

ZEROS OF SOLUTIONS AND OF WRONSKIANS FOR THE DIFFERENTIAL EQUATION $L_n y + p(x)y = 0$

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ABSTRACT. The equation which is studied here is $L_n y + p(x)y = 0$, $a \leq x \leq b$, where L_n is a disconjugate differential operator and $p(x)$ is of a fixed sign. We prove that certain solutions of the equation and corresponding odd-order minors of the Wronskian have an equal number of zeros, and we apply this property to oscillation problems.

1. INTRODUCTION

In a recent work [4], Etgen, Jones, and Taylor investigated ways of factoring the differential equation

$$(1.1) \quad L_n y + p(x)y = 0, \quad a < x < b,$$

where $p(x)$ is continuous and of one sign and L_n is disconjugate, into a product of lower order operators. Their investigation was based on relations between the nonvanishing of certain minors of the Wronskian and disconjugacy properties of equation (1.1). In this paper we extend the results of [4] and study, more generally, how many zeros certain minors of the Wronskian have. We shall show that certain odd order minors of the Wronskian and related solutions of (1.1) have the same number of zeros. This property will be interpreted in terms of the oscillation theory of (1.1).

Let the disconjugate differential operator L_n be

$$L_n y = \rho_n (\rho_{n-1} \cdots (\rho_1 (\rho_0 y)')' \cdots)',$$

with weight functions $\rho_i > 0$, $\rho_i \in C^{n-i}$, $i = 0, \dots, n$. We denote $L_0 y = \rho_0 y$ and $L_i y = \rho_i (L_{i-1} y)'$, $i = 1, \dots, n$, and call $L_0 y, \dots, L_n y$ the *quasiderivatives* of y .

Our results will be formulated in terms of a special basis $\{u_0(x), \dots, u_{n-1}(x)\}$ of the solution space of (1.1), which has been used in [1–5]. Since $u_0(x), \dots, u_{n-1}(x)$ will be defined in the next section by boundary value problems $(2.2)_0 - (2.2)_{n-1}$, we first mention a known property [7] of boundary value problems of

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equation (1.1): If k quasiderivatives of a solution y vanish at the endpoint a and $n - k$ quasiderivatives of y vanish at b , $a < b$, then $n - k$ must be odd if $p(x) > 0$ and even if $p(x) < 0$. Therefore we shall say that an integer k , $0 \leq k \leq n$, is of *admissible parity* if

$$(1.2) \quad (-1)^{n-k} p(x) < 0.$$

In [4], the authors studied the nonvanishing of certain minors of odd order of the Wronskian. Theorems 7, 8 and 13, 14 of [4] motivate our work. For example,

Theorem [4, Theorems 13, 14]. *Let $l < k$ be of admissible parity and $u_{l-1}, \dots, u_{k-2}, u_{k-1}$ elements of the basis which is defined by (2.2). If equation (1.1) is $(k, n - k)$ -disconjugate on $[a, b]$, then*

$$W(u_{l-1}, \dots, u_{k-2}, u_{k-1}) \neq 0, \quad a < x < b.$$

Conversely, if (1.1) is not $(k, n - k)$ -disconjugate on $[a, b]$ and b is not a $(k, n - k)$ -type conjugate point of $x = a$, then $W(u_{l-1}, \dots, u_{k-2}, u_{k-1})$ must have a zero on (a, b) .

Our aim is to generalize this result:

Theorem 1.1. *Let $l < k, q$ be of admissible parity. If $q \geq k$ then for arbitrary constants α, β , not both zero, the solution $\alpha u_{q-1} + \beta u_q$ and the minor $W(u_{l-1}, \dots, u_{k-2}, \alpha u_{q-1} + \beta u_q)$ have an equal number of zeros, all of them simple, in (a, b) . If $q \leq l - 2$, then the numbers of the above-mentioned zeros differ at most by 1.*

Theorem 1.1 has a simple interpretation in view of the results of [2]. In [2] we proved that for every solution y of (1.1), the function

$$z = W(u_{l-1}, \dots, u_{k-2}, y)$$

is a solution of an equation

$$(1.3) \quad M_n z + p(x)z = 0, \quad a < x < b,$$

where $M_n z = \sigma_n (\sigma_{n-1} \cdots (\sigma_1 (\sigma_0 z)') \cdots)'$, $\sigma_0, \dots, \sigma_n$ depend only on u_{l-1}, \dots, u_{k-2} , and

$$\sigma_0 \cdots \sigma_n = \rho_0 \cdots \rho_n.$$

We called $\{u_{l-1}, \dots, u_{k-2}\}$ a *pivotal block* and equation (1.3) the equation *associated* with equation (1.1) and this pivotal block. Let us denote

$$z_i = W(u_{l-1}, \dots, u_{k-2}, u_i), \quad i \notin \{l-1, \dots, k-2\}.$$

In these terms Theorem 1.1 may be stated as follows:

Theorem 1.2. *For arbitrary constants α, β the solution $\alpha u_{q-1} + \beta u_q$ of (1.1) and the solution $\alpha z_{q-1} + \beta z_q = W(u_{l-1}, \dots, u_{k-2}, \alpha u_{q-1} + \beta u_q)$ of the associated equation (1.3) have an equal number of zeros in (a, b) if $q \geq k$, and the number of these zeros differ at most by 1 if $q \leq l - 2$.*

2. DEFINITIONS

Before we begin the actual proof, we list some definitions and results from [1,2]. Let $S(c_0, \dots, c_n)$ denote the number of sign changes in the sequence c_0, \dots, c_n of nonzero numbers. For a solution y of (1.1) we define

$$S(y, x^+) = \lim_{\xi \rightarrow x^+} S(L_0 y(\xi), -L_1 y(\xi), \dots, (-1)^n L_n y(\xi)),$$

$$S(y, x^-) = \lim_{\xi \rightarrow x^-} S(L_0 y(\xi), L_1 y(\xi), \dots, L_n y(\xi)).$$

The quantities $S(y, x^+)$, $S(y, x^-)$ are closely related to boundary value problems of (1.1). For example, if y has a zero of multiplicity q at x_0 , then obviously

$$S(y, x_0^+) \geq q, \quad S(y, x_0^-) \geq q.$$

We summarize some of their useful properties:

Proposition 2.1 [1]. (a) Let x_1, \dots, x_r be the zeros of the quasiderivatives $L_0 y, \dots, L_{n-1} y$ in (a, b) , let $n(x_i)$ be the multiplicity of the zero at x_i , and let $\langle q \rangle$ be the greatest even integer not exceeding q . Then

$$(2.1) \quad S(y, a^+) + \sum_{a < x_i < b} \langle n(x_i) \rangle + S(y, b^-) \leq n.$$

(b) $S(y, x^+)$ is an integer valued, nondecreasing function of x .

