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COMPARISON THEOREMS FOR DISFOCALITY AND DISCONJUGACY OF DIFFERENTIAL EQUATIONS*

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Abstract. Pairs of ordinary differential equations are compared with respect to disfocality and disconjugacy.

1. Introduction. We consider two-term ordinary differential equations of the type

$$y^{(n)}+p(x)y=0, \tag{1}$$

where $p(x)$ has a fixed sign. Much of the work about oscillation and disconjugacy of (1) is done by using the concept of $(k,n-k)$-disfocality: (1) is called $(k,n-k)$-disfocal on an interval I if for every $a,b\in I$, $a<b$, no solution of (1), except the trivial one, satisfies

$$\begin{aligned} y^{(i)}(a)&=0, \quad i=0,\cdots,k-1,\\ y^{(j)}(b)&=0, \quad j=k,\cdots,n-1. \end{aligned} \tag{2}$$

Similarly, (1) is $(k,n-k)$-disconjugate if only the trivial solution satisfies

$$\begin{aligned} y^{(i)}(a)&=0, \quad i=0,\cdots,k-1,\\ y^{(j)}(b)&=0, \quad j=0,\cdots,n-k-1. \end{aligned} \tag{3}$$

The above concepts are applicable to the study of (1) thanks to some of the following properties. First, if $p\geq 0$ (≤ 0) and $n-k$ is even (odd) then (1) is $(k,n-k)$-disfocal and $(k,n-k)$-disconjugate on every interval. Thus it is sufficient to consider the values of k such that

$$(-1)^{n-k}p\leq 0. \tag{4}$$

Next, there are simple relations between disfocality and disconjugacy. $(k,n-k)$-disfocality implies $(k,n-k)$-disconjugacy on every interval and eventual $(k,n-k)$-disfocality (that is disfocality on some ray (c,∞)) is equivalent to eventual $(k,n-k)$-disconjugacy. Finally, $(k,n-k)$-disfocality is elegantly characterized: If $(-1)^{n-k}p\leq 0$, then (1) is $(k,n-k)$-disfocal on I if and only if there exists a solution y of (1) such that

$$\begin{aligned} y^{(i)}&>0, \qquad i=0,\cdots,k-1,\\ (-1)^{j-k}y^{(j)}&>0, \qquad j=k,\cdots,n-1 \end{aligned} \tag{5$_k$}$$

on I. References to these known facts and others may be found in [7], [2].

In [5], Jones proved that if (1) is eventually $(k,n-k)$-disfocal and $k\leq(n+1)/2$, then it is also eventually $(k-2,n-k+2)$-disfocal. In fact, Jones formulated his results in terms of eventual disconjugacy; however, his proof is more natural in the framework of disfocality. By this ordering theorem he reduced substantially the number of possible oscillation types of (1). In the course of his proof, Jones also compared disfocality types of equations of different orders. His proofs are based on the characterization of

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disfocality by a solution which satisfies (5) and inequalities of Kiguradze which such solution satisfies.

This note, motivated by Jones' results, is aimed to compare pairs of differential equations with respect to various types of disfocality and disconjugacy. Our comparison theorems are based on Green's functions inequalities.

2. Disfocality. Green's function $g_{k,n-k}(x,t)$ of the operator d^n/dx^n and the boundary conditions (2) is explicitly known:

$$(6)\quad g_{k,n-k}(x,t)=\begin{cases} -\sum\limits_{i=k}^{n-1}(x-a)^i(a-t)^{n-1-i}/i!(n-i-1)!, & a\le x<t\le b,\\ \sum\limits_{i=1}^{k-1}(x-a)^i(a-t)^{n-1-i}/i!(n-i-1)!, & a\le t\le x\le b.\end{cases}$$

Note that $g_{k,n-k}$ is independent of b and it is defined practically for $a\le x,\ t<\infty$. Also

$$(7)\quad \begin{aligned} &(-1)^{n-k}g^{(i)}_{k,n-k}>0, && i=0,\cdots,k-1,\\ &(-1)^{n-i}g^{(i)}_{k,n-k}\ge 0, && i=k,\cdots,n-1\end{aligned}$$

on (a,b) (with equality for $i\ge k$, $t\le x$). For $i\ge k$, (7) is immediate since $g^{(i)}_{k,n-k}(x,t)=-(x-t)^{n-i-1}/(n-i-1)!$ for $x<t$ and $g^{(k)}_{k,n-k}\equiv 0$ for $x>t$. Integration of $g^{(k)}_{k,n-k}$ from a to x yields (7) for $i\le k-1$.

