$

and the adjoint boundary value problem of (2.3) is the $(n-k, k)$-focal point problem for (2.20), namely,

$$\begin{aligned} M^{(i)}v(a) &= 0, \qquad i = 0,..., n-k-1, \\ M^{(j)}v(b) &= 0, \qquad j = n-k,..., n-1. \end{aligned} \tag{2.21}$$

It is known that (2.1) is $(k, n-k)$-disfocal if and only if (2.20) is $(n-k, k)$-disfocal [8]. Thus, if there is a solution u of (2.1) that satisfies (2.6), then there exists a solution v of (2.20) that satisfies

$$(-1)^{(t-n+k)_+} M^{(t)}v > 0, \qquad t = 0,..., n-1. \tag{2.22}$$

Of course, $G(s, x)$ is the Green's function of (2.20)–(2.21). This explains the invariance of (2.7) under the exchanges $x \leftrightarrow s$, $k \leftrightarrow n-k$, $L^{(n)} \leftrightarrow (-1)^n M^{(n)}$.

If (2.1) is $(k, n-k)$-disfocal on $[0, \infty)$, we consider the dependence of the Green's function for (2.2)–(2.3) on b, when $0 < a < b < \infty$. It is necessary to take $a > 0$, since even if $L^{(n)}$ is disconjugate on $[0, \infty)$, we have $L^{(n)}y = \rho_n(...(\rho_0 y)' ...)'$ with $\rho_i > 0$ only on $(0, \infty)$. Similarly, if (2.1) is $(k, n-k)$-disfocal on $[0, \infty)$, we may deduce only that (2.6) holds on $(0, \infty)$. In order to emphasize the dependence on b we denote the Green's function by $G(x, s, b)$.

THEOREM 2.4. *Let (2.1) be $(k, n-k)$-disfocal on $[0, \infty)$ and let $G(x, s, b)$ be the Green's function for the $(k, n-k)$-focal point problem (2.3). Then $|L_x^{(q)} M_s^{(r)} G(x, s, b)|$, $r, q = 0,..., n$, are strictly increasing functions of b for fixed x and s, $a < x, s < b$. If $a > 0$ then as $b \to \infty$*

$$L_x^{(q)} M_s^{(r)} G(x, s, b) \to L_x^{(q)} M_s^{(r)} G_\infty(x, s), \qquad q, r = 0,..., n,$$

uniformly on compact sets of $[a, \infty)$. The limit function $G_\infty(x, s)$ satisfies, obviously, inequalities similar to (2.7) for $a < x, s < \infty$.

Proof. In order to prove the monotone dependence of $L_x^{(q)} M_s^{(r)} G(x, s, b)$ *on* b, let r and s be fixed and consider the function

$$w(x) = (-1)^{n-k+(r-n+k)_+} [M_s^{(r)} G(x, s, c) - M_s^{(r)} G(x, s, b)]$$

on $[a, b]$, where $a < s < b < c$. Since the $(n-r-1)$-st quasiderivatives of $M_s^{(r)} G(x, s, b)$ and $M_s^{(r)} G(x, s, c)$ have equal jumps at $x = s$, $w(x)$ has n continuous derivatives and it is a solution of (2.1). At the endpoints $x = a, b$ we have by (2.3) and (2.7)

$$L^{(t)}w(a) = 0, \qquad t = 0,..., k-1, \tag{2.23}$$

$$(-1)^{t-k}L^{(t)}w(b) = (-1)^{n-k+(t-k)+(r-n+k)_+} L_x^{(t)}M_s^{(r)}G(b, s, c) > 0, \qquad t = k,..., n-1. \tag{2.24}$$

By using (2.23)–(2.24) we shall show that

$$(-1)^{(t-k)_+} L^{(t)}w > 0, \qquad a < x < b, \quad t = 0,..., n-1. \tag{2.25}$$

This, in turn, will imply by the definition of w that

$$| L_s^{(q)}M_s^{(r)}G(x, s, c)| > | L_x^{(q)}M_s^{(r)}G(x, s, b)|$$

for $c > b$ and the monotone dependence of $G(x, s, b)$ on b will follow.

Indeed, suppose (2.25) is not true. Then there exist $x_0 \in (a, b)$ and $0 \leqslant q_0 \leqslant n-1$ such that

$$(-1)^{(q_0-k)_+} L^{(t)}w(x_0) < 0. \tag{2.26}$$

Let u again be a solution of (2.1) obeying (2.6) on $[a, b]$ and let λ_0 be the maximal value of λ such that

$$(-1)^{(t-k)_+} L^{(t)}(u + \lambda w) \geqslant 0, \qquad a \leqslant x \leqslant b, \quad t = 0,..., n-1 \tag{2.27}$$

It follows by (2.26) and (2.6) that $0 < \lambda_0 < \infty$. We now get a contradiction as in the proof of (2.12). By using (2.27), (2.6) and (2.23) we show that the functions $L^{(t)}(u + \lambda_0 w)$, $t = 0,..., k-1$, are increasing and strictly positive on $[a, b]$. By using (2.27), (2.6) and (2.24), we deduce that the functions $(-1)^{t-k} L^{(t)} \times (u + \lambda_0 w)$, $t = k,..., n-1$, are decreasing and positive on $[a, b]$. This contradicts the definition of λ_0 and proves (2.25) and our first assertion about $G(x, s, b)$.

In order to prove that $G(x, s, b)$ converges pointwise to a limit function $G_\infty(x, s)$ as $b \to \infty$ it is sufficient to exhibit an upper bound, independent of b, for $| G(x, s, b)|$. Since $G(x, s, b)$ is a solution of a linear differential equation on $[a, s) \cup (s, b]$ (in fact, on $[0, s) \cup (s, \infty)$), it will follow that all its quasi-derivatives $L_x^{(q)}G(x, s, b)$, $q = 0,..., n$, converge uniformly on compact sets. Moreover, $L_x^{(q)}G(x, s, b)$ is, as a function of s, a solution of the adjoint equation (2.20) for $s \in (a, x) \cup (x, b)$. This will imply the uniform convergence on compact (x, s)-sets of each $L_x^{(q)}M_s^{(r)}G(x, s, b)$, $q, r = 0,..., n$ as $b \to \infty$.

Now we obtain the required upper bound, independent of b, for $(-1)^{n-k} \times G(x, s, b) > 0$. For convenience of writing we shall do this first for the operator (1.2) for which $L^{(i)}y \equiv y^{(i)}$ and the corresponding boundary value conditions are given in (1.3). First we write an integral equation for G. Integrating

$G^{(n)}(x, s, b) = -p(x)\, G(x, s, b)$ on $[x, b]$ and using the boundary value conditions (1.3) and the discontinuity jump of $G^{(n-1)}(x, s, b)$ at $x = s$, we have

$$-G^{(n-1)}(x, s, b) = \chi_{[a,s]}(x) + \int_x^b [-p(t)]\, G(t, s, b)\, dt.$$

Here $\chi_{(a,s]}$ is the characteristic function of $[a, s]$. Integrating $n - k - 1$ more times on $[x, b]$ and using (1.3), we have

$$\begin{aligned}(n-k-1)!(-1)^{n-k}G^{(k)}(x, s, b)\\ = (s-x)^{n-k-1}\chi_{[a,s]}(x) + \int_x^b (t-x)^{n-k-1}[-p(t)]\, G(t, s, b)\, dt.\end{aligned}$$

Finally we integrate k times on $[a, x]$ and use again (1.3):

$$\begin{aligned}(k-1)!(n-k-1)!(-1)^{n-k}G(x, s, b)\\ = \int_a^x (x-\tau)^{k-1}(s-\tau)^{n-k-1}\chi_{[a,s]}(\tau)\, d\tau\\ + \int_a^x (x-\tau)^{k-1}\left\{\int_\tau^b (t-\tau)^{n-k-1}[-p(t)]\, G(t, s, b)\, dt\right\} d\tau \qquad (2.28)\end{aligned}$$

Since (1.2) is disfocal on $[0, \infty)$ it has according to Lemma 2.1 and by letting $b \to \infty$ a solution u such that

$$(-1)^{n-k+(t-k)_+}L^{(t)}u > 0, \qquad t = 0,\ldots, n-1,$$

on $(0, \infty)$, hence on $[a, \infty)$. Now we write for u an integral inequality similar to (2.28). Integrating $u^{(n)} = -pu$ on $[x, b]$ and using $u^{(n-1)}(b) < 0$, we have

$$-u^{(n-1)}(x) > \int_x^b [-p(t)]\, u(t)\, dt.$$

As above, we integrate this inequality $n - k - 1$ times on $[x, b]$ and k times on $[a, x]$. Omitting $u^{(n-1)}(b),\ldots, u^{(k)}(b)$, $u^{(k-2)}(a),\ldots, u(a)$ whose signs are known (but not $u^{(k-1)}(a)$!), we get

$$\begin{aligned}(k-1)!(n-k-1)!(-1)^{n-k}u(x)\\ > (n-k-1)!(-1)^{n-k}u^{(k-1)}(a)(x-a)^{k-1}\\ + \int_a^x (x-\tau)^{k-1}\left\{\int_\tau^b (t-\tau)^{n-k-1}[-p(t)]\, u(t)\, dt\right\} d\tau. \qquad (2.29)\end{aligned}$$

Note that $(-1)^{n-k}\, G > 0$, $(-1)^{n-k}\, u > 0$ and $(-1)^{n-k}\, p < 0$, hence all the terms in (2.28) and (2.29) are positive. Let us multiply u by a constant K_s such that the first term on the right side of (2.29) will be greater than the corresponding term of (2.28):

$$K_s(n-k-1)!(-1)^{n-k}u^{(k-1)}(a)(x-a)^{k-1}$$
$$\geqslant \int_a^x (x-\tau)^{k-1}(s-\tau)^{n-k-1}\chi_{[a,s]}(\tau)\,d\tau. \tag{2.30}$$