(c) $S(y, x^+)$ and $n - S(y, x^-)$ are integers of admissible parities.

Now we define a basis for the solution space of (1.1) by means of n boundary value problems. For every admissible k we define $u_{k-1}(x)$ as the unique solution of (1.1) which satisfies the system of boundary value conditions

$$(2.2)_{k-1} \quad \begin{aligned} L_t u(a) &= 0, & t = 0, \dots, k-2, k, \\ L_{k-1} u(a) &= 1, \\ L_t u(b) &= 0, & t = 0, \dots, n-k-2, \end{aligned}$$

and $u_k(x)$ as the unique solution which satisfies

$$(2.2)_k \quad \begin{aligned} L_t u(a) &= 0, & t = 0, \dots, k-1, \\ L_k u(a) &= 1, \\ L_t u(b) &= 0, & t = 0, \dots, n-k-2. \end{aligned}$$

Here and throughout the paper the following convention will be used: *Whenever an admissible k and solutions u_{k-1} , u_k are mentioned, omit u_{-1} (if $k=0$) or u_n (if $k=n$) which are not defined at all.*

It is proved in [1,4] that the set $\{u_0, \dots, u_{n-1}\}$ which is defined in this way, exists, it is unique, and it is a basis of solutions of (1.1). This basis had been used extensively in [1-5].

Let the Wronskian be

$$W(u_0, \dots, u_{n-1}) = \det(L_i u_j)_{i,j=0,\dots,n-1}$$

and let its minors be denoted by

$$W \begin{pmatrix} u_{i_1}, \dots, u_{i_r} \\ j_1, \dots, j_r \end{pmatrix} = \det(L_{j_s} u_{i_t}(x))_{s,t=1,\dots,r}.$$

The properties of the basis u_0, \dots, u_{n-1} which will be needed are the following:

Proposition 2.2 [1, 4]. *Let u_0, \dots, u_{n-1} be the basis of solutions as defined by (2.2)₀–(2.2)_{n-1}. If $r \leq q$ are of admissible parities, then for arbitrary constants c_{r-1}, \dots, c_q*

$$(2.3) \quad r \leq S(c_{r-1}u_{r-1} + \dots + c_q u_q, x^+) \leq q \quad \text{on } [a, b),$$

$$(2.4) \quad W \begin{pmatrix} u_{r-1}, \dots, u_q \\ i, \dots, i+q-r+1 \end{pmatrix} \neq 0, \quad i = 0, \dots, n-q+r-2, \quad \text{on } (a, b)$$

(recall the convention about u_{-1}, u_n !).

Most frequently we shall use that for admissible $l < k$,

$$(2.5) \quad W(u_{l-1}, u_l, \dots, u_{k-1}, u_k) \neq 0, \quad a < x < b.$$

The results of [2] about the associated equation are the following:

Proposition 2.3 [2]. *Let $l \leq k$ be two integers of admissible parity. For every solution y of (1.1), the function*

$$z = W(u_{l-1}, \dots, u_k, y)$$

is a solution of the differential equation

$$(2.6) \quad M_n z + p(x)z = 0, \quad a < x < b,$$

where $M_0 z = \sigma_0 z$, $M_i z = \sigma_i (M_{i-1} z)'$, $\sigma_i > 0$ on (a, b) , $i = 0, \dots, n$, and $\sigma_0(x), \dots, \sigma_n(x)$ depend only on u_{l-1}, \dots, u_k and satisfy

$$\sigma_0 \cdots \sigma_n = \rho_0 \cdots \rho_n.$$

If $r \leq q$ are of admissible parity such that $\{l-1, \dots, k\}$, $\{r-1, \dots, q\}$ are disjoint, then the solutions

$$z_j = W(u_{l-1}, \dots, u_k, u_j), \quad j = r-1, \dots, q,$$

of (2.6) satisfy for arbitrary constants c_{r-1}, \dots, c_q

$$(2.7) \quad r \leq S_M(c_{r-1}z_{r-1} + \dots + c_q z_q, x^+) \leq q \quad \text{on } [a, b),$$

(2.8)

$$W_M \begin{pmatrix} z_{r-1}, \dots, z_q \\ i, \dots, i+q-r+1 \end{pmatrix} \neq 0, \quad i = 0, \dots, n-q+r-2, \quad \text{on } (a, b),$$

(Here and elsewhere the subscripts M, L will remind us that the quasiderivatives in question are $M_i z, L_i u$ respectively.)

It should be emphasized that (2.6) is singular at $x = a$ and $x = b$. The quasiderivatives $M_i z$ are given by

(2.9)

$$M_i z = W \begin{pmatrix} u_{l-1}, u_l, \dots, u_k, y \\ i, i+1, \dots, i+k-l+2 \end{pmatrix} / W \begin{pmatrix} u_{l-1}, \dots, u_k \\ i+1, \dots, i+k-l+2 \end{pmatrix},$$

where $i = 0, \dots, n - k + 1 - 3$ and $a < x < b$. When $i + k - 1 + 2 \geq n$, the explicit expression for $M_i z$ is slightly different (see [2]). Since we do not need it here, it is omitted.

3. PROOF OF THEOREM 1.1 FOR $q = k$

The proof of Theorem 1.1 is divided into several short propositions.

