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Minors of the Wronskian of the differential equation $L_n y + p(x)y = 0$. II. Dominance of solutions

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Synopsis

The equation studied here is $L_n y + p(x)y = 0$, where L_n is a disconjugate differential operator and $p(x)$ is of a fixed sign. We define a basis of the solution space and order its elements according to their relative magnitudes near infinity. Our method is independent of the possible oscillation or nonoscillation of the solutions and it is achieved by utilising the fact that some minors of the Wronskian never vanish.

1. Introduction

The differential equations to be studied here are of the form

$$L_n y + p(x)y, \quad 0 \leqq x < \infty; \tag{1.1}$$

here $p(x)$ is continuous and of one sign and L_n is the disconjugate differential operator

$$L_n y = \rho_n(\rho_{n-1} \ldots (\rho_1(\rho_0 y)')' \ldots)'$$

with weight functions $\rho_i > 0$, $\rho_i \in C^{n-i}[0, \infty)$, $i = 0, \ldots, n$. We denote $L_0 y = \rho_0 y$, $L_i y = \rho_i(L_{i-1}y)'$, $i = 1, \ldots, n$ and call $L_0 y, \ldots, L_n y$ the *quasi-derivatives* of y. This type of equation was studied recently in several works **[2–7, 9–12]**.

The purpose of this paper is to order the solutions of equation (1.1) according to their magnitude as $x \to \infty$. We shall define a basis of solutions and show that some elements of this basis are larger, in a certain sense, than others. To make the problem and the potential difficulties clear, let us turn to a simple equation of type (1.1), the Euler equation

$$y^{(8)} + 500x^{-8}y = 0, \quad 1 \leqq x < \infty. \tag{1.2}$$

x^r is a solution of (1.2) if r solves the algebraic equation $r(r-1) \ldots (r-7) + 500 = 0$. Here $0 < r_0 < r_1 < 1$; $2 < \operatorname{Re}\{r_2\} = \operatorname{Re}\{r_3\} < 3$; $4 < \operatorname{Re}\{r_4\} = \operatorname{Re}\{r_5\} < 5$; $6 < r_6 < r_7 < 7$ and if we let $r_2 = \alpha + i\beta \approx 2.316 + 0.968i$, $r_4 = (7-\alpha) + i\beta$, the corresponding solutions of (1.2) are

$$\begin{gathered} y_0 = x^{r_0}, \quad y_1 = x^{r_1}, \quad y_2 = x^{\alpha}\cos(\beta \ln x), \quad y_3 = x^{\alpha}\sin(\beta \ln x), \\ y_4 = x^{7-\alpha}\cos(\beta \ln x), \quad y_5 = x^{7-\alpha}\sin(\beta \ln x), \quad y_6 = x^{r_6}, \quad y_7 = x^{r}_7. \end{gathered} \tag{1.3}$$

Obviously y_0, y_1 are smaller near $x = \infty$ than y_6 and y_7. Since $\operatorname{Re}\{r_2\} = \alpha \in (2, 3)$

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and Re $\{r_4\} = 7 - \alpha \in (4, 5)$, we tend intuitively to say that y_2, y_3 are smaller than y_4, y_5, which in turn are smaller than y_6, y_7. The aim of this paper is to give these comparisons an exact meaning.

It is not difficult to handle the nonoscillatory solutions of equations of type (1.1). If, for example, y is a nonoscillatory solution of

$$y^{(n)} + p(x)y = 0, \quad 0 \leqq x < \infty, \tag{1.4}$$

then there exists an integer k, $0 \leqq k \leqq n$, $(-1)^{n-k}p < 0$, such that

$$\left.\begin{array}{ll} y^{(i)} > 0, & i = 0, \ldots, k-1, \\ (-1)^{i-k}y^{(i)} > 0, & i = k, \ldots, n, \quad c < x < \infty, \end{array}\right\} \tag{1.5}$$

which enables us to estimate y near ∞. Similarly, if (1.1) is written in the canonical form of Trench,

$$\int^{\infty} \rho_i^{-1} = \infty, \qquad i = 1, \ldots, n-1$$

and y is nonoscillatory, then we have

$$\left.\begin{array}{ll} L_i(y) > 0, & i = 0, \ldots, k-1, \\ (-1)^{i-k}L_i y > 0, & i = k, \ldots, n, \quad c < x < \infty. \end{array}\right\} \tag{1.6}$$

Kim **[9, 10]** used (1.5) and (1.6) to classify the nonoscillatory solutions of (1.4) and (1.1). His classification is a particular case of that of **[2]**, where all solutions of (1.1), *both* oscillatory and nonoscillatory, are classified into disjoint sets. Several other authors were interested in related questions. Etgen *et al.* **[5]** compared some sets of oscillatory solutions of (1.1) with other sets of nonoscillatory solutions. Their results were formulated in terms of *dominance* of sets as defined by Dolan and Klaasen **[1]** as follows. The set A of solutions *dominates* another set B if $u \in A$ and $v \in B$ imply that $u + cv \in A$ for every c. Jones **[7]** studied questions of dominance for equation (1.1) and its adjoint. In **[3]** we made several attempts to compare solutions one with another; however, none of them now seems satisfactory. The common point of all the referenced works mentioned above is that in each, to varying degrees, nonoscillatory solutions have some special role.

