Differential and Integral Equations Volume 10, Number 1, January 1997, pp. 137 – 152.

ASYMPTOTIC APPROXIMATION FOR MATRIX DIFFERENTIAL EQUATIONS AND APPLICATIONS

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Abstract. The Liouville-Green (WKB) approximation is generalized to a matrix differential equation with an analytic, symmetric matrix coefficient. Some applications to oscillation problems are given.

1. Introduction. Consider the vector and matrix differential equations

$$\begin{aligned} \vec{y}\,'' &= A(t)\vec{y}, \qquad a \le t \le \infty, \\ Y'' &= A(t)Y, \qquad a \le t \le \infty, \end{aligned} \tag{1}$$

where $A(t)$ is a Hermitian, $n \times n$ matrix function, analytic on $[a, \infty]$ and invertible at ∞. The main objective of this work is to provide a new tool for the investigation of this system. We develop a complex-valued and a real-valued vector and matrix analog to the Liouville-Green (L-G, WKB) asymptotic approximation. This is an interesting endeavor in its own right and its development requires nontrivial modifications to the scalar equation method. Analyticity of $A(t)$ is a technical assumption which makes our estimates simple. Invertibility of $A(t)$ is essential and has an analogue even for the scalar approximation. An example is given in Section 4.

One application of our asymptotic approximation is in oscillation theory. Recall that b is called a *conjugate point* of a if there exists a nontrivial vector solution $\vec{y}$ of (1) such that $\vec{y}(a) = \vec{y}(b) = 0$. Here, in contrast with the scalar second-order equation, the various conjugate points are associated with different solutions. The relation between existence of conjugate points and the eigenvalues of $A(t)$ and $\int^t A$ is extensively studied in the literature. The difference between our approach and those by, e.g., Butler, Byres, Coles, Dosley, Mingarelli ([1–5,10]) is that we obtain an asymptotic formula for Y, Y', complex and real valued, and an asymptotic expression for a determinant that determines the conjugate points. This leads us to some conjectures and to some results which were not obtained in any other way before.

Received for publication in revised form May 1995.
AMS Subject Classifications: 34E10, 34C10.

First we notice that $A(t)$ may have both positive and negative eigenvalues. Our determinant representation shows that (1) has conjugate points approximately at the zeros of the function

$$\prod_{j=1}^{k} \sin \int_a^t |\lambda_j^-(s)|^{1/2}\, ds,$$

where λ_j^- are the *negative* eigenvalues of $A(t)$. Since two negative eigenvalues could give rise asymptotically to the same conjugate points, it is not a trivial matter to count all distinct conjugate points asymptotically. Anyway, this leads us to the conjecture that the total number of conjugate points of a in $[a,t]$ depends asymptotically only on the negative eigenvalues λ_j^- of $A(t)$ as $t \to \infty$ and the positive eigenvalues λ_j^+ do not contribute to oscillation. Compare, for example, with [9] for related eigenvalue problems.

Our method can easily handle more complicated boundary value problems,

$$\Omega_1 Y(a) + \Omega_2 Y'(a) = 0, \quad \tilde{\Omega}_1 Y(b) + \tilde{\Omega}_2 Y'(b) = 0, \quad a < b,$$

where $\Omega_1, \Omega_2, \tilde{\Omega}_1, \tilde{\Omega}_2$ are certain matrices. The analysis of Coles ([4]) will need a substantial modification to handle these boundary conditions.

Another virtue of our approximation is that it could be an intermediate step to the development of an asymptotic formula to problems in, e.g, wave propagation and quantum mechanics, $u_{tt} = A[u](t)$, where A is an infinite-dimensional symmetric or antisymmetric operator. See, e.g., Edelstein ([7]).

2. Preliminaries and linear transformations. With the notation $z = (\vec{y}, \vec{y}\,')^T$, (1) is written as

$$z' = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} z.$$

In fact, it will be more convenient to study the corresponding $2n \times 2n$ matrix system

$$Z' = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} Z. \tag{2}$$

Here and in the sequel $n \times n$ and $2n \times 2n$ matrices are both denoted by capital letters. However, it is easy to distinguish between them according to the context or the block structure. Since $A(t)$ is Hermitian and analytic, there exists according to Rellich's theorem a unitary matrix $U(t)$, analytic in t, such that

$$A = U^* D U, \tag{3}$$

where

$$D(t) = \text{ diag } \{\lambda_1(t), \ldots, \lambda_n(t)\} \tag{4}$$

and $\lambda_1(t), \dots, \lambda_n(t)$ are the real, analytic eigenvalues of $A(t)$. For a global and different method of derivation see Gingold and Hsieh ([8]).