Proposition 3.1. $\alpha u_{q-1} + \beta u_q$ and $\alpha z_{q-1} + \beta z_q$ have no multiple zeros in (a, b) .

Indeed, if we take $r = q$ in (2.3) and (2.6), we get

$$(3.1) \quad S_L(\alpha u_{q-1} + \beta u_q, x^+) \equiv S_M(\alpha z_{q-1} + \beta z_q, x^+) \equiv q, \quad a < x < b.$$

Consequently,

$$S(\alpha u_{q-1} + \beta u_q, a^+) = q,$$

$$\begin{aligned} S(\alpha u_{q-1} + \beta u_q, b^-) &= S(\alpha u_{q-1} + \beta u_q, (b - \varepsilon)^-) \\ &= n - S(\alpha u_{q-1} + \beta u_q, (b - \varepsilon)^+) = n - q \end{aligned}$$

and, by (2.1), the solution $\alpha u_{q-1} + \beta u_q$ of (1.1) may have no multiple zeros in (a, b) . The same argument shows that the solution $\alpha z_{q-1} + \beta z_q$ of (1.3) also has no multiple zeros in (a, b) . \square

From here on we separate the cases $q = k$ and $q \neq k$. In this section we prove the theorem for $q = k$ only, while the case $q > k$ will be discussed in the following section. The reason for this separation will be clarified soon.

Let x_0 be a zero of $\alpha z_{k-1} + \beta z_k$, that is

$$W(u_{l-1}, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k)(x_0) = 0.$$

Then there exists a combination $c_{l-1}u_{l-1} + \dots + c_{k-2}u_{k-2} + c_{k-1}(\alpha u_{k-1} + \beta u_k)$ which has a zero of multiplicity $k - l + 1$ at x_0 . Now, for every $s \in (a, b)$, we take a combination of $u_{l-1}, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k$ which has at $x = s$ a zero of multiplicity at least $k - l$. Such a combination obviously exists and as $W(u_{l-1}, \dots, u_{k-2})(s) \neq 0$ for all $a < s < b$, the combination is essentially unique and may be written, up to a constant

$$(3.2) \quad u(x, s) = \begin{vmatrix} u_{l-1}(x), & \dots, & u_{k-2}(x), & \alpha u_{k-1} + \beta u_k(x) \\ L_0 u_{l-1}(s), & \dots, & L_0 u_{k-2}(s), & L_0(\alpha u_{k-1} + \beta u_k)(s) \\ \vdots & & & \vdots \\ L_{k-l-1} u_{l-1}(s), & \dots, & & L_{k-l-1}(\alpha u_{k-1} + \beta u_k)(s) \end{vmatrix}.$$

More explicitly,

$$(3.3) \quad u(x, s) = c_{l-1}(s)u_{l-1} + \dots + c_{k-2}(s)u_{k-2} + c_{k-1}(s)(\alpha u_{k-1} + \beta u_k)$$

and

$$c_{k-1}(s) = W(u_{l-1}, \dots, u_{k-2})(s) \neq 0.$$

By (2.3), $u(x, s)$ satisfies $S(u, a^+) \geq l$, $S(u, b^-) \geq n - k$. At $x = s$, $u(x, s)$

has a zero of multiplicity $k-l$ and $k-l$ is even, therefore $S(u, a^+) + \langle n(s) \rangle + S(u, b^-) \geq l + (k-l) + (n-k) = n$. It follows by (2.1) that $u(x, s)$ and its quasiderivatives have no multiple zeros in $(a, s) \cup (s, b)$. (If $\alpha u_{k-1} + \beta u_k$ is replaced in (3.2) by $\alpha u_{q-1} + \beta u_q$, $q > k$, the last conclusion is definitely false. This is why we separate the cases $q = k$ and $q > k$). By the same reason it follows also from (2.1) that the multiplicity of the zero at $x = s$ can be either $k-l$ or $k-l+1$ but not more. Now,

$$L_{k-l}u(x, s)|_{x=s} = W(u_{l-1}, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k)(s).$$

Therefore, the multiplicity of the zero of $u(x, s)$ at s is exactly $k-l+1$ if and only if s is a (simple) zero of $\alpha z_{k-1} + \beta z_k$. For every other $s \in (a, b)$, $u(x, s)$ has at $x = s$ a zero of exact multiplicity $k-l$. This characterization will be used to count the zeros of $\alpha z_{k-1} + \beta z_k$ by tracing the zeros of $u(x, s)$. More exactly, in the next propositions the following will be proved: $u(x, s)$ has for $s \rightarrow a^+$ as many simple zeros in (s, b) as $\alpha u_{k-1} + \beta u_k$ has in (a, b) ; as s increases, zeros of $u(x, s)$ leave (s, b) one by one whenever s crosses a zero of $\alpha z_{k-1} + \beta z_k$; when $s \rightarrow b^-$, $u(x, s)$ has no zeros in (s, b) .

Proposition 3.2. *One simple zero of $u(x, s)$ passes from (s, b) to (a, s) when and only when s crosses a zero of $\alpha z_{k-1} + \beta z_k$ in (a, b) from left to right. This is the only way by which the number of the zeros of $u(x, s)$ in (s, b) may change as s varies in (a, b) .*

Let $(\alpha z_{k-1} + \beta z_k)(s_0) = 0$. Then $u(x, s_0)$ has at $x = s_0$ a zero of exact multiplicity $k-l+1$, while for s close to s_0 , $s \neq s_0$, $u(x, s)$ has at $x = s$ a zero of exact multiplicity $k-l$. By continuity arguments, when s is sufficiently close to s_0 , $u(x, s)$ has a unique simple zero, $\zeta(s)$, near s_0 , such that $\zeta(s_0) = s_0$. By the same arguments $L_{k-l}u(x, s)$ has also a unique, simple zero, say $\bar{\zeta}(s)$, near s_0 , and $\bar{\zeta}(s_0) = s_0$. If we apply Rolle's theorem successively to $L_0(x, s), \dots, L_{k-l}(x, s)$, we see that $\bar{\zeta}(s)$ has to be located between $\zeta(s)$ and s .