In the present paper our aim is to compare various solutions *without any reference to their oscillation or nonoscillation*. This approach is based on the fact that for a suitably chosen basis of solutions, some minors of the Wronskian do not vanish, even if the associated solutions vanish infinitely many times. The solutions (1.3), for example, satisfy $W(y_{k-1}, y_k) \neq 0$ for $k = 1, 3, 5, 7$ and we have

$$W(y_i, y_j) = o(W(u_{k-1}u_k)) \quad \text{as} \quad x \to \infty$$

for every $i, j < k$, $k = 3, 5, 7$. Results of similar type will be proved for equation (1.1).

2. Definitions and background results

Nehari **[11]** observed that if k quasiderivatives of a solution of (1.1) vanish at $x = a$ and $n - k$ quasiderivatives vanish at $x = b$, $a < b$, then $(-1)^{n-k}p \leqq 0$.

Accordingly, an integer k, $0 \leqq k \leqq n$, will be called of *admissible parity* for equation (1.1) if

$$(-1)^{n-k} p(x) \leqq 0. \tag{2.1}$$

In the study of boundary value problems for (1.1) we found it useful to count not only the number of quasiderivatives which vanish at a point but also to consider the number of sign changes among the quasiderivatives $L_0 y, \ldots, L_n y$ at a point x. Namely, let $S(c_0, \ldots, c_n)$ denote the number of sign changes in the sequence $c_0, \ldots, c_n$ of nonzero terms. For a solution $y \not\equiv 0$ of (1.1) we define

$$S(y, x^+) = \lim_{\xi \to x^+} S(L_0 y(\xi), -L_1 y(\xi), \ldots, (-1)^n L_n y(\xi)),$$

$$S(y, x^-) = \lim_{\xi \to x^-} S(L_0 y(\xi), L_1 y(\xi), \ldots, L_n y(\xi)).$$

These quantities are closely related to boundary value problems. If, for example, y has at x_0 a zero of multiplicity q then

$$S(y, x_0^+) \geqq q, \; S(y, x_0^-) \geqq q.$$

As $L_n y = (-p/\rho_0) L_0 y$, it follows that $S(y, x^+)$, $n - S(y, x^-)$ are integers of admissible parities. Some of their useful properties are now summarised:

THEOREM 2.1 **[2, 3]**. (a) *Let $x_1, \ldots, x_r$ be the zeros of the quasiderivatives $L_0 y, \ldots, L_{n-1} y$ in (a, b). If $n(x_i)$ is the number of consecutive quasiderivatives (with $L_0 y$ following $L_{n-1} y$) which vanish at x_i and $\langle q \rangle$ denotes the greatest even integer not exceeding q, then*

$$S(y, a^+) + \sum_{a<x_i<b} \langle n(x_i) \rangle + S(y, b^-) \leqq n. \tag{2.2}$$

(b) *$S(y, x^+)$ is an integer valued, non-decreasing function of x on $[0, \infty)$.*

For various applications of $S(y, x^+)$ see, for example, **[5–7, 12]**.

Now we define a basis for the solution space of equation (1.1) by means of boundary value problems. Let $a < b$ be two fixed points in $[0, \infty)$. For every integer k, $0 \leqq k \leqq n$, of admissible parity we define $u_{k-1}(x, b)$ as the unique solution of (1.1) which satisfies

$$\left. \begin{aligned} &L_t u(a) = 0, && t = 0, \ldots, k-2, k, \\ &L_{k-1} u(a) = 1, && \\ &L_t u(b) = 0, && t = 0, \ldots, n-k-2 \end{aligned} \right\} \qquad (2.3)_{k-1}$$

and $u_k(x, b)$ as the unique solution which satisfies

$$\left. \begin{aligned} &L_t u(a) = 0, && t = 0, \ldots, k-2, k-1, \\ &L_k u(a) = 1, && \\ &L_t u(b) = 0, && t = 0, \ldots, n-k-2. \end{aligned} \right\} \qquad (2.3)_k$$

Note. Here and throughout the paper the following convention will be used: Whenever an admissible k and solutions u_{k-1}, u_k are mentioned, omit u_{-1} (if

$k=0$) or u_n (if $k=n$), which are not defined at all. It is proved in [**2**, **6**] that the set $\{u_0(x, b), \ldots, u_{n-1}(x, b)\}$ which is obtained in this way exists, is unique and is a basis of solutions for (1.1). In the next two theorems we summarise some properties of $u_0, \ldots, u_{n-1}$. Let us denote the Wronskian of (1.1) by

$$W(u_0, \ldots, u_{n-1}) = \det(L_i y_j) \quad i, j = 0, \ldots, n-1$$

and its minors by

$$W\begin{pmatrix} u_{i_1}, \ldots, u_{i_r} \\ j_1, \ldots, j_r \end{pmatrix} = \det(L_{i_s} u_{j_t}) \quad s, t = 1, \ldots, r.$$