THEOREM 1. *If $k>l$ then*

$$(8)\quad \frac{l!(n-l-1)!}{k!(n-k-1)!}\le\frac{(-1)^{n-k}g_{k,n-k}(x,t)}{(x-a)^k(t-a)^{n-k-1}}\Big/\frac{(-1)^{n-l}g_{l,n-l}(x,t)}{(x-a)^l(t-a)^{n-l-1}}\le\frac{(l-1)!(n-l)!}{(k-1)!(n-k)!},$$

the quotient bounded in (8) *increases with x and decreases with t and equalities are attained in* (8) *when $x=a$ and $x\to\infty$ respectively.*

Proof. We rewrite the quotient in (8) as a product

$$\left[-\frac{(t-a)g_{k,n-k}}{(x-a)g_{k-1,n-k+1}}\right]\left[-\frac{(t-a)g_{k-1,n-k+1}}{(x-a)g_{k-2,n-k+2}}\right]\cdots\left[-\frac{(t-a)g_{l+1,n-l-1}}{(x-a)g_{l,n-l}}\right]$$

and show that each factor increases. Note that if u/v is continuous and not monotone, then there exists a linear combination of u and v with two zeros. In our case, if $g_{k,n-k}/(x-a)g_{k-1,n-k+1}$ is not monotone, there exists a linear combination $h(x)=(x-a)g_{k-1,n-k+1}+cg_{k,n-k}$ with two zeros in $(0,\infty)$. Since h has a zero of multiplicity k at $x=a$, we obtain by Rolle's theorem that $h^{(k)}$ changes its sign twice in (a,∞). But according to (6), h is a polynomial of degree $k-1$ on (t,∞) and $h^{(k)}\equiv 0$ there. Thus, the two changes of sign of $h^{(k)}$ must be located in (a,t). But $h^{(k)}(x)=(x-a)g^{(k)}_{k-1,n-k+1}+kg^{(k-1)}_{k-1,n-k+1}+cg^{(k)}_{k,n-k}$ and as $g^{(i)}_{k,n-k}=-(x-t)^{n-i-1}/(n-i-1)!$ for $i\ge k$ and $x<t$, it is immediately seen that $h^{(k)}$ does not have two zeros in (a,t). Similarly, none of $h^{(i)}$ has two distinct zeros. This contradiction confirms the monotony of the first factor. By (6), $-(t-a)g_{k,n-k}/(x-a)g_{k-1,n-k+1}$ attains at $x=a$ the value

$(k-1)!(n-k)/k!(n-k-1)!=(n/k)-1$ and it tends to $(n/(k-1))-1$ as $x\to\infty$; hence it increases. This argument applied to each of the $k-l$ factors proves that the quotient in (8) increases on $[a,\infty)$. The bounds are obtained by taking $x=a$ and $x\to\infty$ respectively. The monotony with respect to t is proved similarly.

It is possible to prove the inequalities in (8) by replacing $g_{k,n-k}$ for $x<t$ and $x>t$ by the corresponding polynomials and direct manipulation.

THEOREM 2. *Let* $(-1)^{n-k}p\le 0$ *and suppose* (1) *is* $(k,n-k)$*-disfocal on* $[a,b]$. *If* $l\le k$ *then*

$$y^{(n)}+(-1)^{k-l}\left(\binom{n-1}{k}\Big/\binom{n-1}{l}\right)p(x)y=0 \tag{9}$$

is $(l,n-l)$*-disfocal on* $[a,b]$, *and if* $l\ge k$ *then*

$$y^{(n)}+(-1)^{k-l}\left(\binom{n-1}{k-1}\Big/\binom{n-1}{l-1}\right)p(x)y=0 \tag{10}$$

is $(l,n-l)$*-disfocal there.*

Proof. It is known that if (1) is $(k,n-k)$-disfocal on $[a,b]$ and $(-1)^{n-k}p\le 0$, then the unique solution of (1) which satisfies

$$\begin{aligned} &y^{(i)}(a)=0, && i=0,\cdots,k-2,\\ &y^{(k-1)}(a)=1,\\ &y^{(j)}(b)=0, && j=k,\cdots,n-1, \end{aligned} \tag*{$(11)_{k-1}$}$$

is positive and even satisfies $(5)_k$ on (a,b). Equations (1) and (11) are equivalent to the integral equation

$$y(x)=(x-a)^{k-1}/(k-1)!+\int_a^b g_{k,n-k}(x,t)[-p(t)]\,y(t)\,dt. \tag{12}$$

Put $u(x)=y(x)/[(x-a)^{k-1}/(k-1)!]$. Dividing (12) by $(x-a)^{k-1}/(k-1)!$ we get