We want to show that $\zeta(s) < s$ when $s > s_0$ and $\zeta(s) > s$ when $s < s_0$. Since $\bar{\zeta}(s)$ is between $\zeta(s)$ and s , it suffices to show that $s - \bar{\zeta}(s)$ increases near s_0 . Recall that $\bar{\zeta}(s)$ is the simple zero of $L_{k-l}u(x, s)$ which satisfies $\bar{\zeta}(s_0) = s_0$. By the implicit function theorem

$$\left. \frac{d\bar{\zeta}}{ds} \right|_{s_0} = - \frac{\partial}{\partial s} L_{k-l}u(x, s) \Big/ \frac{\partial}{\partial x} L_{k-l}u(x, s) \Big|_{x=s=s_0}.$$

It follows from (3.2) that $(\partial/\partial s)L_{k-l}(x, s)|_{x=s} = 0$. The denominator

$$\frac{\partial}{\partial x} L_{k-l}u(x, s)|_{x=s=s_0} = \rho_{k-l+1}^{-1}(s_0) L_{k-l+1}u(s_0, s_0)$$

is not zero, otherwise $u(x, s_0)$ would have at $x = s_0$ a zero of multiplicity $k-l+2$, which we already saw to be impossible. Therefore $\bar{\zeta}'(s_0) = 0$, $(d/ds)(s - \bar{\zeta}(s))(s_0) = 1$, and the first part of the proposition is proved.

The number of the zeros of $u(x, s)$ in (s, b) may change when s varies in (a, b) only as described above. Indeed, two zeros cannot merge at a point of (s, b) since $u(x, s)$ has no multiple zeros in (s, b) . Neither can a zero of $u(x, s)$ tend to b as s varies in (a, b) . To show this, observe that by the boundary conditions (2.2), $u(x, s)$ satisfies

$$\begin{aligned} L_t u(b) &= 0, & t &= 0, \dots, n-k-2, \\ L_{n-k-1} u(b) &= c_{k-1}(s)(\alpha L_{n-k-1} u_{k-1} + \beta L_{n-k-1} u_k)(b), \\ L_{n-k} u(b) &= c_{k-1}(s)(\alpha L_{n-k} u_{k-1} + \beta L_{n-k} u_k)(b), \end{aligned}$$

and $c_{k-1}(s) \neq 0$ for $a < s < b$. Thus, the multiplicity of the zero of $u(x, s)$ at b (which is either $n-k-1$ or $n-k$) is equal to that of $\alpha u_{k-1} + \beta u_k$ and therefore is independent of s . Consequently, no additional zero of $u(x, s)$ meets b as s varies in (a, b) . \square

Next we have to discuss the situation as $s \rightarrow a^+$ and $s \rightarrow b^-$.

Proposition 3.3. *When s is sufficiently close to a , $u(x, s)$ has in (s, b) exactly as many zeros as $\alpha u_{k-1} + \beta u_k$ has in (a, b) for all but possibly two values of α/β . When s is sufficiently close to b , $u(x, s)$ has no zeros in (s, b) for all but possibly one value of α/β .*

When $s = a$ or $s = b$, the determinant (3.2) is identically zero (unless $l = 0, 1$). Therefore we normalize $u(x, s)$ and define

$$\bar{u}(x, s) = [c_{l-1}(s)u_{l-1} + \dots + c_{k-2}(s)u_{k-2} + c_{k-1}(s)(\alpha u_{k-1} + \beta u_k)] / \sum_{l-1}^{k-1} |c_i(s)|.$$

By the normalization, each sequence $s_i \rightarrow a^+$ has a subsequence s_{i_j} such that $u(x, s_{i_j})$ converges uniformly to a nontrivial solution. $u(x, s)$ has at $x = a, s$, and b zeros of multiplicities at least $l-1, k-l$, and $n-k-1$, respectively, and $\bar{u}(x, s)$ has the same ones. Hence, $\lim_{s \rightarrow a^+} \bar{u}(x, s)$ (or the limit of a subsequence) has at least $(l-1) + (k-l) = k-1$ zeros at $x = a$. On the other side, the limit solution is spanned by $u_{l-1}, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k$, so it may have a zero of multiplicity $k-1$ at a if and only if, up to a constant factor,

$$\lim_{s \rightarrow a^+} \bar{u}(x, s) = \alpha u_{k-1} + \beta u_k.$$

In order to prove that for s sufficiently close to a , $u(x, s)$ and $\alpha u_{k-1} + \beta u_k$ have an equal number of zeros in (s, b) , it suffices to show that no zero of $u(x, s)$, except s , tends to the endpoints a or b when $s \rightarrow a^+$. This is definitely the case if

$$(3.4) \quad \alpha/\beta \neq 0, \quad \alpha/\beta \neq -L_{n-k-1} u_{k-1}(b)/L_{n-k-1} u_k(b).$$

For (3.4) ensures that

$$(3.5) \quad L_{k-1}(\alpha u_{k-1} + \beta u_k)(a) \neq 0, \quad L_{n-k-1}(\alpha u_{k-1} + \beta u_k)(b) \neq 0$$

and $\alpha u_{k-1} + \beta u_k$ has exactly $k-1$ zeros at a and exactly $n-k-1$ zeros at b . If $u(x, s)$ would have for every s a zero near a (or near b), in addition to its zeros at a , s , and b , then the limit solution $\lim_{s \rightarrow a^+} \bar{u}(x, s) = \alpha u_{k-1} + \beta u_k$ would have more zeros at a (or at b) than is possible by (3.5). Thus, assuming (3.4), the first part of Proposition 3.3 is verified.