Since the right-hand side of (2.30) is not greater than $\int_a^s (x-a)^{k-1} \times (s-\tau)^{n-k-1}\,d\tau = (x-a)^{k-1}(s-a)^{n-k}/(n-k)$, we can take $K_s = [(s-a)^{n-k}/(n-k)!]/|u^{(k-1)}(a)|$. For this K_s we prove that

$$|G(x, s, b)| < K_s\,|u(x)|.$$

The following argument is a standard successive iteration: Define a sequence $\{u_i(x)\}$ by

$$u_0(x) = K_s u(x),$$

$$(k-1)!(n-k-1)!(-1)^{n-k}u_{i+1}(x)$$
$$= \int_a^x (x-\tau)^{k-1}(s-\tau)^{n-k-1}\chi_{[a,s]}(\tau)\,d\tau$$
$$+ \int_a^x (x-\tau)^{k-1}\left\{\int_\tau^b (t-\tau)^{n-k-1}[-p(t)]\,u_i(t)\,dt\right\}d\tau.$$

By (2.28), (2.29) and (2.30) it is easily proved that $(-1)^{n-k}\,u_i > (-1)^{n-k} \times u_{i+1} > 0$, $i = 0, 1,\ldots$, and that the sequence $\{u_i\}$ converges to the unique solution $G(x, s, b)$ of (2.28). Therefore $(-1)^{n-k}\,G(x, s, b) \leqslant K_s(-1)^{n-k}\,u(x)$, i.e.,

$$0 < (-1)^{n-k}\,G(x, s, b) \leqslant [(-1)^{n-k}\,u(x)/|u^{(k-1)}(a)|]\cdot[(s-a)^{n-k}/(n-k)!] \tag{2.31}$$

and the proof of the theorem is completed.

When we study Eq. (2.1) instead of (1.3), we replace (2.28) and (2.29) by

$$(-1)^{n-k}G(x, s, b)$$

$$= \rho_0^{-1}(x)\int_a^x \rho_1^{-1}\int_a^{t_1}\cdots\int_a^{t_{k-1}} \rho_k^{-1}\chi_{[a,s]}\int_{t_k}^s \rho_{k+1}^{-1}\cdots\int_{t_{n-2}}^s \rho_{n-1}^{-1}\,dt_{n-1}\cdots dt_1\cdot\rho_n^{-1}(s)$$

$$+ \rho_0^{-1}(x)\int_a^x \rho_1^{-1}\int_a^{t_1}\cdots\int_a^{t_{k-1}} \rho_k^{-1}\int_{t_k}^b \rho_{k+1}^{-1}\cdots\int_{t_{n-1}}^b \rho_n^{-1}[-p]\,G(t_n, s, b)\,dt_n\cdots dt_1$$

where $\rho_i \equiv \rho_i(t_i)$ and

$$(-1)^{n-k}u(x) > (-1)^{n-k}L^{(k-1)}u(a) \cdot \rho_0^{-1}(x)\int_a^x \rho_1^{-1} \int_a^{t_1} \cdots \int_a^{t_{k-2}} \rho_{k-1}^{-1}\, dt_{k-1} \cdots dt_1$$

$$+ \rho_0^{-1}(x) \int_a^x \rho_1^{-1} \int_a^{t_1} \cdots \int_a^{t_{k-1}} \rho_k^{-1} \int_{t_k}^b \rho_{k+1}^{-1} \cdots \int_{t_{n-1}}^b \rho_n^{-1}[-p]u\, dt_n \cdots dt_1 .$$

Accordingly, we choose

$$K_s = [| L^{(k-1)}u(a) \mid \rho_n(s)]^{-1} \int_a^s \rho_k^{-1} \int_{t_k}^s \rho_{k+1}^{-1} \cdots \int_{t_{n-2}}^s \rho_{n-1}^{-1}\, dt_{n-1} \cdots dt_k .$$

By considering the iterated integral as a multiple integral of $\rho_k^{-1}(t_k) \cdots \rho_{n-1}^{-1}(t_{n-1})$ on the simplex $a \leqslant t_k \leqslant \cdots \leqslant t_{n-1} \leqslant s$, we see that it is equal to $\int_a^s \rho_{n-1}^{-1} \int_a^{t_{n-1}} \rho_{n-2}^{-1} \cdots \int_a^{t_{k+1}} \rho_k^{-1}\, dt_k \cdots dt_{n-1}$. Therefore, if we put

$$\psi_{n-k}(s) = \rho_n^{-1}(s) \int_a^s \rho_{n-1}^{-1} \int_a^{t_{n-1}} \rho_{n-2}^{-1} \cdots \int_a^{t_{k+1}} \rho_k^{-1}\, dt_k \cdots dt_{n-1}$$

and

$$\varphi_k(x) = \rho_0^{-1}(x) \int_a^x \rho_1^{-1} \int_a^{t_1} \rho_2^{-1} \cdots \int_a^{t_{k-1}} \rho_k^{-1}\, dt_k \cdots dt_1 ,$$

we get, in analogy woth (2.31), the following bounds for G_∞ :

COROLLARY 2.5. *For every solution u of* (2.1) *that satisfies*

$$(-1)^{(t-k)_+}L^{(t)}u > 0, \qquad a \leqslant x < \infty, \quad t = 0,..., n-1, \tag{2.6}$$

and for every solution v of (2.20) *that satisfies*

$$(-1)^{(t-n+k)_+}L^{(t)}v > 0, \qquad a \leqslant s < \infty, \quad t = 0,..., n-1, \tag{2.22}$$

we have

$$0 < (-1)^{n-k}\, G_\infty(x, s) \leqslant u(x)\, \psi_{n-k}(s)/L^{(k-1)}u(a) \tag{2.32}$$

and

$$0 < (-1)^{n-k}\, G_\infty(x, s) \leqslant v(s)\, \varphi_k(x)/M^{(n-k-1)}v(a). \tag{2.33}$$

The inequality (2.33) follows from (2.32) by the transformation $x \Leftrightarrow s$, $k \Leftrightarrow n-k$, $L^{(n)} \Leftrightarrow (-1)^n M^{(n)}$ according to Remark 2.3. Note that

$$u = O(\varphi_k), \qquad v = O(\psi_{n-k})$$

as $x \to \infty$. Indeed, $L^{(k)}\varphi_k(x) \equiv 1$ while $0 < L^{(k)}u(x) \leqslant L^{(k)}u(c)$ for $x \geqslant c > a$.

Hence it follows that $0 < u(x) \leqslant A\varphi_k(x)$ on $[0, \infty)$ where $A = \max\{L^{(t)}u(c)/L^{(t)}\varphi_k(c);\ t = 0,..., k\}$. Consequently, the bound (2.32) is better than (2.33) for $x \to \infty$ while the opposite is true for $x \to a$. On the other hand we cannot replace (2.32) and (2.33) by

$$(-1)^{n-k}\, G_\infty(x, s) \leqslant Ku(x)\, v(s)$$

for *every* pair of solutions u, v of (2.1) and (2.20) which satisfy (2.6) and (2.22) respectively. This can be seen from Example 3.10.

The next result describes the effect of a small forcing term on the solutions satisfying (2.6).

THEOREM 2.6. *Let* (2.1) *have a solution u satisfying* (2.6) *on* $[a, \infty)$ (*that is* (2.1) *is* $(k, n-k)$-*disfocal on* $[a-\epsilon, \infty)$) *and let v be any solution of the adjoint equation* (2.20) *which satisfies* (2.22). *If*

$$(-1)^{n-k} f \geqslant 0$$

and

$$\int^\infty v(s)\, |f(s)|\, ds < \infty, \tag{2.34}$$

then the forced (*nonhomogeneous*) *equation*

$$L^{(n)}y + py = f,$$

too, has a solution y satisfying

$$(-1)^{(t-k)_+} L^{(t)} y > 0, \qquad a \leqslant x < \infty, \quad t = 0,..., n-1. \tag{2.35}$$

Indeed, the solution $y(x) = \int_a^\infty G_\infty(x, s) f(s)\, ds$, which exists by (2.33) and (2.34), satisfies (2.35) on (a, ∞) and $y + u$ satisfies it on $[a, \infty)$. Observe that by the previous remarks, (2.34) may be replaced by the rougher condition

$$\int^\infty \psi_{n-k}(s)\, |f(s)|\, ds < \infty. \tag{2.36}$$

For Eq. (1.4), the integral (2.36) reads $\int^\infty s^{n-k}\, |f(s)|\, ds < \infty$.

3. GREEN'S FUNCTIONS FOR OTHER PROBLEMS

In the previous section we studied Green's function associated with the $(k, n-k)$-focal point problem (2.2)–(2.3), which will be denoted now by $G_{\mathrm{foc}}(x, s)$. We proved that if Eq. (2.1) is $(k, n-k)$-disfocal, then $(-1)^{n-k} \times$

$G_{foc}(x, s) > 0$ for $a < x, s < b$ and $G_{foc}(x, s, b) \to G_\infty(x, s)$ as $b \to \infty$. In this section we pose the following question:

Given the $(k, n - k)$*-disfocal equation* (2.1) *and the boundary conditions*

$$L^{(i)}y(a) = 0, \qquad i = 0,..., k - 1, \tag{3.1}$$

for which other boundary conditions at b, *does the Green's function exist and satisfy*

$$(-1)^{n-k} G(x, s) > 0, \qquad a < x, s < b, \tag{3.2}$$

$$G(x, s, b) \to G_\infty(x, s) \qquad \textit{as} \quad b \to \infty \ ? \tag{3.3}$$

We do not attempt to alter the boundary conditions (3.1). First, they are most simple since (3.1) is equivalent to $y^{(i)}(a) = 0$, $i = 0,..., k - 1$. Moreover, (3.1) is essential for the existence of $G(x, s)$. For example, the boundary conditions of Example 3.10 admit Green's functions for $1 < b < \infty$ but for the boundary conditions $y'(1) = y(b) = 0$ the Green's function does not exist if $b = e^2$.