(13)

$$u(x)=1+\int_a^b\left[(-1)^{n-k}((t-a)/(x-a))^{k-1}g_{k,n-k}(x,t)\right]\left[(-1)^{n-k-1}p(t)\right]u(t)\,dt$$

and the integrand is positive by (4) and (7). If $k\ge l$, then by (8) we have

(14)

$$\begin{aligned} u(x)\ge 1+&\left(\binom{n-1}{k}\Big/\binom{n-1}{l}\right)\\ &\cdot\int_a^b\left[(-1)^{n-l}((t-a)/(x-a))^{l-1}g_{l,n-l}(x,t)\right]\left[(-1)^{n-k-1}p(t)\right]u(t)\,dt. \end{aligned}$$

Now, it is known that if inequality (14) has a positive solution u, then the corresponding integral equation

$$v=1+\mathcal{K}v, \tag{15}$$

where $\mathcal{K}$ denotes the integral operator on the right-hand side of (14), has a solution v such that $0 \le v(x) \le u(x)$. This may be verified by defining iterations $v_0 = u$, $v_i = 1 + \mathcal{K}v_{i-1}$. We multiply now (15) by $(x-a)^{l-1}/(l-1)!$ and put $\tilde{y}(x) = v(x)(x-a)^{l-1}/(l-1)!$ to obtain

$$\tilde{y}(x) = (x-a)^{l-1}/(l-1)! + (-1)^{k-l}\left(\binom{n-1}{k}\Big/\binom{n-1}{l}\right) \cdot \int_a^b g_{l,n-l}(x,t)[-p(t)]\,\tilde{y}(t)\,dt,$$

which is equivalent to (9) and the boundary conditions $(11)_{l-1}$. By $(11)_{l-1}$ we see that the solution $\tilde{y}$ of (9) is not only positive but also satisfies $(5)_l$ on (a,b), hence (9) is $(l,n-l)$-disfocal on (a,b). It is disfocal on $[a,b]$ since $\tilde{y}$ satisfies $(11)_{l-1}$ and consequently no solution can satisfy the $(l,n-l)$-focal point boundary value conditions at a and b.

To treat the case $l \ge k$, we exchange the roles of l and k in (8) and use analogously the right-hand side of the inequality so obtained. Another approach is to note that $(k,n-k)$-disfocality of (1) is equivalent to $(n-k,k)$-disfocality of its adjoint and $\binom{n-1}{k-1} = \binom{n-1}{n-k}$.

The results of Jones [5] follow if $\binom{n-1}{k}/\binom{n-1}{l}$ (or $\binom{n-1}{k-1}/\binom{n-1}{l-1}$) is not smaller than 1 and we neglect it in the proof of Theorem 2 and in (9), (10). Thus, the $(k,n-k)$-disfocality of (1), where $(-1)^{n-k}p \le 0$, implies also its $(l,n-l)$ disfocality when

$$l \le k,\ \binom{n-1}{l} \le \binom{n-1}{k} \quad \text{or} \quad l \ge k,\ \binom{n-1}{l-1} \le \binom{n-1}{k-1},$$

that is for $l = 1, 2, \cdots, k-1, n-k+1, \cdots, n-1$ or $l = 1, 2, \cdots, n-k-1, k+1, \cdots n-1$. This can be written as $|l-n/2| > |k-n/2|$ when we do not exclude the values of l such that $l \not\equiv k \pmod 2$ (for which $(l,n-l)$-disfocality of (1) is trivial and is not a consequence of Theorem 2).

Following Jones we summarize:

THEOREM 3. *If* (1) *is* $(k,n-k)$*-disfocal on* $[a,b]$, $(-1)^{n-k}p \le 0$, *then* (1) *is also* $(l,n-l)$*-disfocal on* $[a,b]$ *where* $|l-n/2| > |k-n/2|$ *and* $l \equiv k \pmod 2$. *The equation* $y^{(n)} - py = 0$ *is* $(l,n-l)$*-disfocal when* $|l-n/2| > |k-n/2|$ *and* $l \not\equiv k \pmod 2$.