Now consider $s \rightarrow b^-$. By a similar argument, $\lim_{s \rightarrow b^-} \bar{u}(x, s)$ has at least $l-1$ zeros at a and at least $(k-l) + (n-k-1) = n-l-1$ zeros at b . Hence we must have

$$\lim_{s \rightarrow b^-} \bar{u}(x, s) = \gamma u_{l-1} + \delta u_l.$$

In order to prove that for s close to b , $u(x, s)$ has no zero in (s, b) , it suffices to show that no zero of $u(x, s)$ tends to b when $s \rightarrow b^-$. This happens when

$$(3.6) \quad L_{n-l-1}(\gamma u_{l-1} + \delta u_l)(b) \neq 0.$$

For, if $u(x, s)$ would have a zero near b , when $s \rightarrow b^-$, in addition to the zeros at s and at b , then $\lim_{s \rightarrow b^-} \bar{u}(x, s) = \gamma u_{l-1} + \delta u_l$ would have more than $n-l-1$ zeros at b , in contradiction with (3.6).

When is (3.6) satisfied? By (3.2),

$$u(x, s) = W(u_l, u_{l+1}, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k)(s) u_{l-1}(x) \\ + W(u_{l-1}, u_{l+1}, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k)(s) u_l(x) + \dots.$$

So

$$\gamma/\delta = \lim_{s \rightarrow b^-} \frac{W(u_l, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k)}{W(u_{l-1}, u_{l+1}, \dots, \alpha u_{k-1} + \beta u_k)(s)}$$

and γ/δ is a bilinear function of α/β . Therefore (3.6) holds for all but a single value of α/β :

$$(3.7) \quad \alpha/\beta \neq \alpha_3/\beta_3.$$

This completes the proof of Proposition 3.3. \square

We can summarize the situation as follows: For all values of α/β except the three which are specified in (3.4) and (3.7), $u(x, s)$ has for $s \rightarrow a^+$ as many simple zeros in (s, b) as $\alpha u_{k-1} + \beta u_k$ has in (a, b) ; as s increases, zeros of $u(x, s)$ leave (s, b) one by one whenever s crosses a zero of $\alpha z_{k-1} + \beta z_k$; and when $s \rightarrow b^-$, $u(x, s)$ has no zeros in (s, b) . Consequently, $\alpha z_{k-1} + \beta z_k$ and $\alpha u_{k-1} + \beta u_k$ have equal number of (simple) zeros in (a, b) .

To complete the proof of our theorem in the case $q = k$ it remains only to remove the restrictions (3.4), (3.7). Let α_1/β_1 be one of the three exceptional values of α/β , say $\alpha_1/\beta_1 = -L_{n-k-1}u_{k-1}(b)/L_{n-k-1}u_k(b)$. Assuming $\beta_1 \neq 0$,

$$\alpha u_{k-1} + \beta u_k = \beta/\beta_1(\alpha_1 u_{k-1} + \beta_1 u_k) + \beta(\alpha/\beta - \alpha_1/\beta_1)u_{k-1}.$$

For $\alpha/\beta \neq \alpha_1/\beta_1$, α/β close to α_1/β_1 , $L_{n-k-1}(\alpha u_{k-1} + \beta u_k)(b) \neq 0$ and $\alpha u_{k-1} + \beta u_k$ has at $x = b$ a zero of exact multiplicity $n-k-1$. On the other hand $L_{n-k-1}(\alpha_1 u_{k-1} + \beta_1 u_k)(b) = 0$ and $\alpha_1 u_{k-1} + \beta_1 u_k$ has at $x = b$ a zero

of exact multiplicity $n - k$, for a zero of bigger multiplicity would contradict (2.3). Therefore,

$$(3.8) \quad \begin{aligned} \alpha u_{k-1} + \beta u_k &= \beta / \beta_1 (x - b)^{n-k} [A + O(1)] \\ &+ \beta (\alpha / \beta - \alpha_1 / \beta_1) (x - b)^{n-k-1} [B + O(1)], \end{aligned}$$

where $A, B \neq 0$. It follows that for α / β close to α_1 / β_1 , $\alpha u_{k-1} + \beta u_k$ has a simple zero close to b and when α / β crosses α_1 / β_1 , the simple zero of $\alpha u_{k-1} + \beta u_k$ passes from one side of b to the other. The direction of this passage depends, of course, on the signs of the constants in (3.8).

Now, when $\alpha / \beta \neq \alpha_1 / \beta_1$, it had been already proved that $\alpha u_{k-1} + \beta u_k$ and $\alpha z_{k-1} + \beta z_k$ have the same number of zeros in (a, b) . Therefore, when a zero of $\alpha u_{k-1} + \beta u_k$ passes through b during the change of α / β , a zero of $\alpha z_{k-1} + \beta z_k$ has to do the same. It follows that the number of the zeros of $\alpha u_{k-1} + \beta u_k$ and $\alpha z_{k-1} + \beta z_k$ is equal also for $\alpha / \beta = \alpha_1 / \beta_1$. This completes the proof of Theorem 1.1 for $q = k$.

4. PROOF OF THEOREM 1.1 FOR $q \neq k$

Recall the notation

$$(4.1) \quad z_j = W(u_{l-1}, \dots, u_{k-2}, u_j), \quad j \notin \{l-1, \dots, k-2\},$$

$$(4.2) \quad \alpha z_{q-1} + \beta z_q = W(u_{l-1}, \dots, u_{k-2}, \alpha u_{q-1} + \beta u_q).$$

Theorem 1.1 for $q > k$ will follow from the following

Proposition 4.1. *The functions $\alpha z_{q-1} + \beta z_q$ and*

$$W_M(z_{k-1}, \dots, z_{q-2}, \alpha z_{q-1} + \beta z_q)$$

have an equal number of zeros, all simple, in (a, b) .