THEOREM 2.2 [**2**, **6**]. *Let $u_0(x, b), \ldots, u_{n-1}(x, b)$ be the basis of the solution space of equation* (1.1) *as defined by* $(1.3)_0$–$(1.3)_{n-1}$. *If $l \leqq k$ are two integers of admissible parity then*

$$l \leqq S(c_{l-1}u_{l-1} + \ldots + c_k u_k, x^+) \leqq k \quad on\ [a, b), \tag{2.4}$$

$$W\begin{pmatrix} u_{l-1}, u_l, \ldots, u_k \\ i, i+1, \ldots, i+k-l+1 \end{pmatrix} \neq 0 \quad on\ (a, b) \tag{2.5}$$

(*recall the convention about u_{-1}, u_n given in the Note above*).

These are easy consequences of Theorem 2.1. As $u = c_{l-1}u_{l-1} + \ldots + c_k u_k$ has at least $l-1$ zeros at a, $S(u, a^+) \geqq l-1$. But $S(u, a^+)$ must be of admissible parity, hence $S(u, a^+) \geqq l$. Similarly $S(u, b^-) \geqq n-k$ and for x sufficiently close to b, $x < b$, $S(u, x^+) = n - S(u, x^-) = n - S(u, b^-) \leqq k$. Now (2.4) follows from the monotony of $S(u, x^+)$ on $[a, b]$. If the minor (2.5) vanished at $x_0 \in (a, b)$, there would be a combination $u = c_{l-1}u_{l-1} + \ldots + c_k u_k$ with $k-l+2$ consecutive quasiderivatives vanishing at x_0, contradicting (2.2).

Properties (2.4), (2.5) were utilised in [**2**–**4**, **6**, **7**] and it seems to us that the basis $u_0, \ldots, u_{n-1}$ fits in very well with the study of boundary value problems for (1.1).

From various points of view solutions of equation (1.1) and those of equation (1.4) have similar qualitative behaviour and the introduction of quasiderivatives does not reveal essentially new phenomena. However, in the next theorem one is almost forced to introduce quasiderivatives.

THEOREM 2.3 [**4**]. *Let $u_0(x, b), \ldots, u_{n-1}(x, b)$ be the basis of the solution space of* (1.1) *as defined by* $(2.3)_0$–$(2.3)_{n-1}$ *and $l \leqq k$ two integers of admissible parity. For every solution y of* (1.1), *the function*

$$z = W\begin{pmatrix} y, u_{l-1}, \ldots, u_k \\ 0, \quad 1, \ldots, k-l+2 \end{pmatrix} \Big/ W\begin{pmatrix} u_{l-1}, \ldots, u_k \\ 1, \ldots, k-l+2 \end{pmatrix}$$

is a solution of the differential equation

$$M_n z + p(x) z = 0, \quad a < x < b, \tag{2.6}$$

where $M_0 z = \sigma_0 z$, $M_i z = \sigma_i (M_{i-1}, z)'$, $\sigma_i > 0$ *on* (a, b), $i = 0, \ldots, n$ *and* $\sigma_0(x), \ldots, \sigma_n(x)$ *depend only on* $u_{l-1}, \ldots, u_k$ *and satisfy*

$$\sigma_0 \ldots \sigma_n = \rho_0 \ldots \rho_n.$$

Let $\bar{L}_0 y = \bar{\rho}_0 y,\ \bar{L}_i y = \bar{\rho}_i(\bar{L}_{i-1} y)',\ i = 1, \ldots, n, \ldots,$ *where*

$$\left.\begin{aligned} \bar{\rho}_i &= \rho_i, && i = 0, \ldots, n-1, \\ &= \rho_0 \rho_n |p|^{-1}, && i = n, \\ &= \rho_{i-n}, && i = n+1, \ldots. \end{aligned}\right\}$$

Then

$$M_i z = W_{\bar{L}}\begin{pmatrix} y, & u_{l-1}, \ldots, u_k \\ i, & i+1, \ldots, i+k-l+2 \end{pmatrix} \Big/ W_{\bar{L}}\begin{pmatrix} u_{l-1}, \ldots, u_k \\ i+1, \ldots, i+k-l+2 \end{pmatrix},$$
$$a < x < b, \quad i = 0, \ldots,$$

where $W_{\bar{L}}$ *denotes a minor which consists of quasiderivatives* $\bar{L}_i$. *A basis for the solution space of* (2.6) *is given by*

$$z_j = W\begin{pmatrix} y_j, & u_{l-1}, \ldots, u_k \\ 0, & 1, \ldots, k-l+2 \end{pmatrix} \Big/ W\begin{pmatrix} u_{l-1}, \ldots, u_k \\ 1, \ldots, k-l+2 \end{pmatrix}, \quad j = 0, \ldots, n-1,$$

where

$$\left.\begin{aligned} y_j &= u_j, && j \notin \{l-1, \ldots, k\}, \\ &= \text{a solution of } L_n y + py = pu_j, && j \in \{l-1, \ldots, k\}. \end{aligned}\right\}$$