To obtain boundary conditions for which (3.2) and (3.3) hold, we point out the features of Eq. (2.1) and boundary conditions (2.3) which may be generalized. By comparing the boundary conditions (2.3) and inequalities (2.7) (with $r = 0$), we observe the following: The first k quasi-derivatives (with respect to x) of $G_{foc}(x, s)$ vanish at $x = a$ and the first k pairs of its consecutive quasi-derivatives have the same sign; the next $n - k$ quasi-derivatives vanish at $x = b$ and the next $n - k$ pairs of consecutive quasi-derivatives have opposite signs. We shall see that the natural generalization of the focal point pıoblem is achieved if we choose, in addition to (3.1), a set of boundary conditions at b which assures that $n - k$ pairs of consecutive quasi-derivatives have opposite signs near b. Namely, there are integers $0 \leqslant j_1 < \cdots < j_{n-k} \leqslant n - 1$ such that

$$L^{(j)}y \cdot L^{(j+1)}y < 0, \qquad j = j_1 ,..., j_{n-k} , \ b - \epsilon < x < b. \tag{3.4}$$

Under some slight restrictions it will be proved that for such boundary value problems the Green's function exists and satisfies (3.2)–(3.3).

In order to explain the conditions (3.4) and to prepare tools for the forthcoming discussion, we introduce first some results of [3] about the signs of the quasi-derivatives of solutions. Let $S(c_0 ,..., c_n)$ denote the number of sign changes in the sequence $c_0 ,..., c_n$, whose elements are non-zero real numbers. Moreover, for a solution y of (2.1) on (a, b) we write

$$\begin{aligned} S(y, a+) &= \lim_{x \uparrow a} S(L^{(0)}y(x), -L^{(1)}y(x),..., (-1)^n L^{(n)}y(x)), \\ S(y, b-) &= \lim_{x \uparrow b} S(L^{(0)}y(x), L^{(1)}y(x),..., L^{(n)}y(x)). \end{aligned} \tag{3.5}$$

Since $p \neq 0$, there are neighbourhoods $U_a^+ = \{x \mid a < x < a + \epsilon\}$, $U_b^- = \{x \mid b - \epsilon < x < b\}$ on which $L^{(t)}y \neq 0$, $t = 0,..., n$, and thus the limits (3.5) exist.

The importance of the signs of the quasi-derivatives for the study of boundary value problems is a result of the following simple facts which we state, for convenience, as a lemma:

LEMMA 3.1. (i) *If*

$$L^{(t)}y \cdot L^{(t+1)}y > 0, \qquad x \in U_a^+, \tag{3.6}$$

then the function $L^{(t+1)}y$ changes sign between a and the nearest zero of $L^{(t)}y$ on the right side of a, if such a zero exists. If

$$L^{(t)}y \cdot L^{(t+1)}y < 0, \qquad x \in U_b^-, \tag{3.7}$$

the function $L^{(t+1)}y$ changes sign between b and the nearest zero of $L^{(t)}y$ on the left side of b.

(ii) *If $L^{(t)}y(a) = 0$ then* (3.6) *holds for some U_a^+ and if $L^{(t)}y(b) = 0$ then* (3.7) holds for some U_b^-. *Hence,* (i) *may be considered as a genralization of Rolle's theorem.*

By Lemma 3.1(ii), we see that $S(y, a+)$, $S(y, b-)$ are not smaller, respectively, than the number of the quasi-derivatives of y up to $L^{(n-1)}y$ which vanish at a and b. Moreover, since $L^{(n)}y = -(p/\rho_0)L^{(0)}y$ and since we assume that $(-1)^{n-k}p < 0$, it follows by the definition (3.5) that

$$S(y, a+) \equiv k (\text{mod } 2), \qquad S(y, b-) \equiv n - k (\text{mod } 2). \tag{3.8}$$

For a solution u satisfying (2.6), for example, we have

$$S(u, a+) = k, \qquad S(u, b-) = n - k.$$

Let $x_1, ..., x_r$ be the zeros of $L^{(0)}y, ..., L^{(n-1)}y$ in (a, b) and let the number of the consecutive quasi-derivatives which vanish at x_i be denoted by $n(x_i)$. (In [3] we counted the multiplicities of the zeros in a more complicated way to get sharper results. Here, however, this is not necessary.) We put $\langle q \rangle$ for the greatest even integer which is not greater than q. Then

LEMMA 3.2 [3, Lemma 1]. *Every solution y of* (2.1) *satisfies the condition*

$$N(y) \overset{\text{df}}{\equiv} S(y, a+) + \sum_{a<x_i<b} \langle n(x_i) \rangle + S(y, b-) \leqslant n. \tag{3.9}$$

If $N(y) = n$, then the function $L^{(t+1)}y$ changes its sign exactly once between consecutive zeros of $L^{(t)}y$ in $[a, b]$. In addition, $L^{(t+1)}y$ changes its sign between a and *the first zero of* $L^{(t)}y$ *in* $(a, b]$ *if and only if* (3.6) *holds on some* U_a^+; *in this case* $L^{(t+1)}y$ *changes its sign there exactly once. Similarly,* $L^{(t+1)}y$ *changes its sign between* b *and the last zero of* $L^{(t)}y$ *in* $[a, b)$ *if and only if* (3.7) *holds on some* U_b^- *and in this case there is a single sign change of* $L^{(t+1)}y$ *there, too.*

We outline the proof of Lemma 3.2. Consider the solution y and its zeros on $[a, b]$. We determine by using Rolle's theorem and Lemma 3.1, at least how many zeros, has each of the n derivatives

$$(L^{(0)}y)' = L^{(1)}y/\rho_1, \ldots, (L^{(n-1)}y)' = L^{(n)}y/\rho_n .$$

Finally we compare the zeros of $L^{(n)}y$ to those of $L^{(0)}y$ by the equation $L^{(n)}y = -(p/\rho_0)L^{(0)}y$ and (3.9) follows. $N(y) = n$ if and only if each quasi-derivative has no more sign changes than the minimal number which is necessary by Rolle's theorem and Lemma 3.1. This extablishes the description of the zeros in Lemma 3.2 when $N(y) = n$.

Let us return now to the Green's functions. First we show that if for a given boundary value problem Green's function exists, $(-1)^{n-k} G(x, s) \geqslant 0$, and $G(x, s) \not\equiv 0$ on any interval, then $G(x, s)$ satisfies Lemma 3.2 as a function of x, though it is a solution of (2.1) on $[a, s) \cup (s, b]$ only. Indeed, let us repeat the proof of Lemma 3.2 for $G(x, s)$. A difficulty arises only when the discontinuity point of $L^{(n-1} G$, $x = s$, is located between two zeros of $L^{(n-2)}G$. In this case Rolle's theorem does not apply and due to the jump of $L^{(n-1)}G$ by $1/\rho_n(s)$ at $x = s$, it is possible that

$$L^{(n-1)}G(s - 0, s) < 0 < L^{(n-1)}G(s + 0, s),$$

i.e., $L^{(n-1)}G$ does not admit the value zero between two zeros of $L^{(n-2)}G$. However, this "loss" of a zero of $L^{(n-1)}G$ does not disturb our counting of the zeros of $L^{(n)}G$. Applying $(-1)^{n-k} G \geqslant 0$ and $(-1)^{n-k} p < 0$ to Eq. (2.1) we see that $L^{(n)}G(x, s) \geqslant 0$, $\not\equiv 0$, hence

$$\begin{aligned} L^{(n-1)}G \cdot L^{(n)}G &> 0, \qquad x \in U_s^+, \\ L^{(n-1)}G \cdot L^{(n)}G &< 0, \qquad x \in U_s^-. \end{aligned} \tag{3.10}$$

If we now use Lemma 3.1 and (3.10) to determine how many zeros $L^{(n)}G$ has at least, we achieve the same result as if $L^{(n-1)}G$ would have vanished at $x = s$. Thus, Lemma 3.2 is proved for $G(x, s)$, too. (Note that we used only that $(-1)^{n-k} G > 0$ on some neighbourhood of the diagonal $x = s$). For example, the Green's function for the $(k, n - k)$-focal point problem satisfies

$$S(G_{\text{foc}}, a+) = k, \qquad S(G_{\text{foc}}, b-) = n - k.$$

Thus, we have seen that if $(-1)^{n-k} G > 0$ then $S(G, a+) + S(G, b-) \leqslant N(G) \leqslant n$. The boundary conditions (3.1) imply that $S(G, a+) \geqslant k$, therefore

$S(G, b-) \leqslant n - k$. This motivates our interest in boundary value conditions which together with (3.1) imply $S(y, b-) = n - k$, i.e.,

$$L^{(j)}y \cdot L^{(j+1)}y < 0, \qquad j = j_1, \dots, j_{n-k}, \quad x \in U_b^-, \tag{3.4}$$

holds. Wz do not claim that this is the only choice for which (3.2), (3.3) hold, but it seems to us to be the most interesting one.

Our first boundary value problem is (2.1),

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0, \dots, k-1, \\ [(1-\mu_j)L^{(j)}y + \mu_j L^{(j+1)}y](b) &= 0, \qquad j = j_1, \dots, j_{n-k}, \end{aligned} \tag{3.11}$$

where $0 \leqslant j_1 < \cdots < j_{n-k} \leqslant n-1$, $0 \leqslant \mu_{j_1}, \dots, \mu_{j_{n-k}} < 1$ and $\mu_{n-1} = 0$ if $j_{n-k} = n-1$. The restriction

$$j_{n-k} = n - 1 \Rightarrow \mu_{n-1} = 0 \tag{3.12}$$

is quite natural since one does not expect that a boundary value condition for an nth order differential equation will involve the nth derivative, say $[(1-\mu_{n-1})L^{(n-1)}y + \mu_{n-1}L^{(n)}y](b) = 0$. Hence we let $L^{(n-1)}y \cdot L^{(n)}y < 0$ on U_b^- only if $L^{(n-1)}y(b) = 0$. Later we shall see that (3.12) is not only a reasonable restriction but it is also essential for the existence of the Green's function.