The methods of Theorems 1 and 2 may be adopted to compare disfocality of equations of different order. Compare with [5, Thms. 1–4].

THEOREM 4. a) *If* $k < n < m$ *then*

$$(16)\quad (m-k-1)!/(n-k-1)! \le (-1)^{m-n}(t-a)^{m-n} g_{k,n-k}(x,t)/g_{k,m-k}(x,t) \le (m-k)!/(n-k)!,$$

the quotient bounded in (16) *increases with* x *and equalities are obtained when* $x = a$ *and* $x \to \infty$, *respectively.*

b) *Let* $(-1)^{n-k}p \le 0$ *and suppose* (1) *is* $(k,n-k)$*-disfocal on* $[a,b]$. *If* $m \ge n$ *then*

$$(17)\qquad y^{(m)} + (-1)^{m-n}((m-k-1)!/(n-k-1)!)(x-a)^{n-m}p(x)y = 0$$

is $(k,m-k)$*-disfocal on* $[a,b]$ *and if* $k < m < n$ *then*

$$(18)\qquad y^{(m)} + (-1)^{m-n}((m-k)!/(n-k)!)(x-a)^{n-m}p(x)y = 0$$

is $(k,m-k)$*-disfocal there.*

In order to prove (16) we show that $h(x)=g_{k,n-k}-cg_{k,n-k-1}$ has not two zeros in (a,∞). Next, by (16), the integral equation (12) implies

$$y(x)\geq(x-a)^{k-1}/(k-1)!+(-1)^{m-n}((m-k-1)!/(n-k-1)!)$$
$$\cdot\int_a^b g_{k,m-k}(x,t)\left[-(t-a)^{n-m}p(t)\right]u(t)\,dt$$

and (17) follows. Note that $g_{k,m-k}(x,t)(t-a)^{n-m}$ has no singularity at $t=a$ even if $n<m$.

By composing Theorems 2 and 4 the following results are obtained.

THEOREM 5. *Let* $(-1)^{n-k}p\leq 0$ *and suppose* $y^{(n)}+py=0$ *is* $(k,n-k)$*-disfocal on* $[a,b]$. *Then the equation*

$$y^{(m)}+(-1)^{(m-l)+(n-k)}A_{k,l}(x-a)^{n-m}p(x)y=0 \tag{19}$$

is $(l,m-l)$*-disfocal on* $[a,b]$ *where*

$$A_{k,l}=\begin{cases}\dfrac{l!(m-l-1)!}{k!(n-k-1)!} & \text{if } m\geq n,\ k\geq l,\\[2mm] \dfrac{(n-k)(l-1)!(m-l)!}{(m-k)(k-1)!(n-k)!} & \text{if } m\geq n,\ k\leq l,\\[2mm] \dfrac{(l-1)!(m-l)!}{(k-1)!(m-k)!} & \text{if } m\leq n,\ k\leq l,\\[2mm] \dfrac{(m-l)l!(m-l-1)!}{(n-l)k!(n-k-1)!} & \text{if } m\leq n,\ k\geq l.\end{cases} \tag{20}$$

To prove this for $m\geq n$ we pass from the given $(k,n-k)$- to a $(k,m-k)$- and finally to a $(l,m-l)$-disfocal equation. For the case $m\leq n$ we follow the scheme $(k,n-k)\to(l,n-l)\to(l,m-l)$.

3. Disconjugacy. Now we turn to disconjugacy and $(k,n-k)$-disconjugacy. Green's function $G_{k,n-k}(x,t)$ of the operator d^n/dx^n and the boundary conditions (3) is obtained from $g_{k,n-k}$ when we replace $(x-a)$ and $(t-a)$ by $(x-a)(b-t)/(b-a)$ and $(b-x)(t-a)/(b-a)$, respectively [7]. Consequently, we see from (8) that the quotient

$$H(x,t)=\frac{(-1)^{n-k}G_{k,n-k}(x,t)}{(x-a)^{k-1}(b-x)^{n-k}(t-a)^{k-1}(b-t)^{n-k}}\Bigg/$$
$$\frac{(-1)^{n-l}G_{l,n-l}(x,t)}{(x-a)^{l-1}(b-x)^{n-l}(t-a)^{l-1}(b-t)^{n-l}}$$