Solutions $u_{l-1}, u_l, \ldots, u_k$ *will be called the pivotal block for equation* (2.6). *If* $r \leqq q$ *are of admissible parity such that* $\{l-1, \ldots, k\}$, $\{r-1, \ldots, q\}$ *are disjoint, then* $z_{r-1}, \ldots, z_q$ *satisfies*

$$r \leqq S_M(c_{r-1} z_{r-1} + \ldots + c_q z_q, x^+) \leqq q \quad \text{on } [a, b),$$

$$W_M\begin{pmatrix} z_{r-1}, & z_r, \ldots, z_q \\ & i, i+1, \ldots, i+q-r+1 \end{pmatrix} \neq 0 \quad \text{on } (a, b).$$

The subscript M will remind us that the quasiderivatives in question are $M_i z$.

Theorems 2.2, 2.3 concern the solutions $u_0(x, b), \ldots, u_{n-1}(x, b)$ on (a, b). When we study the behaviour of solutions near infinity, it is convenient to introduce another basis. Let

$$u_i(x) = \lim_{b \to \infty} u_i(x, b), \quad i = 0, \ldots, n-1, \tag{2.7}$$

where b tends to infinity through a suitably chosen sequence. The solutions $u_0(x), \ldots, u_{n-1}(x)$ which are defined by (2.7) are suitable for the study of equation (1.1) on (a, ∞). The next theorems summarise their properties.

THEOREM 2.4 **[2]**. *A basis for equation* (1.1) *is given by* $u_0(x), \ldots, u_{n-1}(x)$ *which are defined by* (2.7), *and for every* $l \leqq k$ *of admissible parity these satisfy*

$$l \leqq S(c_{l-1} u_{l-1}(x) + \ldots + c_k u_k(x), x^+) \leqq k \quad a \leqq x < \infty, \tag{2.8}$$

$$W\begin{pmatrix} u_{l-1}(x), \ldots, u_k(x) \\ i, \ldots, i+k-l+1 \end{pmatrix} \neq 0 \quad a < x < \infty. \tag{2.9}$$

THEOREM 2.5. *Let $u_0(x), \ldots, u_{n-1}(x)$ be defined by* (2.7) *and $l \leqq k$ be of admissible parity. For every solution y of* (1.1) *the function*

$$z = W\begin{pmatrix} y, & u_{l-1}, & \ldots, & u_k \\ 0, & 1, & \ldots, & k-l+2 \end{pmatrix} \Big/ W\begin{pmatrix} u_{l-1}, & \ldots, & u_k \\ 1, & \ldots, & k-l+2 \end{pmatrix}$$

is a solution of a differential equation

$$M_n z + p(x) z = 0, \quad a < x < \infty. \tag{2.10}$$

where $M_0 z = \sigma_0 z$, $M_i z = \sigma_i (M_{i-1} z)'$, $\sigma_i > 0$ on (a, ∞), $i = 0, \ldots, n$.

If $r \leqq q$ are admissible integers and $\{l-1, \ldots, k\}$, $\{r-1, \ldots, q\}$ are disjoint, then

$$M_i z_j = W_{\bar{L}}\begin{pmatrix} u_j, & u_{l-1}, & \ldots, & u_k \\ i, & i+1, & \ldots, & i+k-l+2 \end{pmatrix} \Big/ W_{\bar{L}}\begin{pmatrix} u_{l-1}, & \ldots, & u_k \\ i+1, & \ldots, & i+k-l+2 \end{pmatrix}, \quad j = r-1, \ldots, q$$

satisfy

$$r \leqq S_M(c_{r-1} z_{r-1} + \ldots + c_q z_q, x^+) \leqq q, \quad a \leqq x < \infty. \tag{2.11}$$

Proof. The first part of the theorem, namely that z is a solution of (2.10), is proved exactly as was Theorem (2.3) in **[4]**, except that inequality (2.4),

$$l \leqq S(c_{l-1} u_{l-1}(x, b) + \ldots + c_k u_k(x, b), x^+) \leqq k, \qquad a \leqq x < b.$$

has to be replaced in the proof by inequality (2.8). In order to prove (2.11) we put $z(x) = c_{r-1} z_{r-1}(x) + \ldots + c_q z_q(x)$. By the monotony of $S_M(z, x^+)$ it suffices to verify (2.11) on a dense subset of (a, ∞), say, the points x_0 such that $M_i z(x_0) = M_i(c_{r-1} z_{r-1} + \ldots + c_q z_q)(x_0) \neq 0$ for all $i = 0, \ldots, n$. Consider

$$M_i z(x_0) = W_{\bar{L}}\begin{pmatrix} c_{r-1} u_{r-1} + \ldots + c_q u_q, & u_{l-1}, & \ldots, & u_k \\ i, & i+1, & \ldots & \end{pmatrix} \Big/ W_{\bar{L}}\begin{pmatrix} u_{l-1}, & \ldots, & u_k \\ i+1, & \ldots & \end{pmatrix}(x_0)$$

and define

$$M_i z(x_0, b) \stackrel{\text{def}}{=} W_{\bar{L}}\begin{pmatrix} \sum_{r-1}^{q} c_j u_j(x_0, b), & u_{l-1}(x_0, b), & \ldots, & u_k(x_0, b) \\ i, & i+1, & \ldots, & i+k-l+2 \end{pmatrix} \Big/ W_{\bar{L}}\begin{pmatrix} u_{l-1}(x_0, b), & \ldots \\ i+1, & \ldots \end{pmatrix}.$$