If $j_q, j_{q+1}, \dots, j_{q+r}$ are consecutive integers, say $j_q, j_q + 1, \dots, j_q + r$, and $\mu_{j_q+r} = 0$ then the conditions

$$\begin{gathered} [(1-\mu_j)L^{(j)}y + \mu_j L^{(j+1)}y](b) = 0, \qquad j = j_q, j_q + 1, \dots, j_q + r - 1, \\ L^{(j_q+r)}y(b) = 0 \end{gathered}$$

are equivalent to

$$L^{(j)}y(b) = 0, \qquad j = j_q, j_q + 1, \dots, j_q + r.$$

If one wants to avoid several representations of the same problem, it may be assumed that

$$\begin{gathered} \{j_q, j_{q+1}, \dots, j_{q+r}\} = \{j_q, j_q + 1, \dots, j_q + r\} \text{ and } \mu_{j_{q+r}} = 0 \\ \Rightarrow \mu_{j_q} = \mu_{j_q+1} = \cdots = \mu_{j_q+r} = 0. \end{gathered} \tag{3.13}$$

The $(k, n-k)$-focal point problem belongs clearly to this type of boundary value problem. Another problem of particular interest of this type is the $(k, n-k)$-conjugate point problem

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0, \dots, k-1, \\ L^{(j)}y(b) &= 0, \qquad j = 0, \dots, n-k-1. \end{aligned} \tag{3.14}$$

The first result deals with the existence of Green's function for (3.11).

THEOREM 3.3. *Let* (2.1) *be* $(k, n-k)$*-disfocal on* $[a, b]$. *The boundary value problem* (2.2), (3.11), *where* $\mu_{n-1} = 0$ *if* $j_{n-k} = n-1$, *has a Green's function.*

Conversely, for any $0 \leqslant j_1 < \cdots < j_{n-k-1} \leqslant n-2$, $0 \leqslant \mu_{j_1}, \ldots, \mu_{j_{n-k-1}} < 1$, *there exists an* $(n-k)$*th boundary condition with* $j_{n-k} = n-1$ *and* $0 < \mu_{n-1} < 1$ *for which Green's function does not exist.*

We shall prove a slightly different version which does not use the boundary conditions (3.11) explicitly but only the signs of the quasi-derivatives:

Consider the conditions

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0, \ldots, k-1, \\ L^{(j)}y \cdot L^{(j+1)}y &< 0, \qquad j = j_1, \ldots, j_{n-k}, \quad x \in U_b^-, \end{aligned} \tag{3.15}$$

and suppose in addition that

$$\{j_{q+1-k}, \ldots, j_{n-k}\} = \{q, \ldots, n-1\} \Rightarrow L^{(q)}y(b) = \cdots = L^{(n-1)}y(b) = 0. \tag{3.16}$$

Then no solution of (2.1) *satisfies* (3.15) *and* (3.16).

Obviously, (3.11), (3.12) imply (3.15), (3.16) but not the converse: if $L^{(j)}y \cdot L^{(j+1)}y < 0$ on U_b^- but $L^{(j)}y(b) \neq 0$, $L^{(j+1)}y(b) = 0$, then this condition is included in (3.15) but cannot be written as in (3.11). This fact will be needed in the proof of Lemma 3.4.

Proof. If a solution y satisfies (3.15) then $S(y, a+) \geqslant k$, $S(y, b-) \geqslant n-k$ and according to Lemma 3.2,

$$S(y, a+) = k, \qquad S(y, b-) = n-k \qquad N(y) = n.$$

Let us determine the signs of the quasi-derivatives near a. By (3.15) we know that $L^{(t)}y \cdot L^{(t+1)}y > 0$ on U_a^+ for $t = 0, \ldots, k-1$. Since $S(y, a+) = k$, these are the only pairs of consecutive quasi-derivatives with the same sign on U_a^+, hence we have (if we assume that $y > 0$ on U_a^+)

$$(-1)^{(t-k)_+} L^{(t)}y > 0, \qquad t = 0, \ldots, n, \quad x \in U_a^+.$$

Let x_t be the maximal number in $(a, b]$ such that

$$(-1)^{(t-k)_+} L^{(t)}y > 0, \qquad a < x < x_t. \tag{3.17}$$

Here, of course, $x_n = x_0$.

Suppose now that y satisfies the additional restriction (3.16). We shall prove that for each t, $k \leqslant t \leqslant n-1$, $L^{(t)}y$ has a zero in $(a, b]$, i.e.,

$$L^{(t)}y(x_t) = 0, \qquad t = k, \dots, n-1. \tag{3.18}$$

(Note that for $t = 0, \dots, k-1, n$ it is possible that $x_t = b$ and $L^{(t)}y(b) \neq 0$.) Indeed, if $t \geqslant k$ and $\{j_{t+1-k}, \dots, j_{n-k}\} = \{t, \dots, n-1\}$, then $L^{(t)}y(b) = 0$ by (3.16). If $\{j_{t+1-k}, \dots, j_{n-k}\} \neq \{t, \dots, n-1\}$ then $j_{t+1-k} < t$ and consequently $j_1 \leqslant k-1$. Therefore $L^{(j_1)}y(a) = 0$ and since $L^{(j_1)}y \cdot L^{(j_1+1)}y < 0$ on U_b^-, we deduce by Lemma 3.1 that $L^{(j_1+1)}y$ changes sign on (a, b). Similarly the $t - j_1$ quasi-derivatives $L^{(j_1)}y, \dots, L^{(t-1)}y$, $j_1 < k \leqslant t$, or part of them satisfy the $t - j_1 + 1$ conditions

$$L^{(i)}y(a) = 0, \qquad i = j_1, \dots, k-1,$$
$$L^{(j)}y \cdot L^{(j+1)}y < 0, \qquad j = j_1, \dots, j_{t+1-k}, \quad x \in U_b^-,$$

and by a repeated use of Lemma 3.1 we conclude that $L^{(t)}y$ changes sign on (a, b). This proves (3.18).

Since Eq. (2.1) is $(k, n-k)$-disfocal on $[a, b]$, there is according to Lemma 2.1 a solution u satisfying

$$(-1)^{(t-k)_+} L^{(t)}u > 0, \qquad a \leqslant x \leqslant b, \qquad t = 0, \dots, n-1. \tag{3.19}$$

Let λ_0 be the maximal value of λ such that

$$(-1)^{(t-k)_+} L^{(t)}(u - \lambda y) \geqslant 0, \qquad a \leqslant x \leqslant x_t, \quad t = 0, \dots, n-1.$$

By (3.19) and (3.17) it is clear that $0 < \lambda_0 < \infty$ and

$$(-1)^{(t-k)_+} L^{(t)}(u - \lambda_0 y) \geqslant 0, \qquad a \leqslant x \leqslant x_t, \quad t = 0, \dots, n. \tag{3.20}$$

Note that the inequality for $t = n$ is implied by that for $t = 0$, equation (2.1) and (2.4).

We shall derive now a contradiction to the definition of λ_0 thus completing the proof of the theorem. The argument is analogous to that used in the proof of Theorem 2.1. First we show that each function $L^{(t)}(u - \lambda_0 y)$, $t = 0, \dots, n-1$, is monotone on $[a, x_t]$. If $x_{t+1} \geqslant x_t$ this is self-evident since by the $(t+1)$-th inequality of (3.20) we have

$$(-1)^{(t+1-k)_+} L^{(t+1)}(u - \lambda_0 y) \geqslant 0 \tag{3.21}$$

on $[a, x_{t+1}]$ and of course on $[a, x_t]$. But if $x_{t+1} < x_t$, we have to show that (3.21) holds on (x_{t+1}, x_t), too. In order to prove this, note that since $N(y) = n$ $L^{(t)}y \neq 0$ on (a, x_t), $L^{(t+1)}y$ may change sign on (a, x_t) at most once (Lemma 3.2).

Moreover, since $x_{t+1} \in (a, x_t) \subset (a, b)$, x_{t+1} is according to (3.9) a simple zero of $L^{(t+1)}y$. Now, $(-1)^{(t+1-k)_+} L^{(t+1)}y > 0$ on (a, x_{t+1}), therefore $(-1)^{(t+1-k)_+} \times L^{(t+1)}y < 0$ on (x_{t+1}, x_t). This, together with (3.19) and $\lambda_0 > 0$, implies (3.21) on (x_{t+1}, x_t) and $L^{(t)}(u - \lambda_0 y)$ is monotone on $[a, x_t]$ even if $x_{t+1} < x_t$.

More precisely, we see by (3.20) that for $0 \leqslant t \leqslant k-1$, $L^{(t)}(u - \lambda_0 y)$ is increasing on $[a, x_t]$ and since by (3.11) and (3.19) $L^{(t)}(u - \lambda_0 y)(a) = L^{(t)}u(a) > 0$, this function is strictly positive on $[a, x_t]$. For $k \leqslant t \leqslant n-1$, $(-1)^{t-k} L^{(t)}(u - \lambda_0 y)$ is by (3.20) a decreasing function on $[a, x_t]$; by (3.18) and (3.19) we have $(-1)^{t-k} L^{(t)}(u - \lambda_0 y)(x_t) = (-1)^{t-k} L(t)u(x_t) > 0$. Thus $(-1)^{t-k} L^{(t)}(u - \lambda_0 y)$ is strictly positive, too, on $[a, x_t]$. This contradicts the definition of λ_0 and proves that no solution y of (2.1) may satisfy (3.15) and (3.16). This establishes also that Green's function for (3.11) exists. The last assertion of the theorem will be proved after the next lemma.

Remarks. (i) Since the $(k, n-k)$-conjugate point problem (3.14) is a particular case of (3.11), the last theorem provides another proof to the known fact that $(k, n-k)$-disfocality of (2.1) implies its $(k, n-k)$-disconjugacy.