where $k>l$, increases with x and is bounded by $l!(n-l-1)!/k!(n-k-1)!$ and $(l-1)!(n-l)!/(k-1)!(n-k)!$. Indeed, if we denote the quotient in (8) by $h(x,t)$ then $H(x,t)=h(a+(x-a)(b-t)/(b-a),\ a+(b-x)(t-a)/(b-a))$ and $H_x=h_x\cdot(b-t)/(b-a)-h_t\cdot(t-a)/(b-a)>0$ since $h_x>0$ and $h_t<0$. Similarly, by (16),

$$(m-k-1)!/(n-k-1)!\leq(-1)^{m-n}((b-x)(a-t)/(b-a))^{m-n}$$
$$\cdot G_{k,n-k}(x,t)/G_{k,m-k}(x,t)$$
$$\leq(m-k)!/(n-k)!$$

when $m \geq n$. Also the integral equation

$$y(x)=(x-a)^{k-1}(b-x)^{n-k}/(b-a)^{n-k}(k-1)!+\int_a^b G_{k,n-k}(x,t)[-p(t)]\,y(t)\,dt$$

is equivalent to (1) and the boundary value conditions

$$\begin{aligned} &y^{(i)}(a)=0, && i=0,\cdots,k-2,\\ &y^{(k-1)}(a)=1, && \\ &y^{(j)}(b)=0, && j=0,\cdots,n-k-1 \end{aligned}$$

and (1) is $(k,n-k)$-disconjugate on $[a,b]$ iff this y is positive on (a,b). Repeating the proof of Theorem 2 with $u(x)=y(x)/[(x-a)^{k-1}(b-x)^{n-k}/(b-a)^{n-k}(k-1)!]$, we obtain

THEOREM 6. *Theorems* 2 *and* 3 *remain valid if the term disfocality is replaced everywhere by disconjugacy. Theorem* 5 *remains valid if the term disfocality is replaced by disconjugacy and the factor* $(x-a)^{n-m}$ *in* (19) *is replaced by* $((x-a)(b-x)/(b-a))^{n-m}$.

Recall that while eventual $(k,n-k)$-disfocality and eventual $(k,n-k)$-disconjugacy are equivalent, disfocality on $[a,b]$ implies disconjugacy there but is not implied by it. Therefore the last results are not direct consequences of Theorems 2 and 3.

Since $(k,n-k)$-disconjugacy (disfocality) on $[a,b]$ is equivalent to the absence of $(k,n-k)$-type conjugate (focal) point $\eta_{k,n-k}(a)(\zeta_{k,n-k}(a))$ on $[a,b]$, we can restate Theorems 3 and 6 as

THEOREM 6′. *If* $|l-n/2|>|k-n/2|$, $l\equiv k(\mathrm{mod}\,2)$ *then*

$$a<\eta_{k,n-k}(a)\leq\eta_{l,n-l}(a)\leq\infty,$$
$$a<\zeta_{k,n-k}(a)\leq\zeta_{l,n-l}(a)\leq\infty.$$

Various works deal with relations between disconjugacy (nonoscillation) of (1) on $[a,\infty)$ and that of various second order equations. For example, see [3], [6], [4]. In Theorem 6 of [2] we proved that if (1) is eventually disconjugate and $n>m$, then $y^{(n)}+((m-1)!/(n-1)!)p(x)(x-a)^{n-m}y=0$ is eventually disconjugate, too. (In fact $(m-1)!/(n-1)!$ is replaced by the smaller constant $m!/n!$ but the bigger constant is immediately available without any change in the proof). Now we improve this result and extend it.

THEOREM 7. *Let* (1) *be disconjugate on* $[a,b]$. *If* $m<n$, *then the two equations*

$$\text{(21)}\qquad y^{(m)}\pm\frac{[n/2][m/2]![(m+1)/2]!}{(n-[m/2])[n/2]![(n+1)/2]!}\left(\frac{(x-a)(b-x)}{b-a}\right)^{n-m}p(x)y=0$$

are disconjugate on $[a,b]$ *and if* $m>n$, *then the equations*

$$\text{(22)}\qquad y^{(m)}\pm\frac{[(n-1)/2][m/2]![(m-1)/2]!}{(m-[(n-1)/2])[n/2]![(n-1)/2]!}\left(\frac{(x-a)(b-x)}{b-a}\right)^{n-m}p(x)y=0$$

are disconjugate. (*Here* [ ] *denotes the integer part function*).