To carry out the forthcoming three simple transformations, we define powers of $A(t)$ by

$$A^\alpha(t) = U^*(t) D^\alpha(t) U(t),$$

where $D^\alpha = \operatorname{diag}\{\lambda_1^\alpha, \dots, \lambda_n^\alpha\}$ is chosen as one of the possible branches. Since $A(t)$ is invertible and $\lambda_i(t) \neq 0$, this choice causes no difficulties. Now let us define

$$T_1 = \begin{pmatrix} A^{-1/4} & A^{-1/4} \\ A^{1/4} & -A^{1/4} \end{pmatrix}, \qquad T_1^{-1} = \frac{1}{2} \begin{pmatrix} A^{1/4} & A^{-1/4} \\ A^{1/4} & -A^{-1/4} \end{pmatrix}.$$

Our first change of variables, $Z = T_1 Z_1$, transforms (2) into

$$Z_1' = \left[T_1^{-1} \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} T_1 - T_1^{-1} T_1' \right] Z_1.$$

This is, by direct calculation,

$$Z_1' = \left[\begin{pmatrix} A^{1/2} & 0 \\ 0 & -A^{1/2} \end{pmatrix} + R_1 \right] Z_1 \tag{5}$$

with $R_1 = -T_1^{-1} T_1'$. A second change of variables,

$$Z_1 = T_2 Z_2, \qquad T_2 = \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix},$$

takes (5) into

$$Z_2' = \left[\begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix} + R_2 \right] Z_2, \tag{6}$$

where

$$R_2 = T_2^{-1} R_1 T_2 - T_2^{-1} T_2' = -(T_1 T_2)^{-1} (T_1 T_2)'. \tag{7}$$

Note that

$$T_1 = \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} D^{-1/4} & D^{-1/4} \\ D^{1/4} & -D^{1/4} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix},$$

and

$$T_1 T_2 = \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} D^{-1/4} & 0 \\ 0 & D^{1/4} \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}. \tag{8}$$

Equation (6) will be considered as a small perturbation of the diagonal system

$$W' = \begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix} W, \tag{9}$$

whose solutions are

$$W = \begin{pmatrix} \exp\int D^{1/2} & 0 \\ 0 & \exp\int -D^{1/2} \end{pmatrix} C.$$

To reduce (6) to (9) we look for a transformation

$$Z_2 = (I_{2n} + P)W. \tag{10}$$

(The notation I_{2n} is reserved for the $2n \times 2n$ identity matrix while I denotes the $n \times n$ one.) Substituting (10) into (6) and comparing with the expected (9), one sees that $I_{2n} + P$ must satisfy

$$(I_{2n} + P)' = \left[\begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix} + R_2\right](I_{2n} + P) - (I_{2n} + P)\left[\begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix}\right]; \tag{11}$$

that is,

$$P' = \begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix} P - P \begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix} + R_2(I_{2n} + P). \tag{12}$$

This sequence of formal manipulations is justified if we can show that (12) has a solution $P(t)$ which can be estimated in a satisfactory way near $t = \infty$. If this can be done, we obtain a fundamental solution $Z = T_1T_2(I_{2n} + P)W$. If we put

$$I_{2n} + P = \begin{pmatrix} I + P_{11} & P_{12} \\ P_{21} & I + P_{22} \end{pmatrix}$$

and T_1T_2 as in (8), then

$$Z = \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} D^{-1/4} & 0 \\ 0 & D^{1/4} \end{pmatrix} \begin{pmatrix} I + P_{11} + P_{21} & I + P_{12} + P_{22} \\ I + P_{11} - P_{21} & -I - P_{22} + P_{12} \end{pmatrix} \times \begin{pmatrix} \exp\int D^{1/2} & 0 \\ 0 & \exp\int -D^{1/2} \end{pmatrix} C. \tag{13}$$