For sufficiently large values of b, $M_i z(x_0)$ and $M_i z(x_0, b)$ are arbitrarily close and since $M_i z(x_0) \neq 0$, they have the same signature. As we know from Theorem 2.3 that $r \leqq S_M(z(x, b), x_0^+) \leqq q$ for $a \leqq x_0 < b$, it follows also that $r \leqq S_M(z, x_0^+) \leqq q$.

3. The dominance property

Our main result is the following:

THEOREM 3.1. *If k is of admissible parity and $i, j < k$ then*

$$W(u_i, u_j) / W(u_{k-1}, u_k) \tag{3.1}$$

is bounded as $x \to \infty$.

Proof. Let $u_0(x), \ldots, u_{n-1}(x)$ be the solutions of (1.1) which are defined by (2.7) and let us choose $\{u_0, \ldots, u_k\}$ as the pivotal block. According to Theorem 2.5 we obtain an associated equation

$$M_n z + p(x)z = 0, \qquad a < x < \infty, \tag{3.2}$$

and its basis of solutions $z_0, \ldots, z_{n-1}$ which are given by

$$z_j = W\binom{y_j, u_0, \ldots, u_k}{0, 1, \ldots, k+1} \Big/ W\binom{u_0, \ldots, u_k}{1, \ldots, k+1}, \quad j = 0, \ldots, n-1, \tag{3.3}$$

where

$$\left.\begin{aligned} y_j &= u_j, & j \notin \{k+1, \ldots, n-1\}, \\ &= \text{a solution of } L_n y + py = pu_j, & j \in \{0, \ldots, k\}. \end{aligned}\right\}$$

From [4, (2.17)] we have, for $j_1, j_2 \in \{0, \ldots, k\}$,

$$W_M\binom{z_{j_1}, z_{j_2}, z_{k+1}, \ldots, z_{n-1}}{i, i+1, \ldots, i+n-k}$$
$$= ((-1)^n \operatorname{sgn}(p))^{i+2} W_{\tilde{L}}\binom{u_{j_1}, u_{j_2}}{i, i+1} \Big/ W_{\tilde{L}}\binom{u_0 \qquad , \ldots, \qquad u_k}{i+n-k+l+1, \ldots, i+n-1}.$$

Hence

$$\frac{W_L(u_i, u_j)}{W_L(u_{k-1}, u_k)} = \frac{W_M(z_i, z_j, z_{k+1}, \ldots, z_{n-1})}{W_M(z_{k-1}, z_k, z_{k+1}, \ldots, z_{n-1})}, \qquad i, j \leqq k,$$

and we must prove only that the solutions of (3.2) satisfy

$$\limsup_{x \to \infty} |W_M(z_i, z_j, z_{k+1}, \ldots, z_{n-1})$$
$$/W_M(z_{k-1}, z_k, z_{k+1}, \ldots, z_{n-1})| < \infty, \; 0 < i, j < k. \tag{3.4}$$

The proof of (3.4) is based on the idea of [**3**, Theorem 10]. Firstly, we show a property of the adjoint of equation (3.2).

LEMMA 3.2. *Let*

$$\tilde{M}_n \zeta + (-1)^n p(x) \zeta = 0, \tag{3.5}$$

be the adjoint of equation (3.2), *where* $\tilde{M}_0 \zeta = \rho_n \zeta$, $\tilde{M}_i \zeta = \rho_{n-i}(\tilde{M}_{i-1}\zeta)'$, $i = 1, \ldots, n$ *and let* $\zeta_0(x), \ldots, \zeta_{n-1}(x)$ *be the solutions of* (3.5) *which are defined by*

$$((-1)^i \tilde{M}_{n-i-1} \zeta_{n-j-1}(x))^T = (M_i z_j(x))^{-1}. \tag{3.6}$$

Then for every l of admissible parity and $l > k$

$$n - l + 2 \leqq S_{\tilde{M}}(c_{n-l+1}\zeta_{n-l+1} + \ldots + c_{n-1}\zeta_{n-1}\zeta_{n-1}, x^+), \; a \leqq x < \infty. \tag{3.7}$$

Proof. In [**3**, Lemma 2] we associated a basis of equation (1.1) with a basis of the adjoint equation by an identity similar to (3.6). However, to prove (3.7) we must apply methods other than those of [**3**], since here $\{z_i\}$, $\{\zeta_i\}$ are defined not as solutions of boundary value problems but rather as minors, by (3.3) and (3.6).