Suppose momentarily that Proposition 4.1 had been proved. By the discrete Wronskian identity [6, p. 61] applied to (4.1) and (4.2) we get

$$\begin{aligned} &W_M(z_{k-1}, \dots, z_{q-2}, \alpha z_{q-1} + \beta z_q) \\ &= W_L(u_{l-1}, \dots, u_{k-2}, u_{k-1}, \dots, u_{q-2}, \alpha u_{q-1} + \beta u_q) \\ &\quad \times W_L \begin{pmatrix} u_{l-1}, \dots, u_{k-2} \\ 1, \dots, k-l \end{pmatrix} \dots W_L \begin{pmatrix} u_{l-1}, \dots, u_{k-2} \\ q-k, \dots, q-l+1 \end{pmatrix}. \end{aligned}$$

Since

$$W \begin{pmatrix} u_{l-1}, \dots, u_{k-2} \\ i, \dots, i+k-l+1 \end{pmatrix} \neq 0 \quad \text{in } (a, b),$$

it follows from Proposition 4.1 that $\alpha z_{q-1} + \beta z_q$ and $W(u_{l-1}, \dots, u_{q-2}, \alpha u_{q-1} + \beta u_q)$ have an equal number of zeros in (a, b) . However, the last minor is exactly the one which is obtained when one chooses $\{u_{l-1}, \dots, u_{q-2}\}$ as the pivotal block and so we encounter the situation which had been studied in the previous section! It follows by the result of the previous section that

$W(u_{l-1}, \dots, u_{q-2}, \alpha u_{q-1} + \beta u_q)$ has in (a, b) as many zeros as $\alpha u_{q-1} + \beta u_q$ has, all of them simple. So Proposition 4.1 implies Theorem 1.1.

Thus, we have only to prove Proposition 4.1. Now, Proposition 4.1 reminds us of Theorem 1.1 with $q = k$, when the solutions $u_{l-1}, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k$ of (1.1) are replaced by the solutions $z_{k-1}, \dots, z_{q-2}, \alpha z_{q-1} + \beta z_q$ of (2.5). However, there is an essential difference: while u_{l-1}, \dots, u_k are defined as solutions of boundary value problems, z_{k-1}, \dots, z_q are defined by determinants. Nevertheless, Proposition 4.1 will be verified by repeating most of the arguments of the previous section literally. Difficulties arise only when we consider zeros of solutions z_j near the endpoints a and b . Since a and b are singular points for equation (2.6), the multiplicity of the zero of a function $M_i z$ is not necessarily equal to the number of quasiderivatives $M_i z, M_{i+1} z, \dots$ which vanish there; and when zeros approach $x = a$ or b , we cannot simply sum up the number of vanishing quasiderivatives without a careful examination. Therefore we begin with a result about boundary values of minors at a and b .

Lemma [2, Lemma 3.1]. *If $l \leq k$ are of admissible parity, $0 \leq \alpha_{l-1} < \dots < \alpha_k \leq n - 1$, then the function*

$$W \begin{pmatrix} u_{l-1}, \dots, u_k \\ \alpha_{l-1}, \dots, \alpha_k \end{pmatrix}$$

has exactly $\sum_{i=1}^k (i - \alpha_i)_+$ zeros at $x = a$ and exactly $\sum_{i=1}^k (n - k + l + i - \alpha_i)_+$ zeros at $x = b$. If $r > k$ and $\alpha_k < \alpha_r \leq n - 1$, then

$$W \begin{pmatrix} u_{l-1}, \dots, u_k, u_r \\ \alpha_{l-1}, \dots, \alpha_k, \alpha_r \end{pmatrix}$$

has at least $\sum_{i=1}^k (i - \alpha_i)_+ + (r - \alpha_r)_+$ zeros at $x = a$ and at least $(n - \hat{r} - 1 - \alpha_{l-1}) + \sum_{i=1}^k (n - k + 1 + i - \alpha_i)_+ + (n - l - 1 - \alpha_r)_+$ zeros at $x = b$, where \hat{r} equals r or $r + 1$ according to whether or not r is of admissible parity.

After this preparation we turn to the series of arguments which will lead to the proof of Proposition 4.1. In (3.2) we replace $u_{l-1}, \dots, u_{k-2}, \alpha u_{k-1} + \beta u_k$ by $z_{k-1}, \dots, z_{q-2}, \alpha z_{q-1} + \beta z_q$ and define

$$z(x, s) = \begin{vmatrix} z_{k-1}(x), & \dots, & z_{q-2}(x), & \alpha z_{q-1} + \beta z_q(x) \\ M_0 z_{k-1}(s), & \dots, & M_0 z_{q-2}(s), & M_0(\alpha z_{q-1} + \beta z_q)(s) \\ \vdots & & & \vdots \\ M_{q-k-1} z_{k-1}(s), & \dots, & & M_{q-k-1}(\alpha z_{q-1} + \beta z_q)(s) \end{vmatrix}$$

$$= d_{k-1}(s)z_{k-1} + \dots + d_{q-2}(s)z_{q-2} + d_{q-1}(s)(\alpha z_{q-1} + \beta z_q).$$

Here $d_{q-1}(s) = W_M(z_{k-1}, \dots, z_{q-2})(s)$ and by the discrete Wronskian identity

$$d_{q-1}(s) = W_L(u_{l-1}, \dots, u_{k-2}, u_{k-1}, \dots, u_{q-2})(s) \\ \times W_L \begin{pmatrix} u_{l-1}, \dots, u_{k-2} \\ 1, \dots, k-l \end{pmatrix} \dots W_L \begin{pmatrix} u_{l-1}, \dots, u_{k-2} \\ q-k-1, \dots, q-l \end{pmatrix} (s) \neq 0, \\ a < s < b.$$

Proposition 4.2. *One simple zero of $z(x, s)$ passes from (s, b) to (a, s) when and only when s crosses a zero of $W_M(z_{k-1}, \dots, z_{q-2}, \alpha z_{q-1} + \beta z_q)$ from left to right. This is the only way by which the number of the zeros of $z(x, s)$ in (s, b) may change as s varies in (a, b) .*

The proof is parallel to that of Proposition 3.2. In order to show that no zero of $z(x, s)$ tends to b as s varies, we need the boundary values of $z(x, s)$ at $x = b$. This may be done as follows: By the Lemma, if $i + k - l < n$, the function

$$M_i z_r = W \begin{pmatrix} u_{l-1}, \dots, u_{k-2}, u_r \\ i, \dots, i+k-l \end{pmatrix} / W \begin{pmatrix} u_{l-1}, \dots, u_{k-2} \\ i+1, \dots, i+k-l \end{pmatrix},$$

has at $x = b$ a zero of multiplicity at least

$$[(n-\hat{r}-1-i)_+ + (n-k+1-i-1)_+ (k-l)] - [(n-k+1-i-1)_+ (k-l)] = (n-\hat{r}-1-i)_+.$$