3. Asymptotic decomposition. We intend to prove the existence of a continuous transformation (10) such that P is small in some norm and $I_{2n} + P$ is invertible on some $[a, \infty)$. By variations of parameters the general solution of an equation $P' = FP + PG + H$ is given by

$$P(t) = W(t)\left[C + \int^t W^{-1}(s)H(s)\left(V^T(s)\right)^{-1} ds\right] V^T(t), \tag{14}$$

where W is a basic solution of $W' = FW$ and V of $V' = G^T V$. In the case of equation (12),

$$F = -G = \begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix}, \ W(t) = \exp K(t), \quad V(t) = V^T(t) = \exp(-K(t)),$$

with

$$K(t) \equiv \begin{pmatrix} \int D^{1/2} & 0 \\ 0 & -\int D^{1/2} \end{pmatrix},$$

are all $2n \times 2n$ diagonal matrices and $H = R_2(I_{2n} + P)$. If we take in (14) $C = 0$, it becomes the integral equation

$$\begin{aligned} P(t) = &\int^t \exp(K(t) - K(s)) R_2 \exp(-K(t) + K(s))\, ds \\ &+ \int^t \exp(K(t) - K(s)) R_2 P(s) \exp(-K(t) + K(s))\, ds, \end{aligned} \tag{15}$$

where the lower limit of integration in the (j, k)th term, $l_{j,k}$, may be determined individually for each term. Since all we need is one suitable "small" solution $P(t)$ of (12), we loose no generality by taking the most convenient choice.

Now we estimate the first term of (15).

Lemma. *It is possible to choose the lower limits of integration as $l_{j,k} = b < \infty$ or $l_{j,k} = \infty$, respectively, so that all the $2n \times 2n$ terms of*

$$J(t) \equiv \int^t \exp(K(t) - K(s)) R_2 \exp(-K(t) + K(s))\, ds \tag{16}$$

are $\mathcal{O}(t^{-\mu})$ as $t \to \infty$ for some positive μ. Moreover, for a sufficiently large b, all $|J_{jk}(t)|$ will be uniformly small on $[b, \infty)$.

Proof.

$$\begin{aligned} \exp(K(t) - K(s)) = \operatorname{diag}\big\{ &\exp \int_s^t \lambda_1^{1/2}, \ldots, \exp \int_s^t \lambda_n^{1/2}, \\ &\exp \int_s^t -\lambda_1^{1/2}, \ldots, \exp \int_s^t -\lambda_n^{1/2} \big\}. \end{aligned}$$

A typical term of (16) is

$$J_{j,k} = \int_{l_{j,k}}^t \big(\exp \int_s^t \big[\pm\lambda_j^{1/2}(\tau) \mp \lambda_k^{1/2}(\tau) \big]\, d\tau \big) r_{j,k}(s)\, ds, \qquad j, k = 1, \ldots, 2n. \tag{17}$$

Let

$$\Delta_{j,k}(t) = \pm\lambda_j^{1/2}(t) - (\pm\lambda_k^{1/2}(t)), \qquad j,k = 1,\dots,2n.$$

Here and later on it will be agreed that the sign which precedes $\lambda_i^{1/2}$ is $(+)$ if $i \in \{1,\dots,n\}$ and it is $(-)$ if $i \in \{n+1,\dots,2n\}$.

First we show that it is possible to choose $l_{j,k} < \infty$ or $l_{j,k} = \infty$ so that the corresponding kernels

$$\exp \int_s^t \Delta_{j,k}(\tau)\, d\tau, \qquad j,k = 1,\dots,2n, \tag{18}$$

will be bounded on $l_{j,k} \le s \le t < \infty$ or on $a \le t \le s < \infty = l_{j,k}$, respectively. $\lambda_i(t)$ is real analytic on $[a,\infty]$ and $\lambda_i(t) \ne 0$ since it is assumed that $A(t)$ is invertible. Therefore each $\Delta_{j,k}(t)$ is analytic there, too. Consequently, there exists a ray $[b_{jk},\infty)$ on which either

$$\mathcal{R}e\Delta_{j,k}(t) > 0 \tag{19}$$

or

$$\mathcal{R}e\Delta_{j,k}(t) < 0 \tag{20}$$

or

$$\mathcal{R}e\Delta_{j,k}(t) \equiv 0. \tag{21}$$

For example, if $\lambda_j,\ \lambda_k < 0$, then (21) holds. If, say, $\lambda_j > 0$ and $\lambda_k < 0$, then either (19) or (20) holds, according to the sign which precedes $\lambda_j^{1/2}$. If $\lambda_j^{1/2},\ \lambda_k^{1/2} > 0$, $\Delta_{j,k}$ is real valued and analytic on $[a,\infty]$ and it is either identically zero or it does not vanish on some neighborhood $[b_{jk},\infty),\ b_{jk} > 0$, of $t = \infty$. Again one of (19)-(20)-(21) holds. These three conditions are standard in the analytic theory of singular differential equations, e.g. Wasow ([11]) and Eastham ([6]).