(ii) The weaker assumption of $(k, n-k)$-disconjugacy guarantees the existence of Green's function for (3.14). However, $(k, n-k)$-disfocality is the weakest condition under which all boundary conditions of the type (3.11) have Green's functions.

The Green's function for (3.11) exists if (3.12) holds due to the fact that no nontrivial solution of (2.1) satisfies the n boundary conditions of (3.11). In the following discussions we shall frequently use a solution which satisfies only $n-1$ of the boundary conditions (3.11), the condition for $j = j_l$, say, being deleted.

LEMMA 3.4. *Equation* (2.1) *has an essentially unique solution y which satisfies the $n-1$ boundary conditions*

$$\begin{gathered} L^{(i)}y(a) = 0, \quad i = 0,\dots, k-1, \\ [(1-\mu_j)L^{(j)}y + \mu_j L^{(j+1)}y](b) = 0, \quad j = j_1, \dots, \hat{j}_l, \dots, j_{n-k} \end{gathered} \tag{3.22}$$

($\hat{j}_l$ indicates that j_l is omitted). Suppose that (2.1) *is $(k, n-k)$-disfocal on $[a, b]$ and $\mu_{n-1} = 0$ if $j_{n-k} = n-1$. Then*

$$y(x) \neq 0, \qquad a < x < b. \tag{3.23}$$

Moreover, let $\{r+1, r+2, \dots, n-1\}$ be the longest sequence of consecutive integers which is included in $\{j_1, \dots, \hat{j}_l, \dots, j_{n-k}\}$. (If $n-1 \notin \{j_1, \dots, \hat{j}_l, \dots, j_{n-k}\}$, take $r = n-1$). Then

$$L^{(t)}y \cdot L^{(t+1)}y < 0, \qquad t = j_1, \dots, \hat{j}_l, \dots, j_{n-k}, r, \ x \in U_b^-, \tag{3.24}$$

and these are the only pairs of consecutive quasi-derivatives with opposite signs on U_b^-. *Also*

$$L^{(t)}y(b) \neq 0, \qquad t \neq j_1, ..., \hat{j}_l, ..., j_{n-k}. \tag{3.25}$$

Proof. Clearly, there exists a solution y of (2.1) which satisfies any $n-1$ given linear, homogeneous boundary value conditions. If there are two independent solutions which satisfy (3.22), there will exist a linear combination $y_3 = c_1 y_1 + c_2 y_2$ which has an extra quasi-derivative that vanishes at a. This solution satisfies $N(y_3) = n$, $S(y_3, a+) = k+1$, $S(y_3, b-) = n-k-1$, contradicting (3.8). This establishes the uniqueness of y.

Let (2.1) be $(k, n-k)$-disfocal on $[a, b]$. If y has a zero in (a, b), we achieve a contradiction as in the proof of Theorem 3.3. Indeed, by replacing the nth boundary condition which we had in Theorem 3.3 by the supposed zero of y in (a, b), we deduce that for every t, $k \leqslant t \leqslant n-1$, $L^{(t)}y$ has a zero in $(a, b]$. Now, we repeat the rest of the proof of Theorem 3.3 literally. The contradiction that is reached implies that $y \neq 0$ on (a, b).

By boundary conditions (3.22) we have $S(y, a+) \geqslant k$, $S(y, b-) \geqslant n-k-1$. Hence by Lemma 3.2 and (3.8) we conclude that $S(y, a+) = k$, $S(y, b-) = n-k$. Note that $n-k-1$ pairs of quasi-derivatives which satisfy $L^{(t)}y \cdot L^{(t+1)}y < 0$ on U_b^- are implied by (3.22). Pick r so that the $(n-k)$th such pair is $L^{(r)}y \cdot L^{(r+1)}y < 0$. Then the conditions

$$L^{(i)}y(a) = 0, \qquad i = 0, ..., k-1,$$
$$L^{(j)}y \cdot L^{(j+1)}y < 0, \qquad j = j_1, ..., \hat{j}_l, ..., j_{n-k}, r, \quad x \in U_b^-,$$

are satisfied by $y \not\equiv 0$. Thus, the above conditions violate (3.16) while the boundary conditions (3.22) obey (3.12). This may happen only if $\{r, j_{r+1-k}, ..., j_{n-k}\} = \{r, r+1, ..., n-1\}$ and $L^{(r)}y(b) \neq 0$. Thus r is as defined above and (3.24) is proved. Furthermore, since $S(y, b-) = n-k$, no other pairs of quasi-derivatives have opposite signs on U_b^-. Consequently (3.25) follows and the lemma is proved.

If $l = n-k$ and $0 \leqslant j_1 < \cdots < j_{n-k-1} \leqslant n-2$ then $r = n-1$ and $L^{(n-1)}y(b) \cdot L^{(n)}y(b) < 0$. There exists $0 < \tilde{\mu}_{n-1} < 1$ such that $[(1-\tilde{\mu}_{n-1}) \times L^{(n-1)}y + \tilde{\mu}_{n-1}L^{(n)}y](b) = 0$. The boundary value problem which results by adjoining the last boundary value condition to the $n-1$ conditions of (3.22), has a nontrival solution and it demonstrates the last assertion of Theorem 3.3.

Our main result for the Green's function of (2.2), (3.11) is the following.

THEOREM 3.5. *Let* (2.1) *be* $(k, n-k)$*-disfocal on* $[a, b]$. *For the boundary value problem* (3.11), (*where* $\mu_{n-1} = 0$ *if* $j_{n-k} = n-1$), *the Green's function satisfies*

$$(-1)^{n-k}\, G(x, s) > 0, \qquad a < x, \quad s < b. \tag{3.26}$$

Proof. The theorem will be proved for all boundary value problems of type (3.11) by passing from one such problem to another through changing one boundary condition at each step. Namely, suppose $G(x, s)$ is the Green's function for (3.11) and it satisfies (3.26). We shall prove that if $j_l - j_{l-1} > 1$ (we shall assume $l > 1$, the case $l = 1$ is similar) then the Green's function associated with the boundary conditions

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0,\dots, k-1, \\ [(1-\mu_j)L^{(j)}y + \mu_j L^{(j+1)}y](b) &= 0, \qquad j = j_1,\dots, j_{l-1}, j_l - 1, j_{l+1},\dots, j_{n-k}, \end{aligned} \tag{3.27}$$

is of the form $\tilde{G}(x, s) = G(x, s) - \lambda(s)\, y(x)$ where y is a solution of (2.1) which satisfies (3.22), and that $(-1)^{n-k}\, \tilde{G}(x, s) > 0$. Since for the $(k, n-k)$-focal point problem (2.3) the inequality (3.26) is already known and since every problem of type (3.11) can be obtained from (2.3) by a finite number of such steps, the theorem will follow.

Let s be fixed. First we determine the signs of some quasi-derivatives of $G(x, s)$ on U_b^-. As $(-1)^{n-k}\, G > 0$, we know that G satisfies Lemma 3.2. Hence, by the boundary conditions (3.11) and by (3.9) we have

$$S(G, a+) = k, \qquad S(G, b-) = n-k. \tag{3.28}$$

Applying (3.11), we obtain that

$$(-1)^{n-k+l-1} L^{(t)}G > 0, \qquad j_{l-1} < t \leqslant j_l, \quad x \in U_b^-. \tag{3.29}$$

In particular, as $j_l - j_{l-1} > 1$, we shall use that

$$(-1)^{n-k+l-1}\, L^{(j_l-1)}G(b, s) > 0. \tag{3.30}$$

It also follows that

$$(-1)^{(n-k)+(t-k)_+}\, L^{(t)}G > 0, \qquad t = 0,\dots, n, \quad x \in U_a^+. \tag{3.31}$$

For the solution y which satisfies (3.22), we know that $y \neq 0$ on (a, b). Hence, we may take

$$(-1)^{n-k}\, y > 0, \qquad a < x < b. \tag{3.32}$$

Recall that according to (3.24), $L^{(t)}y \cdot L^{(t+1)}y < 0$ on U_b^- precisely for $t = j_1,\dots, \hat{j}_l,\dots, j_{n-k}, r$, where $\{r+1,\dots, n-1\}$ is the longest sequence included in $\{j_1,\dots, \hat{j}_l,\dots, j_{n-k}\}$. By the definition of r, obviously $j_l < r+1$ and we have by (3.32), (3.24) and (3.25)

$$(-1)^{n-k+l-1}\, L^{(t)}y(b) > 0, \qquad j_{l-1} < t \leqslant j_l. \tag{3.33}$$

Let λ_0 be the maximal value of λ such that

$$(-1)^{n-k}[G - \lambda y] \geqslant 0, \qquad a \leqslant x \leqslant b.$$

By the boundary conditions (3.11) and (3.22) it is clear that G and y have at $x = a, b$ zeros of the same multiplicites (recall that $j_l - j_{l-1} > 1$), hence $\lambda_0 > 0$. Furthermore, by (3.26) and (3.32) we have $\lambda_0 < \infty$. Let $w = G - \lambda_0 y$. We show as in the proof of Theorem 2.1 that $w \not\equiv 0$ on any interval. Indeed, if for example, $w \equiv 0$ on a subinterval of $[a, s)$ then being a solution of a linear differential equation, it vanishes identically on all $[a, s)$ and

$$\begin{aligned} L^{(t)}w(s) &= 0, \qquad t = 0,\ldots, n-2, \\ L^{(n+1)}w(s+0) &= 1/\rho_n(s). \end{aligned} \tag{3.34}$$

If $n - k > 1$, (3.34) is impossible since the solution w would satisfy $S(w, s+) + S(w, b-) \geqslant (n-1) + (n-k) > n$, contradicting Lemma 3.2 on $[s, b]$. If $n - k = 1$, then by the definition of λ_0, $(-1)[G - \lambda_0 y] \geqslant 0$, i.e. $w \leqslant 0$, while by the initial conditions (3.34) we have $w > 0$ on U_s^+. This contradiction shows that $w \not\equiv 0$ on every interval. Hence, as $(-1)^{n-k}w \equiv (-1)^{n-k}[G - \lambda_0 y] \geqslant 0$, $\not\equiv 0$, we may deduce by previous arguments that the piecewise solution w satisfies Lemma 3.2 on $[a, b]$.