This implies, by the way, that the $M_i z_r$, $i = 0, \dots, n-k+l-1$, are continuous at $x = b$ in spite of the singularity of equation (2.5) at $x = b$. Now $z(x, s) = d_{k-1} z_{k-1} + \dots + d_{q-2} z_{q-2} d_{q-1} (\alpha z_{q-1} + \beta z_q)$ satisfies

$$M_i z(b) = 0, \quad i = 0, \dots, n-q-2, \\ M_{n-q-1} z(b) = d_{q-1}(s) (\alpha M_{n-q-1} z_{q-1} + \beta M_{n-q-1} z_q)(b), \\ M_{n-q} z(b) = d_{q-1}(s) (\alpha M_{n-q} z_{q-1} + \beta M_{n-q} z_q)(b).$$

Since $d_{q-1}(s) \neq 0$ for $a < s < b$, the vanishing (or nonvanishing) of the functions $M_{n-q-1} z$, $M_{n-q} z$ at b is independent of s . On the other hand, $M_{n-q-1} z(x, s)$, $M_{n-q} z(x, s)$ are continuous near $x = b$ for $a < s < b$ and if a zero of $z(x, s)$ would tend to b as s varies, an additional quasiderivative would vanish at $x = b$, in contradiction to the previous remark. Consequently, no zero of $z(x, s)$ may tend to b when s varies in (a, b) . \square

Proposition 4.3. *When s is sufficiently close to a , $z(x, s)$ has in (s, b) exactly as many zeros as $\alpha z_{q-1} + \beta z_q$ has in (a, b) . When s is sufficiently close to b , $z(x, s)$ has no zeros in (s, b) .*

We normalize $z(x, s)$ and define

$$\bar{z}(x, s) = [d_{k-1}(s) z_{k-1} + \dots + d_{q-2}(s) z_{q-2} + d_{q-1}(s) (\alpha z_{q-1} + \beta z_q)] / \sum_{k-1}^{q-1} |d_i(s)|.$$

By the normalization, $\bar{z}(x, s)$ (or a subsequence) converges uniformly on any compact interval to a nontrivial solution. In order to understand what happens

when $s \rightarrow a^+$, we need to know about the behavior of

$$M_i z_r = W \left(\begin{matrix} u_{l-1}, \dots, u_{k-2}, u_r \\ i, \dots, i+k-l \end{matrix} \right) / W \left(\begin{matrix} u_{l-1}, \dots, u_{k-2} \\ i+1, \dots, i+k-l \end{matrix} \right)$$

near $x = a$. If $i+k-l < n$, then, by the Lemma, $M_i z_r$ has at $x = a$ a zero of multiplicity at least $[(l-1-i)_+(k-l) + (r-i-k+l)_+] - [(l-1-i-1)_+(k-l)]$. If $i \leq l-2$, the multiplicity of the zero of $M_i z_r$ at $x = a$ is at least $r-i$ and if $l-1 \leq i \leq n-k+l$, the multiplicity is at least $[r-(i+k-l)]_+$. Anyway, $M_i z_r$ is continuous near $x = a$ at least for $i = 0, \dots, n-k+l$. (Note the difference between the multiplicity of the zero of $M_{l-2} z_r$ and that of $M_{l-1} z_r = \sigma_{l-1}(M_{l-2} z_r)'$. This is due to the singularity of σ_{l-1} at a). For $i = r-k+l$ we expand the numerator of $M_{r-k+l} z_r(a)$ according to its last column and get

$$W \left(\begin{matrix} u_{l-1}, \dots, u_{k-2}, u_r \\ r-k+l, \dots, r \end{matrix} \right) (a) = W \left(\begin{matrix} u_{l-1}, \dots, u_{k-2} \\ r-k+l, \dots, r-1 \end{matrix} \right) (a) L_r u_r(a) \neq 0.$$

Therefore

$$(4.3) \quad \begin{aligned} M_i z_r(a) &= 0, & i &= 0, \dots, r-k+l-1, \\ M_{r-k+l} z_r(a) &\neq 0. \end{aligned}$$

Since $\bar{z}(x, s)$ is a combination of z_{k-1}, \dots, z_q ,

$$M_i \bar{z}(a) = 0, \quad i = 0, \dots, l-2.$$

Now $\bar{z}(x, s)$ also has $q-k$ zeros at $x = s$, therefore by Rolle's theorem each of the functions $M_{l-1} \bar{z}(x, s), \dots, M_{l-1+q-k-1} \bar{z}(x, s)$ has one or more zeros in (a, s) . As $\bar{z}(x, s)$ (or a subsequence) converges uniformly when $s \rightarrow a^+$, the limit solution and its quasiderivatives of orders $0, \dots, q-k+l-2$ vanish at a . On the other hand, $\lim_{s \rightarrow a^+} \bar{z}(x, s)$ is a combination of $z_{k-1}, \dots, z_{q-2}, \alpha z_{q-1} + \beta z_q$, so by (4.3) it may have $q-k+l-1$ vanishing quasiderivatives at $x = a$ if and only if, up to a constant factor,

$$\lim_{s \rightarrow a^+} \bar{z}(x, s) = \alpha z_{q-1} + \beta z_q.$$

In order to prove that for s sufficiently close to a , $z(x, s)$ and $\alpha z_{q-1} + \beta z_q$ have an equal number of zeros in (s, b) , it suffices to show that no zero of $z(x, s)$, except s , tends to the endpoints a or b when $s \rightarrow a^+$. This is definitely the case if

$$(4.4) \quad \alpha/\beta \neq 0, \quad \alpha/\beta \neq -M_{n-q-1} z_{q-1}(b)/M_{n-q-1} z_q(b).$$

For (4.4) ensures that

$$(4.5) \quad M_{q-k+l-1}(\alpha z_{q-1} + \beta z_q)(a) \neq 0, \quad M_{n-q-1}(\alpha z_{q-1} + \beta z_q)(b) \neq 0.$$

But if $z(x, s)$ would have an additional zero near a (or near b) for every s , then the limit solution $\lim_{s \rightarrow a^+} \bar{z}(x, s) = \alpha z_{q-1} + \beta z_q$ would violate (4.5).