Recall that z is a solution of (3.2) if $z=(M_0z,\dots,M_{n-1}z)^T$ is a solution of the system $z'=Az$, where $a_{i,i+1}=1/\sigma_{i+1}$, $i=0,\dots,n-2$, $a_{n-1,0}=-p/\sigma_0\sigma_n$. By the relation between the system $z'=Az$ and its adjoint $\zeta'=-A^T\zeta$, if $z_0,\dots,z_{n-1}$ is a basis for (3.2) then $\zeta_0,\dots,\zeta_{n-1}$ which are defined by (3.6) form a basis for (3.5). Let

$$B=(M_iz_j),\qquad B^{-1}=((-1)^i\tilde{M}_{n-i-1}\zeta_{n-j-1})^T.$$

Then

$$\left(\tilde{M}_{n-1}\Big(\sum_{n-l+1}^{n-1}c_t\zeta_t\Big),-\tilde{M}_{n-2}\Big(\sum_{n-l+1}^{n-1}c_t\zeta_t\Big),\dots,(-1)^{n-1}\tilde{M}_0\Big(\sum_{n-l+1}^{n-1}c_t\zeta_t\Big)\right)$$
$$=(c_{n-1},\dots,c_{n-l+1},0,\dots,0)B^{-1}.\quad(3.8)$$

To calculate $S(\sum_{n-l+1}^{n-1}c_t\zeta_t,x^+)$, count the number of sign changes in the sequence on the left-hand side of (3.8), which, for short will be denoted by $y_1,\dots,y_n$. But

$$(y_1,\dots,y_n)B=(c_{n-1},\dots,c_{n-l+1},0,\dots,0),$$

therefore

$$(y_1,\dots,y_n)\begin{pmatrix}M_0z_{l-1},\dots,M_0z_{n-1}\\ \\ M_{n-1}z_{l-1},\dots,M_{n-1}z_{n-1}\end{pmatrix}=(0,\dots,0).\qquad(3.9)$$

We shall use the following result [**8**, Chap. 5, Th. 2.1]:

THEOREM 3.3. *Let U be a $n\times m$ matrix, $n>m$, such that all its $n\times m$ minors are nonzero and have the same sign. If $xU=0$, then the number of sign changes among the nonzero components of x is not less than m.*

If this result can be applied to the system (3.9), it will follow that among $y_1,\dots,y_n$ there are at least $n-l+1$ changes of sign and so $S_{\tilde{M}}(\sum_{n-l+1}^{n-1}c_t\zeta_t,x^+)\geqq n-l+1$. However, as $S_{\tilde{M}}(\zeta,x^+)$ must be an integer of admissible parity for equation (3.5), we must have $S_{\tilde{M}}(\sum_{n-l+1}^{n-1}c_t\zeta_t,\ x^+)\geqq n-l+2$, as claimed in (3.7). So, all we need to confirm is that (3.9) fulfills the conditions of the above mentioned results, that is

$$W_M\begin{pmatrix}z_{l-1},\dots,z_{n-1}\\ i_1,\dots,i_{n-l+1}\end{pmatrix}\neq0,\qquad 0\leqq i_1<\dots<i_{n-l+1}\leqq n-1,\qquad(3.10)$$

and all these minors have the same sign on (a,∞).

If a minor in (3.10) vanishes at x_1 then there exists $z=\sum_{l-1}^{n-1}d_tz_t$ such that

$$M_tz(x_1)=0,\qquad t=i_1,\dots,i_{n-l+1}.$$

Hence, $S_M(z,x_1^-)\geqq n-l+1$. On the other hand, $l>k$ and the sets $\{0,\dots,k\}$, $\{l,\dots,n-1\}$ are disjoint. So in our case (with pivotal block $\{u_0,\dots,u_k\}$) (2.11) implies that $l\leqq S_M(\sum_{l-1}^{n-1}d_tz_t,x^+)$ for all $x\in(a,\infty)$. Now, $S_M(z,x_0^+)\geqq l$ and $S_M(z,x_1^-)\geqq n-l+1$ on $[x_0,x_1]\subset(a,\infty)$ contradict (2.2).

In order to prove that all the minors in (3.10) have the same sign, it suffices to show this for two minors with $n-l$ common rows such that in the remaining $(n-l+1)$-th row the corresponding indices of the quasiderivatives differ by 1. If two such minors have opposite signs, there exists $\lambda > 0$ such that

$$W_M\binom{z_{l-1}, \ldots, z_{n-1}}{i_1, \ldots, i_p, \ldots, i_{n-l+1}}(x_1) + \lambda W_M\binom{z_{l-1}, \ldots, z_{n-1}}{i_1, \ldots, i_p+1, \ldots, i_{n-l+1}}(x_1) = 0.$$

Hence there exists $z = \sum_{l-1}^{n-1} a_t z_t$ such that

$$\left.\begin{aligned} &M_t z(x_1) = 0, \qquad t = i_1, \ldots, i_p - 1, i_p + 1, \ldots, i_{n-l+i}, \\ &(M_{i_p} z + \lambda M_{i_p+1} z)(x_1) = 0. \end{aligned}\right\}$$

Since $\lambda > 0$, it follows that $M_t z M_{t+1} z\,|_{x_1-\varepsilon} < 0$ for $t = i_1, \ldots, i_p, \ldots, i_{n-l+1}$ and $S_M(z, x_1^-) \geqq n - l + 1$. But we have already seen above that this is impossible, so Lemma 3.2 is proved.