Recall that we expect the Green's function for (3.27) to be of the form $G - \lambda y$. In (3.29) we determined the signs of some quasi-derivatives of G near b. Now we determine the signs of the quasi-derivatives of $w = G - \lambda_0 y$. By (3.11) and (3.22) we have $S(w, a+) \geqslant k$, $S(w, b-) \geqslant n - k - 1$. Hence, by (3.8) and (3.9) we have in fact $S(w, a+) = k$, $S(w, b-) = n - k$. Again, we know that $L^{(t)}w \cdot L^{(t+1)}w < 0$ on U_b^- for $t = j_1, \ldots, \hat{j}_l, \ldots, j_{n-k}$ and we have to identify only the $(n-k)$th pair of quasi-derivatives with opposite signs. By the definition of λ_0, $w = G - \lambda_0 y$ must have in $[a, b]$ at least one zero more than $G - \lambda y$, $0 \leqslant \lambda < \lambda_0$, has, where the zeros are counted according to their multiplicities. But w cannot have a $(k+1)$-th zero at a since $S(w, a+) = k$. Also w cannot have a simple zero in (a, b) since $(-1)^{n-k} w \geqslant 0$, nor a double zero, by (3.9). Consequently w has at b a zero of order one or more than $G - \lambda y$, $\lambda \in [0, \lambda_0)$, has. If G has at b a zero precisely of multiplicity h, i.e., $\{j_1, \ldots, j_h\} = \{0, 1, \ldots, h-1\}$, $j_{h+1} \geqslant h+1$, then w has a zero of multiplicity $h+1$ at b. Thus

$$L^{(t)}w \cdot L^{(t+1)}w < 0, \qquad t = j_1, \ldots, j_h, h, j_{h+1}, \ldots, \hat{j}_l, \ldots, j_{n-k},$$

on U_b^- and those are the only pairs with opposite signs. It follows that $(-1)^{n-k+l} L^{(t)}w(b) \geqslant 0$, $j_{l-1} < t \leqslant j_l$, i.e.,

$$(-1)^{n-k+l} L^{(j_l - 1)}[G - \lambda_0 y](b) \geqslant 0. \tag{3.35}$$

with equality only if $j_l - 1 = h$.

Finally, put $\lambda_1 = L^{(j_l-1)}G(b, s)/L^{(j_l-1)}y(b)$. By (3.30), (3.33) and (3.35) we have $0 < \lambda_1 \leqslant \lambda_0$, hence $(-1)^{n-k}[G - \lambda_1 y] \geqslant 0$. But by the choice of λ_1, $G - \lambda_1 y$ is the Green's function for (3.27) with $\mu_{j_l-1} = 0$. Using these boundary conditions, we see by the previous arguments that

$$\begin{gathered}(-1)^{n-k+l-1}L^{(j_l-1)}[G - \lambda_1 y] > 0, \qquad x \in U_b^-,\\ (-1)^{n-k+l}L^{(j_l)}[G - \lambda_1 y](b) > 0.\end{gathered} \tag{3.36}$$

Now we are ready to construct the Green's function for (3.27). By (3.29), (3.30) and by (3.36) we see that the monotone bilinear function of λ, $m(\lambda) = L^{(j_l)}[G - \lambda y](b)/L^{(j_l-1)}[G - \lambda y](b)$, maps $[0, \lambda_1]$ on $[-\infty, 0]$. Therefore, for every $0 \leqslant \mu < 1$ there is a unique λ_μ, $0 < \lambda_\mu \leqslant \lambda_1$, dependent on s, such that $m(\lambda_\mu) = -(\lambda - \mu)/\mu$, *i.e.*, the function $G_\mu = G - \lambda_\mu y$ satisfies

$$[(1 - \mu)L^{(j_l-1)}G_\mu + \mu L^{(j_l)}G_\mu](b) = 0.$$

For $\mu = \mu_{j_l-1}$ given in (3.27), G_μ is the Green's function for (3.27).

By (3.32) and as $\lambda_\mu > 0$, we have

$$(-1)^{n-k}\, G > (-1)^{n-k}\, G_\mu\,, \qquad a < x,\ s < b,$$

and since $\lambda_\mu \leqslant \lambda_1 \leqslant \lambda_0$, we have by the definition of λ_0 and by the above considerations about the zeros of $w = G - \lambda_0 y$ that

$$(-1)^{n-k}\, G_\mu = (-1)^{n-k}[G - \lambda_\mu y] > 0, \qquad a < x, s < b.$$

This completes the proof of the theorem.

By the above proof it follows that $\lambda_{\mu_1} > \lambda_{\mu_2}$ whenever $\mu_1 < \mu_2$. By this observation and by (3.32) we have

COROLLARY 3.6. *Assume* (2.1) *is* $(k, n - k)$*-disfocal on* $[a, b]$. *Let* G *be the Green's function of* (3.11) *and* $\tilde{G}$ *be the Green's function of*

$$\begin{gathered}L^{(i)}y(a) = 0, \qquad i = 0,..., k - 1,\\ [(1 - \tilde{\mu}_j)L^{(j)}y + \tilde{\mu}_j L^{(j+1)}y](b) = 0, \quad j = \tilde{j}_1,..., \tilde{j}_{n-k},\end{gathered}$$

where both problems comply with (3.13). *If* $j_l \geqslant \tilde{j}_l$ *and* $\mu_{j_l} \geqslant \tilde{\mu}_{j_l}$ *whenever* $j_l = \tilde{j}_l$. $l = 1,..., n - k$, *and at least one inequality is strict, then*

$$(-1)^{n-k}\, G(x, s) > (-1)^{n-k}\, \tilde{G}(x, s) > 0, \qquad a < x,\ \ s < b.$$

COROLLARY 3.7. *Assume* (2.1) *is* $(k, n - k)$*-disfocal on* $[a, b]$. *Let* G *be the*

Green's function for (3.11) *and* G_{conj} *be the Green's function for the* $(k, n-k)$-*conjugate point problem* (3.14). *Then*

$$(-1)^{n-k} G_{\text{foc}} \geqslant (-1)^{n-k} G \geqslant (-1)^{n-k} G_{\text{conj}} > 0, \qquad a < x, s < b.$$

Now we turn to the question of the convergence of $G(x, s, b)$ as $b \to \infty$. For $G_{\text{conj}}(x, s, b)$ we have

THEOREM 3.8. *Let* (2.1) *be* $(k, n-k)$-*disfocal on* $[0, \infty)$. *Then for fixed* x, s, $a < x, s < b$, $(-1)^{n-k} G_{\text{conj}}(x, s, b)$ *is a strictly increasing function of* b *and if* $a > 0$ *it converges as* $b \to \infty$.

Proof. By Corollary 3.7, $(-1)^{n-k} G_{\text{conj}}(x, s, b)$ is positive and bounded from above by $(1-)^{n-k} G_{\text{foc}}(x, s, b)$ which is itself bounded by $(-1)^{n-k} G_\infty(x, s)$. Since the last bound is independent of b, it is sufficient to prove that $(-1)^{n-k} G_{\text{conj}}(x, s, b)$ increases with b for each fixed (x, s). This assertion will be proved as the analogous fact for G_{foc} was in the proof of Theorem 2.2.

Consider $w(x) = G_{\text{conj}}(x, s, c) - G_{\text{conj}}(x, s, b)$. Note that w is a solution of Eq. (2.1) and obviously

$$L^{(t)} w(a) = 0, \qquad t = 0, \ldots, k-1. \tag{3.37}$$

By the $(k, n-k)$-conjugate point boundary value conditions we have

$$(-1)^{n-k+t} L^{(t)} G_{\text{conj}}(x, s, c) > 0, \qquad t = 0, \ldots, n-k, \quad x \in U_c^-$$

and

$$L^{(t)} G_{\text{conj}}(b, s, b) = 0, \qquad t = 0, \ldots, n-k-1.$$

Therefore, for $c > b$ and c sufficiently close to b, we have

$$(-1)^{n-k+t} L^{(t)} w(b) > 0, \qquad t = 0, \ldots, n-k-1.$$

Thus, we found $n-k-1$ pairs of quasi-derivatives with opposite signs at b:

$$L^{(t)} w(b) \cdot L^{(t+1)} w(b) < 0, \qquad t = 0, \ldots, n-k-2.$$

Now w satisfies $n-1$ boundary conditions of the type (3.22) and by Lemma 3.4, $w \neq 0$ on (a, b). Since $(-1)^{n-k} w(b) > 0$, we have $(-1)^{n-k} w > 0$ on $(a, b]$, i.e.,

$$(-1)^{n-k} G_{\text{conj}}(x, s, c) > (-1)^{n-k} G_{\text{conj}}(x, s, b) > 0$$

on $(a, b]$ for $c > b$, c sufficiently close to b. Since $G_{\text{conj}}(x, s, c)$ is obviously a continuous function of c, this proves our assertion.

Until now we studied Eq. (2.1) when the factorization

$$L^{(n)} y = \rho_n(\rho_{n-1}(\cdots (\rho_0 y)' \cdots)')'$$

was given. However, this factorization is not unique. If we choose another factorization, say

$$L^{(n)}y = \tilde{\rho}_n(\tilde{\rho}_{n-1}(\cdots (\tilde{\rho}_0 y)' \cdots)')',$$

the boundary conditions, the $(k, n - k)$—disfocality and Green's function will be altered. For example, the equation

$$x^{-2}(x^4(x^{-2}y)')' + 2x^{-2}y = 0$$

is not $(1, 1)$-disfocal on any *ray* (a, ∞), $a > 0$, since the solution $y = x - a$ satisfies $y(a) = (x^{-2}y)'(2a) = 0$. On the other hand, the same equation can be written as

$$x(x^{-2}(xy)')' + 2x^{-2}y = 0$$

and this equation is $(1, 1)$-disfocal on $(0, \infty)$ according to Lemma 2.1 since the solution $y = 1$ satisfies $y > 0$, $(xy)' > 0$. By the way, both equations are identical to $y'' = 0$. The only boundary value problem which is independent of the choice of the factorization is the $(k, n - k)$-conjugate point problem since (3.14) is equivalent to the boundary conditions (1.1) for every $\rho_0, \dots, \rho_n$.