Analogous results hold when we replace $[a,b]$ *by* $[a,\infty)$ *and* $(x-a)(b-x)/(b-a)$ *by* $x-a$.

Proof. An equation of type (1) is disconjugate when it is $(i,n-i)$-disconjugate for every i, $1\leq i\leq n-1$, such that $(-1)^{n-i}p\leq 0$. In order to prove that this is the case, it is sufficient, by Theorems 3 and 6, to show that it is $(i, n-i)$-disconjugate either for $i=[n/2]$ or for $[n/2]+1$, according to the parity of n and the signature of p. Indeed, one of these values of i has the required parity and for it $|i-n/2|$ is minimal. In our case we have to show that (21) (or (22)) is $(l, m-l)$-disconjugate either for $l=[m/2]$ or for $l=[m/2]+1$. But there are many possibilities to deduce this from the disconjugacy of (1). According to Theorems 6 and 3, $(k, n-k)$-disconjugacy of (1), when $1 \leq k \leq n-1$, $(-1)^{n-k} p \leq 0$, implies $(l, m-l)$-disconjugacy of

$$y^{(m)}+(-1)^{m-l+n-k} A_{kl} p(x)((x-a)(b-x)/(b-a))^{n-m} y=0 \tag{23}$$

where A_{kl} is given in (20). Thus, we are in possession of several values of k by each of which we can deduce $(l, m-l)$-disconjugacy of an mth order equation. Since we did not assume what is the sign of $p(x)$ and the parity of n, we have to check two values of k of different parities and select from the two so-obtained mth order equations the one with the smaller coefficient A_{kl}. Since l will be either $[m/2]$ or $[m/2]+1$, it is easily seen that the best choice of a pair of values of k is $[(n-1)/2]$, $[(n+1)/2]$. First let $m<n$. If $m \leq n-2$, then $l \leq [m/2]+1 \leq [(n-1)/2] \leq k$ and by (20), $A_{kl}=(m-l)!l!/(n-l)k!(n-k-1)!$ is smaller for $k=[(n+1)/2]$. Among the two candidates for l, A_{kl} is smaller for $l=[m/2]$ and its value is

$$C_{mn}=\frac{[m/2]![(m+1)/2]!}{(n-[m/2])[(n+1)]![n/2-1]!}$$

(use $[n/2]+[(n+1)/2]=n$). This proves that both equations in (21) are disconjugate if $m \leq n-2$. If $m=n-1$, the result is still valid since by direct calculation we get an equation in which the numerical coefficient is even larger than that in (21).

When $m>n$, we select among $k=[(n-1)/2]$, $[(n+1)/2]$ the first one and from $l=[m/2]$, $[m/2]+1$ we choose the second to obtain (22). For $n=2$, only the + sign and (1, 1)-disconjugacy are acceptable.

Let C_{mn} be defined by the numerical coefficient in (21) or (22), according as $m<n$ or $m>n$. Theorem 7 may be stated also as follows:

THEOREM 7′. *In order that $y^{(n)}+py=0$ be disconjugate on $[a, \infty)$, it is necessary that both $y^{(m)} \pm C_{mn}(x-a)^{n-m} p(x) y=0$ be disconjugate on $[a, \infty)$ and sufficient that one of $y^{(m)} \pm C_{nm}^{-1}(x-a)^{n-m} p(x) y=0$ be disconjugate on the interval (for $m=2$ take only + sign).*

Theorem 7 improves known results for $n \geq 4$ but is worse than those for $n=3$, $m=2$. Obviously the numerical constants in Theorem 7 are not the best possible. To estimate how good these constants are, we may compare the constants in the necessary and the sufficient conditions of Theorem 7′. For $n>m$, for example,

$$\frac{C_{mn}}{C_{nm}^{-1}}=\frac{[(m-1)/2](m-[m/2])(m-[(m-1)/2])}{[(n+1)/2](n-[m/2])(n-[(m-1)/2])}<1,$$

and the distance between this ratio and 1 give an idea how far are our results from the optimum.

When the sign of p and the parities of m and n are known, we can obviously get specific results which are better than the last theorem.