We return now to the proof of Theorem 3.1 and follow some of the ideas of [**3**, Theorem 10]. Take a nontrivial combination of the solutions $\zeta_{n-k}, \ldots, \zeta_{n-1}$ of equation (3.5),

$$\zeta = \sum_{n-k}^{n-1} c_t \zeta_t,$$

which satisfies the $k-1$ boundary value conditions

$$\tilde{M}_t \zeta(b) = 0, \quad t = 0, \ldots, k-2. \tag{3.11}$$

This solution $\zeta = \zeta(x, b)$ must satisfy

$$S_{\tilde{M}}(\zeta, x^+) \equiv n - k, \qquad a \leqq x < b. \tag{3.12}$$

First, $S_{\tilde{M}}(\zeta, b^-) \geqq k - 1$ by (3.11). Hence for x sufficiently close to b, $x < b$, $S_{\tilde{M}}(\zeta, x^+) = n - S_{\tilde{M}}(\zeta, x^-) = n - S_{\tilde{M}}(\zeta, b^-) \leqq n - k + 1$. Since $S_{\tilde{M}}(\zeta, x^+)$ must have an admissible parity with respect to equation (3.5), we have in fact $S_{\tilde{M}}(\zeta, x^+) \leqq n - k$ for $x < b$. On the other hand, we have by (3.7) (with $l = k + 2$ and $c_{n-k-1} = 0$) that $S_{\tilde{M}}(\zeta, x^+) \geqq n - k$ for $x \geqq a$. The last two inequalities and the monotony of $S_{\tilde{M}}(\zeta, x^+)$ verify (3.12).

We have $c_{n-k} \neq 0$, for if $c_{n-k} = 0$ then $\zeta = \sum_{n-k+1}^{n-1} c_t \zeta_t$ and so by (3.7) it satisfies $S_{\tilde{M}}(\zeta, x^+) \geqq n - k + 2$, in contradiction to (3.12). Thus, $c_{n-k} \neq 0$ and our $\zeta = \zeta(x, b)$ which satisfies (3.11) may be written as

$$\zeta(x, b) = \zeta_{n-k}(x) + \sum_{n-k+1}^{n-1} c_t(b) \zeta_t(x). \tag{3.13}$$

The boundary value conditions (3.11) at $x = b$ yield

$$\tilde{M}_i \zeta_{n-k}(b) + \sum_{n-k+1}^{n-1} c_t(b) \tilde{M}_i \zeta_t(b) = 0, \qquad i = 0, \ldots, k-2$$

and the solutions of this system are

$$c_t(b) = -\frac{W_{\tilde{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_{n-k}, \zeta_{t+1}, \ldots, \zeta_{n-1})}{W_{\tilde{m}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_t, \zeta_{t+1}, \ldots, \zeta_{n-1})}(b),$$
$$t = n - k + 1, \ldots, n - 1. \tag{3.14}$$

Similarly, we may construct a combination of $\zeta_{n-k-1}, \zeta_{n-k+1}, \ldots, \zeta_{n-1}$

$$\hat{\zeta}(x, b) = \zeta_{n-k-1}(x) + \sum_{n-k+1}^{n-1} d_t(b)\zeta_t(x) \tag{3.15}$$

which also satisfies boundary value conditions (3.11) and identity (3.12). The $d_t(b)-s$ are given by

$$d_t(b) = -\frac{W_{\tilde{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_{n-k-1}, \zeta_{t+1}, \ldots, \zeta_{n-1})}{W_{\tilde{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \quad \zeta_t, \zeta_{t+1}, \ldots, \zeta_{n-1})}(b),$$
$$t = n-k+1, \ldots, n-1. \tag{3.16}$$