However, Trench [14] proved that a disconjugate operator $L^{(n)}$ admits a canonical factorization such that

$$\int^{\infty} \rho_i^{-1}\, dx = \infty, \qquad i = 1, \dots, n - 1,$$

and this representation is essentially unique. Note that no restriction is posed on ρ_0 and ρ_n. This canonical form has an important role in connection with disfocality and disconjugacy of (2.1) [3] as well as in the present problem.

THEOREM 3.9. *Suppose that*

$$\int^{\infty} \rho_i^{-1}\, dx = \infty, \qquad i = 1, \dots, n - 1. \tag{3.38}$$

Under the assumptions of Theorem 3.8 *with* $a > 0$, *the Green's function* $G(x, s, b)$ *corresponding to each of the boundary value problems of type* (3.11) *satisfies*

$$L^{(t)}G(x, s, b) \to L^{(t)}G_{\infty}(x, s), \qquad t = 0, \dots, n - 1$$

as $b \to \infty$. *Here* $G_{\infty}(x, s)$ *is the same limit function which corresponds to the* $(k, n - k)$*-focal point problem.*

Proof. First we demonstrate that (3.38) is necessary. The equation

$$(x^2y')' + x^{-2}y = 0$$

is not written in the canonical form since $\int^\infty \rho_1^{-1} = \int^\infty x^{-2} < \infty$. It is (1, 1)-disfocal on $(2/\pi, \infty)$ according to Lemma 2.1 since the solution $y = \cos(x^{-1})$ satisfies $y > 0$, $x^2y' > 0$. For the boundary conditions $y(1) = y(b) = 0$ we have $G_{\text{conj}}(x, s, b) = -\sin(s^{-1} - b^{-1}) \sin(1 - x^{-1})/\sin(1 - b^{-1})$ for $1 \leqslant x < s \leqslant b$, $G_{\text{conj}}(x, s, b) = G_{\text{conj}}(s, x, b)$, $1 \leqslant s < x \leqslant b$. For the focal point problem $y(1) = (x^2y')(b) = 0$ we have $G_{\text{foc}}(x, s, b) = -\cos(s^{-1} - b^{-1}) \times \sin(1 - x^{-1})/\cos(1 - b)$, $1 \leqslant x < s \leqslant b$, and G_{conj}, G_{foc} converge to different limits as $b \to \infty$.

Now we turn to the proof of the theorem. By Corollary 3.7 it suffices to prove that $G_{\text{conj}}(x, s, b) \to G_\infty(x, s)$ as $b \to \infty$. In fact, we shall show that for every b there exists $\beta < b$ such that

$$(-1)^{n-k}\, G_{\text{foc}}(x, s, \beta) < (-1)^{n-k}\, G_{\text{conj}}(x, s, b) < (-1)^{n-k}\, G_{\text{foc}}(x, s, b) \quad (3.39)$$

for $a < x < \beta$ and $\beta \to \infty$ as $b \to \infty$. This will establish the theorem.

The following arguments are similar to those of the proof of [3, Lemma 6]. Let s be fixed. By the boundary conditions (3.14) we deduce that $L^{(t)}G_{\text{conj}}$, $t = 1,\dots, n-1$, change their signs in (a, b). Let x_t, $t = 0,\dots, n$, be the maximal number in $(a, b]$ such that $L^{(t)}G_{\text{conj}}$ has a fixed sign on (a, x_t). Obviously $x_0 = x_n = b$ and in view of the above remark, $a < x_t < b$ for $t = 1,\dots, n-1$. We avoid calling x_t the first zero of $L^{(t)}G_{\text{conj}}$ in $(a, b]$ since $L^{(n-1)}G_{\text{conj}}$ may change its sign at the discontinuity jump without vanishing.

Since $L^{(t)}G_{\text{conj}}(a, s, b) = 0$, $t = 0,\dots, k-1$, we have by Rolle's theorem $a < x_{t+1} < x_t$, i.e.,

$$a < x_k < x_{k-1} < \cdots < x_0 = b. \quad (3.40)$$

By (3.31) we know that $L^{(t)}G_{\text{conj}} \cdot L^{(t+1)}G_{\text{conj}} < 0$ on U_a^+ for $t = k,\dots, n-1$. Thus, since $N(G_{\text{conj}}) = n$, it follows by Lemma 3.2 that for $t = k,\dots, n-1$, $L^{(t+1)}G_{\text{conj}}$ does not change its sign between a and the first zero of $L^{(t)}G_{\text{conj}}$. Hence $x_{t+1} > x_t$ for $t = k,\dots, n-1$ and

$$a < x_k < x_{k+1} < \cdots < x_n = b. \quad (3.41)$$

Put $\beta = x_k - \epsilon$, $\epsilon > 0$. Since $a < \beta < \min\{x_t\}$ and since $L^{(t)}G_{\text{conj}}$ cannot vanish in (a, b) without changing its sign, we have by (3.31)

$$(-1)^{n-k+(t-k)_+} L^{(t)}G_{\text{conj}}(x, s, b) > 0, \qquad t = 0,\dots, n, \quad a < x \leqslant \beta.$$

To prove (3.39) consider the function

$$w(x) = (-1)^{n-k}[G_{\text{conj}}(x, s, b) - G_{\text{foc}}(x, s, \beta)]$$

on $[a, \beta]$. Note that w is a solution of (2.1) and it satisfies

$$L^{(t)}w(a) = 0, \qquad t = 0,\dots, k-1,$$

$$(-1)^{t-k} L^{(t)}w(\beta) > 0, \qquad t = k,\dots, n-1.$$

But these conditions are identical to (2.23), (2.24), hence they imply $(-1)^{(t-k)+}L^{(t)}w > 0$, $t = 0,\dots, n$, on $(a, \beta]$. This proves the first inequality of (3.39) while the second results from Corollary 3.7.

It remains to prove that $x_k \to \infty$ as $b \to \infty$. Suppose on the contrary that there is a sequence of values of b which tends to ∞ such that $x_k = x_k(b)$ is bounded. On the other hand $x_n \equiv b \to \infty$, hence by (3.41) there is q, $k \leqslant q \leqslant n-1$, such that x_q is bounded and x_{q+1} is unbounded as $b \to \infty$. By choosing a subsequence of b's, we may take $x_{q+1} \to \infty$.

By (3.31) we know that $(-1)^{n-q} L^{(q)}G_{\text{conj}} > 0$ on (a, x_q) and $(-1)^{n-q-1} \times L^{(q+1)}G_{\text{conj}} > 0$ on (a, x_{q+1}). Since $L^{(q)}G$ changes its sign at x_q and since $x_{q+1} > x_q$ by (3.41), we see that $(-1)^{n-q-1}L^{(q)}G_{\text{conj}}$, $(-1)^{n-q-1}L^{(q+1)}G_{\text{conj}} > 0$ on (x_q, x_{q+1}). When $b \to \infty$, x_q is bounded, say by c_1, and $x_{q+1} \to \infty$; also by Theorem 3.8 $G_{\text{conj}}(x, s, b) \to v(x)$ as $b \to \infty$ and v is a solution of (2.1) on $[a, s) \cup (s, \infty)$. Hence $(-1)^{n-q-1}L^{(q)}v$, $(-1)^{n-q-1}L^{(q+1)}v > 0$ on $[c_1, \infty)$. Thus $(-1)^{n-q-1}L^{(q)}v$ increases on $[c_1, \infty)$ and by (3.38) we have

$$\begin{aligned}(-1)^{n-q-1}&[L^{(q-1)}v(x) - L^{(q-1)}v(c_1)] \\ &= (-1)^{n-q-1}\int_{c_1}^{x} L^{(q)}v/\rho_q \geqslant (-1)^{n-q-1}L^{(q)}v(c_1)\int_{c_1}^{x}\rho_q^{-1} \to \infty\end{aligned}$$

as $x \to \infty$. We conclude that

$$(-1)^{n-q-1}L^{(t)}v > 0, \qquad t = 0,\dots, q+1 \tag{3.42}$$

on some interval $[c_2, \infty)$.

On the other hand x_k is bounded as $b \to \infty$ but $x_0 \equiv b$. Therefore, by (3.40) there is r, $0 < r \leqslant k$, such that x_r is bounded and x_{r-1} is unbounded as $b \to \infty$ through a suitable sequence. By (3.31), $(-1)^{n-k}L^{(r-1)}G_{\text{conj}} > 0$ on (a, x_{r-1}) and $(-1)^{n-k}L^{(r)}G_{\text{conj}} > 0$ on (a, x_r) and changes its sign at x_r. Thus $L^{(r)}G_{\text{conj}} \cdot L^{(r-1)}G_{\text{conj}} < 0$ on (x_r, x_{r-1}) and when $b \to \infty$ we find that $L^{(r)}v \cdot L^{(r-1)}v < 0$ on $[c_3, \infty)$, contradicting (3.42). This proves that $\beta = x_k(b) - \epsilon \to \infty$ and completes the proof of the theorem.

EXAMPLE 3.10. The equation

$$y'' + 1/4x^{-2}y = 0$$

is (1, 1)-disfocal on $(0, \infty)$ since the solution $y = x^{1/2}$ satisfies $y, y' > 0$, $y'' < 0$. For the boundary conditions

$$y(1) = 0,$$
$$[(1-\mu)\,y + 2b\mu y'](b) = 0, \qquad 0 \leqslant \mu \leqslant 1,$$

there corresponds the Green's function

$$G_\mu(x, s, b) = -(sx)^{1/2} \log x[1 - \log s/(\log b + 2\mu)], \qquad 1 \leqslant x < s \leqslant b$$

and $G_\mu(x, s, b) = G_\mu(s, x, b)$, $1 \leqslant s < x \leqslant b$. This Green's function illustrates all our previous results.