Now we determine $l_{jk},\ j,k = 1,\dots,2n$, so that (18) will be bounded for $s \in [l_{jk},t]$. Let us denote $b = \max\{b_{jk}\} > 0$. If (19) or (21) holds, we choose $l_{jk} = \infty$ and the integral (17) is considered on $t \le s < \infty \equiv l_{jk}$. If (20) holds, take $l_{jk} = b$ and the integration of (17) is on $l_{jk} \equiv b \le s \le t$. If $\mathcal{R}e\Delta_{j,k}(t) = \sum_0^\infty \alpha_i t^{-i}$ happens to be $\mathcal{O}(t^{-2})$, then $\mathcal{R}e\Delta_{j,k} \in \mathbf{L}[a,\infty)$ and (18) is bounded for either choice of l_{jk}.

According to the above determinations, $J_{j,k} \to 0$ as $t \to \infty$. Recall that $A(t) = A(\infty) + \mathcal{O}(t^{-1})$, and so is $U(t)$. Therefore, according to (7), $R_1 = -T_1^{-1}T_1' = \mathcal{O}(t^{-2})$ and $R_2 = \mathcal{O}(t^{-2})$ and their derivatives are $\mathcal{O}(t^{-3})$. If $\mathcal{R}e\Delta_{j,k}(t) \ge 0$ and so $l_{jk} = \infty$, then

$$\left|\exp\left(-\int_t^s \Delta_{j,k}(\tau)\, d\tau\right)\right| \le 1, \qquad t \le s < l_{j,k} = \infty$$

and

$$\left|J_{j,k}(t)\right| \le \int_t^\infty |r_{jk}(s)|\, ds = \int_t^\infty \mathcal{O}(s^{-2})\, ds = \mathcal{O}(t^{-1}). \tag{22}$$

If $l_{jk} = b$, that is when $\mathcal{R}e\Delta_{j,k}(t) < 0$, then the estimate of (17) is different. Let $\mathcal{R}e\Delta_{j,k}(t) = \alpha + \beta t^{-1} + \mathcal{O}(t^{-2}) < 0$ on $[b, \infty)$. If $\alpha \neq 0$ then there exists $\alpha',\ \alpha < \alpha' < 0$, such that $\mathcal{R}e\Delta(t) \leq \alpha' < 0$ on $[b, \infty]$ and

$$\begin{aligned}
\left|J_{j,k}(t)\right| &= \int_b^t e^{\alpha'(t-s)}\mathcal{O}(s^{-2})\,ds \\
&= -\frac{1}{\alpha'}e^{\alpha'(t-s)}\mathcal{O}(s^{-2})\Big|_{s=b}^t + \int_b^t \frac{1}{\alpha'}e^{\alpha'(t-s)}\mathcal{O}(s^{-3})\,ds \\
&\leq \mathcal{O}(t^{-2}) + \mathcal{O}(e^{\alpha' t}) + \int_b^{t/2} e^{\alpha'(t-s)}\mathcal{O}(s^{-3})\,ds + \int_{t/2}^t e^0\mathcal{O}(s^{-3})\,ds \\
&\leq \mathcal{O}(t^{-2}) + e^{\alpha'(t/2)}\int_b^\infty \mathcal{O}(s^{-3})\,ds = \mathcal{O}(t^{-2}). \qquad (23)
\end{aligned}$$

If $\alpha = 0$ but $\beta < 0$, then there exists $\beta',\ \beta < \beta' < 0$, such that $\mathcal{R}e\Delta_{j,k}(t) \leq \beta' t^{-1} < 0$ on $[b, \infty],\ b > 0$. Now

$$\begin{aligned}
\left|J_{j,k}(t)\right| &\leq \int_b^{\sqrt{t}} \exp\Big(\int_s^t \beta'\tau^{-1}\,d\tau\Big)\mathcal{O}(s^{-2})\,ds + \int_{\sqrt{t}}^t e^0\mathcal{O}(s^{-2})\,ds \qquad (24)\\
&\leq e^{\beta'(\log t - \log\sqrt{t})}\int_b^\infty \mathcal{O}(s^{-2})ds + \mathcal{O}(t^{-1})\Big|_{\sqrt{t}}^t = \mathcal{O}(t^{\beta'/2}) + \mathcal{O}(t^{-1/2}) \to 0.
\end{aligned}$$

If $\alpha = \beta = 0$, that is, $\mathcal{R}e\Delta_{j,k}(t) = \mathcal{O}(t^{-2})$, then by a previous remark we take $l_{jk} = \infty$. Thus, in each case $J_{j,k}(t) = \mathcal{O}(t^{-\mu})$ as $t \to \infty$ for some $0 < \mu < 1/2$.