Next, we show that $\{\zeta(x, b) \mid a < b < \infty\}$ is a bounded subset of the solution space. That is, for an arbitrary fixed $x_0 \in (a, \infty)$,

$$N(b) \overset{\text{def}}{=} \sum_{i=0}^{n-1} |\tilde{M}_i\zeta(x_0, b)|$$

remains bounded as $b \to \infty$. Indeed, as the solutions $\zeta(x, b)/N(b)$ are normalised at x_0, every sequence $b_i \to \infty$ has a subsequence $b_i' \to \infty$ such that $\zeta(x, b_i')/N(b_i')$, as well as its quasiderivatives, converges to a nontrivial solution ζ^* uniformly on compact sets. If $N(b_i') \to \infty$, then, by (3.13), ζ^* will be a linear combination of $\zeta_{n-k+1}, \ldots, \zeta_{n-1}$ only, and not ζ_{n-k}. Therefore, by (3.7),

$$S_{\tilde{M}}(\zeta^*, x^+) \geqq n-k+2, \qquad a < x < \infty.$$

But this is impossible. Take a point $x_1 \in (a, \infty)$ such that $\tilde{M}_t\zeta^*(x_1) \neq 0$ for all $t = 0, \ldots, n-1$ (almost all points are such). Since $\tilde{M}_t\zeta(x_1, b_i')/N(b_i') \to \tilde{M}_t\zeta^*(x_1) \neq 0$ when $b_i' \to \infty$, we must have $\operatorname{sgn} \tilde{M}_t\zeta(x_1, b_i')/N(b_i') = \operatorname{sgn} M_t\zeta^*(x_i)$, $t = 0, \ldots, n$, for sufficiently large values of b_i'. So $S_{\tilde{M}}(\zeta(x, b_i'), x_1^+) = S_{\tilde{M}}(\zeta^*, x_1^+) \geqq n-k+2$, contradicting (3.12). This proves that $N(b)$ is bounded.

But if the $\zeta(x, b)$ already form a bounded set of solutions as $b \to \infty$, then every sequence $b_i \to \infty$ has a subsequence $b_i' \to \infty$ such that the sequence $\zeta(x, b_i')$ converges to a nontrivial solution. Apply this to the representation (3.13) of $\zeta(x, b)$. It follows that for every $b_i \to \infty$, each of the sequences $\{c_t(b_i)\}_{i=1}^{\infty}$, $t = n-k+1, \ldots, n-1$, must have a convergent subsequence, hence $c_t(b)$ must be bounded as $b \to \infty$:

$$\limsup_{b\to\infty} \frac{W_{\tilde{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_{n-k}, \zeta_{t+1}, \ldots, \zeta_{n-1})}{W_{\tilde{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_t, \zeta_{t+1}, \ldots, \zeta_{n-1})} < \infty,$$
$$t = n-k+1, \ldots, n-1. \tag{3.17}$$

A similar argument applied to $\hat{\zeta}(x, b)$ and $d_t(b)$, $t = n-k+1, \ldots, n-1$, shows that

$$\limsup_{b\to\infty} \frac{W_{\tilde{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_{n-k+1}, \zeta_{t+1}, \ldots, \zeta_{n-1})}{W_{\tilde{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_t, \zeta_{t+1}, \ldots, \zeta_{n-1})} < \infty,$$
$$t = n-k+1, \ldots, n-1. \tag{3.18}$$

The minors of the inverse matrices

$$B^{-1} = ((-1)^i\tilde{M}_{n-i-1}\zeta_{n-j-1})^T, \qquad B = (M_iz_j),$$

satisfy

$$\frac{W_{\bar{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_{n-k}, \zeta_{t+1}, \ldots, \zeta_{n-1})}{W_{\bar{M}}(\zeta_{n-k+1}, \ldots, \zeta_{t-1}, \zeta_t, \zeta_t, \zeta_{t+1}, \ldots, \zeta_{n-1})} = \frac{W_M(z_{n-t-1}, z_k, \ldots, z_{n-1})}{W_M(z_{k-1}, z_k, \ldots, z_{n-1})}.$$

So, putting $i = n - t - 1$, (3.17) becomes

$$\limsup_{b\to\infty} |W_M(z_i, z_k, \ldots, z_{n-1})/W_M(z_{k-1}, z_k, \ldots, z_{n-1})| < \infty, \quad i = 0, \ldots, k$$

and (3.18) becomes

$$\limsup_{b\to\infty} |W_M(z_{k-1}, z_i, z_{k+1}, \ldots, z_{n-1})/W_M(z_{k-1}, z_k, \ldots, z_{n-1})| < \infty, \quad i = 0, \ldots, k.$$

Therefore, when we divide the identity [**8**, p. 7]

$$\begin{vmatrix} W_M(z_i, z_k, z_{k+1}, \ldots, z_{n-1}) & W_M(z_{k-1}, z_i, z_{k+1}, \ldots, z_{n-1}) \\ W_M(z_j, z_k, z_{k+1}, \ldots, z_{n-1}) & W_M(z_{k-1}, z_j, z_{k+1}, \ldots, z_{n-1}) \end{vmatrix} = W_M(z_i, z_j, z_{k+1}, \ldots, z_{n-1})W_M(z_{k-1}, z_k, z_{k+1}, \ldots, z_{n-1})$$

by $W_M(z_{k-1}, z_k, z_{k+1}, \ldots, z_{n-1})^2$, we find that (3.4) is bounded for $i, j \leqq k$. Thus, Theorem 3.1 is proved.

It would be interesting to know when the quotients $W(u_i, u_j)/W(u_{k-1}, u_k)$, $i, j < k$, are not only bounded but even tend to zero as $x \to \infty$.

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