Thus, we have proved the first half of Proposition 4.3. The other half, concerning $s \rightarrow b^-$, is proved by a similar modification of the corresponding part of the proof of Proposition 3.3 and it will be omitted.

The singular values of α/β are treated as in §3. This completes the proof of Theorems 1.1 and 1.2 for every $q \geq k$. \square

For the case $q \leq l-2$, Theorem 1.1 is proved by similar arguments. Proposition 3.3, for example, will be replaced by the following: *When s is sufficiently close to b , $u(x, s)$ has in (a, s) exactly as many zeros as $\alpha u_{l-3} + \beta u_{l-2}$ has in (a, b) . When s is sufficiently close to a , $u(x, s)$ may have in (a, s) at most one zero.*

In the analogue of Proposition 4.1, the zeros of $\alpha z_{q-1} + \beta z_q$ will be compared with those of $W_M(\alpha z_{q-1} + \beta z_q, z_{q+1}, \dots, z_{l-2})$. The differences between the cases $q \geq k$ and $q \leq l-2$ are due to the asymmetry between the boundary conditions (2.2) at $x = a$ and at $x = b$.

5. SOME APPLICATIONS

The number of zeros of $\alpha u_{q-1} + \beta u_q$ is closely related to oscillatory properties of equation (1.1). Recall that the $(q, n-q)$ -type conjugate point of $x = a$ is the smallest value of c , $c > a$, such that the boundary value problem

$$(5.1) \quad \begin{aligned} L_i y(a) &= 0, & i &= 0, \dots, k-1, \\ L_j y(c) &= 0, & j &= 0, \dots, n-k-1, \end{aligned}$$

has a nontrivial solution. More generally, the i th $(k, n-k)$ -type conjugate point of a , $i \geq 1$, is the i th value of c in (a, ∞) such that (5.1) has a nontrivial solution. It is known that the number of zeros of u_q in (a, b) is equal to the number of $(q, n-q)$ -type conjugate points of $x = a$ which are in (a, b) . Hence,

Corollary. *The number of the zeros of $z_q = W(u_{l-1}, \dots, u_k, u_q)$ in (a, b) is equal to the number of the $(q, n-q)$ -type conjugate points of a for equation (1.1) which are in (a, b) . For every other combination $\alpha u_{q-1} + \beta u_q$, the number of zeros of $W(u_{l-1}, \dots, u_k, \alpha u_{q-1} + \beta u_q)$ may differ from the number mentioned above by at most 1.*

Indeed, for two independent solutions $\alpha u_{q-1} + \beta u_q$ and $\gamma u_{q-1} + \delta u_q$, $W(\alpha u_{q-1} + \beta u_q, \gamma u_{q-1} + \delta u_q) = (\alpha\delta - \beta\gamma)W(u_{q-1}, u_q) \neq 0$ on (a, b) , so their zeros separate each other in (a, b) and their number may differ by at most 1. Hence, the same holds for the number of zeros of the minors $W(u_{l-1}, \dots, u_k, u_q)$ and $W(u_{l-1}, \dots, u_k, \alpha u_{q-1} + \beta u_q)$.

Up to now, we have discussed equation (1.1) only on a finite interval (a, b) . Let us consider it now on (a, ∞) . First we point out the dependence of the solutions u_i of (2.2) _{i} , $i = 0, \dots, n-1$, on b and denote them from now on by $u_i(x, b)$. Next, we let $b \rightarrow \infty$ and define

$$\tilde{u}_i(x) \stackrel{\text{def}}{=} \lim_{b \rightarrow \infty} u_i(x, b),$$

when $b \rightarrow \infty$ along a suitable sequence of values. $\{\tilde{u}_0(x), \dots, \tilde{u}_{n-1}(x)\}$ which

are defined so, are a basis [1], and they satisfy Proposition 2.2 [2, 4]. In particular,

$$S(\alpha\tilde{u}_{q-1} + \beta\tilde{u}_q, x) \equiv q, \quad a < x < \infty,$$

and all zeros of $\alpha\tilde{u}_{q-1} + \beta\tilde{u}_q$ on (a, ∞) are simple. Moreover, $\tilde{u}_0, \dots, \tilde{u}_{n-1}$ also satisfy Proposition 2.3 for $b = \infty$ [3, Theorem 2.5].

Recall that by Proposition 2.1, $S(y, x^+)$ is an integer valued, nondecreasing function of x and the values of $S(y, x^+)$ are integers of admissible parities. Accordingly, we defined S_k in [1] as the set of solutions of (1.1) on (a, ∞) such that $S(y, x^+) \equiv k$ on some ray (x_0, ∞) . The partition of the solution space into the sets S_k for admissible values of k is discussed in [1, 5]. One property of S_k that we are going to use here is that the elements of S_k are either all oscillatory or all nonoscillatory solutions of (1.1).

Now we consider the sets S_q for equations (1.1) and (2.6) and denote them by $S_q(L)$ and $S_q(M)$, respectively. By inequalities (2.3) and (2.7) applied to (a, ∞) , we get that

$$\alpha\tilde{u}_{q-1} + \beta\tilde{u}_q \in S_q(L) \quad \text{and} \quad \alpha\tilde{z}_{q-1} + \beta\tilde{z}_q \in S_q(M).$$

But when we let $b \rightarrow \infty$, we obtain from our present results that \tilde{u}_q and \tilde{z}_q have on (a, ∞) an equal number, finite or infinite, of simple zeros. Hence

Corollary. *The solutions of (1.1) in $S_q(L)$ are oscillatory if and only if the solutions of (2.6) in $S_q(M)$ are oscillatory.*

Equation (1.1) is $(q, n - q)$ -disconjugate on (a, ∞) if and only if equation (2.6) is $(q, n - q)$ -disconjugate on (a, ∞) .

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