We can compare by similar method $y^{(n)}+py=0$ and $y^{(n)}-py=0$. If n is odd, these equations are adjoint and they are disconjugate together. However when n is even we obtain a nontrivial result. Following the proof of Theorem 7 we take in (23) $l=n/2$ and

$k=n/2+1$ if $(-1)^{n/2-1}p<0$ and $l=n/2+1$, $k=n/2$ if $(-1)^{n/2}p>0$. We obtain

THEOREM 8. *Let $y^{(2r)}+py=0$ be disconjugate. If $(-1)^r p\leq 0$, then $y^{(2r)}-py=0$ is disconjugate too. If $(-1)^r p>0$, then $y^{(2r)}-((r-1)/(r+1))py=0$ is disconjugate there.*

For $2r=4$ this result is far from being strict: We know that $y^{(4)}+Ax^{-4}y=0$ is disconjugate on $(0,\infty)$ for $0\leq A\leq 1$ while $y^{(4)}-Ax^{-4}y=0$ is disconjugate for $0\leq A\leq 9/16$.

4. Estimates of solutions.

THEOREM 9. *Suppose $(-1)^{n-k}p\leq 0$ and (1) is $(k,n-k)$-disfocal on $[a,b]$ and let $y_{k-1}(x,b)$ be the unique solution of (1) which satisfies the boundary conditions $(11)_{k-1}$. If $|l-n/2|>|k-n/2|$ and $(-1)^{n-l}p(x)\leq 0$, then the corresponding solution $y_{l-1}(x,b)$ of (1), $(11)_{l-1}$ satisfies $(5)_l$ and*

$$(24)\qquad 0<\frac{y_{l-1}(x,b)}{(x-a)^{l-1}/(l-1)!}\leq\frac{y_{k-1}(x,b)}{(x-a)^{k-1}/(k-1)!},$$

$$(25)\qquad 0\leq(-1)^{p-1}\frac{d^p}{dx^p}\left(\frac{y_{l-1}(x,b)}{(x-a)^{l-1}/(l-1)!}\right)\frac{\binom{n-1}{k+p-1}}{\binom{n-1}{l+p-1}}$$

$$\leq(-1)^{p-1}\frac{d^p}{dx^p}\left(\frac{y_{k-1}(x,b)}{(x-a)^{k-1}/(k-1)!}\right)\qquad p=1,\cdots,n-k-1,$$

on $[a,b]$.

In fact, we have already proved (24). In the proof of Theorem 2 it was seen that $u(x)=y(x)/[(x-a)^{k-1}/(k-1)!]$ and $v(x)=\tilde{y}(x)/[(x-a)^{l-1}/(l-1)!]$ satisfy $0\leq v(x)\leq u(x)$. If $|l-n/2|>|k-n/2|$ and we replace in the proof of Theorem 2 $\binom{n-1}{k}/\binom{n-1}{l}$ (or $\binom{n-1}{k-1}/\binom{n-1}{l-1}$) by the smaller number 1, we obtain (24) for the solutions y_{k-1}, y_{l-1} of (1).

To prove (25), we need an extension of Theorem 1.

THEOREM 10. a) *For $q=0,\cdots,l$, $p=0,\cdots,n-l-1$ we have*

$$(26)\qquad (-1)^{(p-q)_+}\frac{\partial^p}{\partial x^p}\left(\frac{(-1)^{n-l}g_{l,n-l}(x,t)}{(x-a)^{l-q}}\right)\geq 0$$

where $i_+=\max\{i,0\}$.

b) *If $k>l$, then for $q=0,\cdots,l$, $p=0,\cdots,n-l-1$, the ratios*

$$(27)\qquad (-1)^{(p-q)_+}\frac{\partial^p}{\partial x^p}\left(\frac{(-1)^{n-k}g_{k,n-k}(x,t)}{(x-a)^{k-q}(t-a)^{n-k+q-1}}\right)\Big/$$

$$(-1)^{(p-q)_+}\frac{\partial^p}{\partial x^p}\left(\frac{(-1)^{n-l}g_{l,n-l}(x,t)}{(x-a)^{l-q}(t-a)^{n-l+q-1}}\right)$$

increase from $\binom{n-1}{k+(p-q)_+}/\binom{n-1}{l+(p-q)_+}$ to $\binom{n-1}{k+(p-q)_+-1}/\binom{n-1}{l+(p-q)_+-1}$ as x varies from a to ∞.