4. More Boundary Value Problems

In most of the considerations of Section 3 we did not use the boundary conditions explicitly but only the facts

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0,\ldots, k-1, \\ L^{(j)}y \cdot L^{(j+1)}y &< 0, \qquad j = j_1, \ldots, j_{n-k}, \quad x \in U_b^-, \end{aligned} \tag{3.15}$$

where

$$\{j_{q+1-k}, \ldots, j_{n-k}\} = \{q, \ldots, n-1\} \Rightarrow L^{(q)}y(b) = \cdots = L^{(n-1)}y(b) = 0. \tag{3.16}$$

This observation will lead us to more boundary conditions for which the results of Section 3 hold.

A matrix $(c_{jt})_{j=1,\ldots,m}^{t=1,\ldots,n}$, $n > m$, is called *sign consistent of order m* if each of its $m \times m$ minors has the same sign and at least one of them is nonzero. If, in addition, all the minors are nonzero the matrix is called *strictly sign consistent of order m.*

Let $S^+(c_1, \ldots, c_n)[S^-(c_1, \ldots, c_n)]$ denote the maximal [minimal] number of sign changes in the sequence $c_1, \ldots, c_n$ achievable by appropriate assignment of signs to the zero entries (if any).

Lemma 4.1 [5, p. 294]. *If the matrix $(c_{jt})_{j=1,\ldots,m}^{t=1,\ldots,n}$, $n > m$, is sign consistent of order m, then for every solution of*

$$\sum_{t=1}^{n} c_{jt} z_t = 0, \qquad j = 1, \ldots, m,$$

we have

$$S^+(z_1, \ldots, z_n) \geqslant m.$$

If the matrix is strictly sign consistent of order m then $S^-(z_1, \ldots, z_n) \geqslant m$.

THEOREM 4.2. *Let* (2.1) *be* $(k, n-k)$*-disfocal on* $[a, b]$ *and let the matrix* $(c_{jt})_{j=k,\ldots,n-1}^{t=0,\ldots,n-1}$ *be sign consistent of order* $n-k$. *Then for the boundary conditions*

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0, \ldots, k-1, \\ \sum_{t=0}^{n-1} c_{jt} L^{(t)} y(b) &= 0, \qquad j = k, \ldots, n-1, \end{aligned} \tag{4.1}$$

the Green's function exists and satisfies

$$0 < (-1)^{n-k} G_{\mathrm{conj}}(x, s) \leqslant (-1)^{n-k} G(x, s) \leqslant (-1)^{n-k} G_{\mathrm{foc}}(x, s). \tag{4.2}$$

If (2.1) *is* $(k, n-k)$*-disfocal on* $[0, \infty)$, *if*

$$\int^\infty \rho_i^{-1} = \infty, \qquad i = 1, \ldots, n-1,$$

and $a > 0$ *then*

$$L^{(t)}G(x, s, b) \to L^{(t)}G_\infty(x, s), \qquad t = 0, \ldots, n,$$

as $b \to \infty$, *where* $G_\infty(x, s)$ *is the same limit function which corresponds to the focal point problem.*

Proof. If there is a nontrivial solution y of (2.1) which satisfies the boundary conditions (4.1), then by Lemma 4.1,

$$S^+(L^{(0)}y(b), \ldots, L^{(n-1)}y(b)) \geqslant n-k. \tag{4.3}$$

The contribution of a sequence of quasi-derivatives $L^{(q-1)}y(b) \neq 0, L^{(q)}y(b) = \cdots = L^{(r-1)}y(b) = 0, L^{(r)}y(b) \neq 0$, $1 \leqslant q \leqslant r \leqslant n-1$, to (4.3) is $r-q$ or $r-q+1$ according to whether $(-1)^{r-q} L^{(q-1)}y(b) \cdot L^{(r)}y(b)$ is positive or negative. But this is also the number of pairs of consecutive quasi-derivatives with opposite signs on U_b^-: $L^{(t)}y \cdot L^{(t+1)}y < 0$ on U_b^- for $t = q, \ldots, r-1$ or for $t = q-1, q, \ldots, r-1$ according to whether $(-1)^{r-q} L^{(q-1)}y(b) L^{(r)}y(b)$ is positive or negative. Similarly, a sequence $L^{(q-1)}y(b) \neq 0$, $L^{(q)}y(b) = \cdots = L^{(n-1)}y(b) = 0$, $1 < q \leqslant r-1$, contributes $n-q$ to (4.3) and $L^{(t)}yL^{(t+1)}y < 0$ on U_b^- for $t = q, \ldots, n-1$ or for $t = q-1, q, \ldots, n-1$. Therefore, by (4.3) we can choose $0 < j_1 < \cdots < j_{n-k} \leqslant n-1$ such that (3.15) and (3.16) hold. But this is impossible and the existence of a unique Green's function for (4.1) is proved.

To prove (4.2), we use a standard method of [5]. For $\epsilon > 0$ let the matrix $(c_{jt}(\epsilon))$ be strictly sign consistent of order $n-k$ such that $(c_{jt}(\epsilon)) \to (c_{jt})$ as $\epsilon \to 0$ and let $G_\epsilon(x, s)$ be the Green's function for the boundary conditions

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0,\dots, k-1, \\ \sum_{t=0}^{n-1} c_{jt}(\epsilon) L^{(t)}y(b) &= 0, \qquad j = k,\dots, \in -1. \end{aligned} \tag{4.4}$$

For a fixed s_0, $G_\epsilon(x, s_0)$ satisfies (4.4) and by Lemma 4.1 we have $S^-(L^{(0)}G_\epsilon(b, s_0),\dots, L^{(n-1)}G_\epsilon(b, s_0)) \geqslant n-k$. So we can break up this sequence into at least $n-k+1$ groups

$$L^{(0)}G_\epsilon(b, s_0),\dots, L^{(\nu_1)}G_\epsilon(b, s_0); L^{(\nu_1+1)}G_\epsilon(b, s_0),\dots, L^{(\nu_2)}G_\epsilon(b, s_0);\dots, L^{(n-1)}G_\epsilon(b, s_0)$$

such that all the elements of a group have the same sign, the signs in consecutive groups are opposite and in each group there is at least one nonzero term. For $l = 1,\dots, n-k$ pick from the $(l+1)$st group a quasiderivative $L^{(j_l+1)}G_\epsilon(b, s_0) \neq 0$ such that $L^{(j_l)}G_\epsilon(b, s_0) L^{(j_l+1)}G_\epsilon(b, s_0) \leqslant 0$. Hence there exist $0 \leqslant \mu_{j_1},\dots, \mu_{j_{n-k}} < 1$ such that $G_\epsilon(x, s_0)$ satisfies the boundary conditions

$$\begin{aligned} L^{(i)}y(a) &= 0, \qquad i = 0,\dots, k-1, \\ (1-\mu_j) L^{(j)}y(b) + \mu_j L^{(j+1)}y(b) &= 0, \qquad j = j_1,\dots, j_{n-k}. \end{aligned} \tag{4.5}$$

Obviously $j_1,\dots, j_{n-k}, \mu_{j_1},\dots, \mu_{j_{n-k}}$ depend on s_0 and ϵ. As $\nu_{n-k} \leqslant j_{n-k} + 1 \leqslant n-1$, we have $j_{n-k} < n-1$ and the boundary conditions (4.5) are of the type (3.11). Now, by the uniqueness of the Green's function, $G_\epsilon(x, s_0)$ is identical to $\tilde{G}(x, s_0)$, where $\tilde{G}(x, s)$ is the Green's function associated with the boundary conditions (4.5). Of course, for $s_1 \neq s_0$, $G_\epsilon(x, s_1)$ will in general be compared with a Green's function of another problem. However, as all these Green's functions satisfy the assertion of Corollary 3.7, $G_\epsilon(x, s)$ satisfies (4.2).

Now let $\epsilon \to 0$. Being a bounded, piecewise solution of a linear differential equation, $G_\epsilon(x, s)$ converges to a limit as $\epsilon \to 0$ through a suitable sequence and the limit function $G_0(x, s)$ satisfies (4.2), too. On the other hand $G_0(x, s)$ satisfies the continuity properties which define a Green's function and the boundary conditions (4.1), since $(c_{jt}(\epsilon)) \to (c_{jt})$. Thus $G_0(x, s)$ is the required Green's function and the proof of the theorem is completed.

Note that it is necessary to use in the proof the modified boundary conditions (4.4), since the Green's function $G(x, s_0)$ for the original problem (4.1) may not satisfy at b any boundary conditions of type 3.11 (even though (3.28) holds).

Remark 4.4. For the sake of simplicity we have assumed throughout this work that $p > 0$ or $p < 0$. However, our results need only minor modifications if we assume that $p \geqslant 0$ or $p \leqslant 0$ (see [3], [4]) and even if $p \equiv 0$.

For example, if $(-1)^{n-k} p \leqslant 0$ we obtain that $L^{(t)}G_{\text{foc}}(x, s) > 0$ if either $a < x < s$ or $0 \leqslant t \leqslant k - 1$ but only $(-1)^{t-k} L^{(t)}G_{\text{foc}}(x, s) \geqslant 0$ of $x < s < b$ and $k \leqslant t \leqslant n$. If $p \equiv 0$, we deal with the disconjugate oherator $L^{(n)}$ which is obviously $(k, n - k)$-disfocal. In this case we have, for example, $L^{(t)}G_{\text{foc}}(x, s) \equiv 0$ for $s < x < b$ and $t = k, \ldots, n - 1$ and $G_{\text{foc}}(x, s, b)$ is independent of b. This is the analogue of Theorem 2.4. Our other results, too, seem to be interesting even for this long-studied operator. For G_{foc} and G_{conj} for the operator d^n/dx^n, see [9].

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