Once we know Theorem 10, it is easy to prove (25). Recall that $u=y_{k-1}(x,b)/[(x-a)^{k-1}/(k-1)!]$ satisfies (13) and $v=y_{l-1}/[(x-a)^{l-1}/(l-1)!]$ satisfies a similar integral equation. Differentiation of (13) yields

$$u^{(p)}(x)=\int_a^b \frac{\partial^p}{\partial x^p}\left\{(t-a)^{n-1}\frac{(-1)^{n-k}g_{k,n-k}}{(x-a)^{k-1}(t-a)^{n-k}}\right\}\{(-1)^{n-k-1}p(t)\}u(t)\,dt. \tag{28}$$

Equation (25) follows immediately if we apply Theorem 10(b) with $q=1$ and the inequality $u\geq v>0$ to (28).

Proof of Theorem 10. In [1, Thm. 1] it is shown that if for some j, k, $j\geq k\geq 0$, a function y fulfills

$$y(a),\cdots,y^{(k-1)}(a)\geq 0,\qquad (-1)^{j-k}y^{(j)}(x)\geq 0\quad\text{on } [a,\infty), \tag{29}$$

then

$$(-1)^{j-k}\left(y/(x-a)^k\right)^{(j-k)}\geq 0\quad\text{on } [a,\infty). \tag{30}$$

Equation (26) is a particular case of (30) with $k=l-q$, $j=l-q+p$ and $y(x)=(-1)^{n-l+p-(p-q)_+}g_{l,n-l}(x,t)$. Indeed, (29) holds since $y(a)=\cdots=y^{(l-q-1)}(a)=0$ and $(-1)^{j-k}y^{(j)}=(-1)^{n-l-(p-q)_+}g_{l,n-l}^{(l+p-q)}$ is positive by (7), either if $p-q\geq 0$ or $p-q<0$. Here we used $l\leq q$, $l+p-q\leq n-1$. Consequently, by (30),

$$\begin{aligned}0\leq(-1)^{j-k}\left(y/(x-a)^k\right)^{(j-k)}&=(-1)^p\left((-1)^{n-l+p-(p-q)_+}g_{l,n-l}/(x-a)^{l-q}\right)^{(p)}\\&=(-1)^{(p-q)_+}\frac{\partial^p}{\partial x^p}\left((-1)^{n-l}g_{l,n-l}/(x-a)^{l-q}\right)\end{aligned}$$

and (26) is proved.

The proof of part (b) makes use of the identity

$$\left(\frac{d}{dx}\right)^i x^j\left(\frac{d}{dx}\right)^{j-i}x^{-i}y\equiv x^{j-i}\left(\frac{d}{dx}\right)^j y. \tag{31}$$

This equality between two differential operators of order j may be verified, for example, by applying both sides to the functions x^α, $-\infty<\alpha<\infty$.

We return now to the monotony of (27). If

$$\frac{\partial^p}{\partial x^p}\left(g_{k,n-k}/(x-a)^{k-q}\right)\Big/\frac{\partial^p}{\partial x^p}\left(g_{k-1,n-k+1}/(x-a)^{k-q+1}\right) \tag{32}$$

is not monotone, then there is a linear combination

$$\frac{\partial^p}{\partial x^p}\left(g_{k-1,n-k+1}/(x-a)^{k-q+1}\right)+c\frac{\partial^p}{\partial x^p}\left(g_{k,n-k}/(x-a)^{k-q}\right)$$

with two zeros in (a,∞). When we multiply this combination by $(x-a)^{p-q+k}$, $(p-q+k\geq p\geq 0)$, we see that the function

$$(x-a)^{p-q+k}\left(\frac{\partial}{\partial x}\right)^p(x-a)^{-(k-q)}\left((x-a)g_{k-1,n-k+1}+cg_{k,n-k}\right)$$

has two zeros in (a,∞) and at least $(p-q+k)+(q-p)_+$ zeros, hence not less than k zeros, at $x=a$. Consequently, its $(k-q)$th derivative, $(q\leq l<k)$,

$$\left(\frac{d}{dx}\right)^{k-q}(x-a)^{p-q+k}\left(\frac{d}{dx}\right)^p(x-a)^{-(k-q)}h(x),$$

changes its sign at least twice in (a, ∞). But according to (31) this means that $(x-a)^p(d/dx)^{k-q+p}h(x)$ changes its sign twice in (a, ∞) while we have seen in the proof of Theorem 1 that $h(x)=(x-a)g_{k-1,n-k+1}+cg_{k,n-k}$ and its derivatives cannot have such zero distribution in (a, ∞). This proves the monotony of (32), and in turn establishes Theorem 11.

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