A reevaluation of the above calculation shows that each term is uniformly small on some ray $[b, \infty)$ either because $\int_b^\infty |r_{jk}| = \int_b^\infty \mathcal{O}(s^{-2})$ is a small number or because $\int_{t/2}^t |r_{jk}|$ and $\int_{\sqrt{t}}^t |r_{jk}|$ are small for $t \geq b$.

Theorem. *There exists a solution $P(t)$ of (12) such that $P(t) = \mathcal{O}(t^{-\mu})$ as $t \to \infty$.*

Proof. We can fix the lower limit b in the integral (16) so that not only $J_{j,k}(t) \to 0$ as $t \to \infty$ but even so that $\left|J_{j,k}(t)\right| \leq \rho < 1$ on $[b, \infty)$. On the space of continuous, bounded matrix-valued functions on $[a, \infty)$ equipped with the norm $\|P\| = \max_{j,k}\big(\sup_{b\leq t<\infty} |P_{jk}(t)|\big)$, we define the linear operator

$$\mathcal{L}\big[P\big](t) \equiv \int^t \exp(K(t) - K(s))R_2(s)P(s)\exp(-K(t) + K(s))\,ds. \qquad (25)$$

According to the choice of b it is clear that the norm of the operator $\mathcal{L}$ satisfies $\|\mathcal{L}\| \leq \rho$. By this notation (16) is $\big(J_{j,k}\big) \equiv \mathcal{L}\big[I\big]$ and equation (15) is $P = \mathcal{L}\big[I + P\big]$. Iterating this,

$$P = \mathcal{L}\big[I + \mathcal{L}\big[I + P\big]\big] = \mathcal{L}\big[I\big] + \mathcal{L}^2\big[I\big] + \mathcal{L}^2\big[P\big],$$

and eventually

$$P = \sum_{m=1}^{\infty} \mathcal{L}^m\big[I\big].$$

This formal solution is actually a solution thanks to the convergence of the series and $||P|| \le 1/(1-\rho)$. In fact we can say more about the rate of convergence of $P(t)$ as $t \to \infty$. The integral equation (15) is $P(t) = \mathcal{L}\big[I\big] + \mathcal{L}\big[P\big]$. By (25) we get an elementwise estimate

$$|P_{j,k}(t)| \le \big(\mathcal{L}\big[I\big] + \|P\|\mathcal{L}\big[I\big]\big)_{j,k}, \tag{26}$$

and the required estimate follows from $\mathcal{L}\big[I\big] = \mathcal{O}(t^{-\mu})$ and $\|P\| \le 1/(1-\rho)$.

4. An example. Let A be the 2×2 symmetric matrix

$$A = \begin{pmatrix} \eta(t) & \omega(t) \\ \omega(t) & \eta(t) \end{pmatrix},$$

where $\omega(t) = \omega_0 + \omega_1 t^{-1} + \dots$, $\omega_0 > 0$ and $\eta(t) = \eta_1 t^{-1} = \dots$. For this case $D(t) = \operatorname{diag}\{-\omega + \eta, \omega + \eta\}$, $D^{1/2}(t) = \operatorname{diag}\{i(\omega - \eta)^{1/2}, (\omega + \eta)^{1/2}\}$ and

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since U is a constant matrix, the product $T_1 T_2$ contains only one variable factor and $R_2 = -(T_1T_2)^{-1}(T_1T_2)'$ simplifies to

$$-\left[\begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix}\begin{pmatrix} D^{-1/4} & 0 \\ 0 & D^{1/4} \end{pmatrix}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}\right]^{-1}\begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix}\begin{pmatrix} D^{-1/4} & 0 \\ 0 & D^{1/4} \end{pmatrix}' \\ \times \begin{pmatrix} I & I \\ I & -I \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & D^{-1}D' \\ D^{-1}D' & 0 \end{pmatrix}.$$

(16) becomes

$$J(t) = \frac{1}{4}\int^t \begin{pmatrix} \exp\int_s^t D^{1/2} & 0 \\ 0 & \exp\int_s^t -D^{1/2} \end{pmatrix}\begin{pmatrix} 0 & D^{-1}D' \\ D^{-1}D' & 0 \end{pmatrix} \\ \times \begin{pmatrix} \exp\int_s^t -D^{1/2} & 0 \\ 0 & \exp\int_s^t D^{1/2} \end{pmatrix} ds \\ = \frac{1}{4}\int^t \begin{pmatrix} 0 & D^{-1}D'\exp\big(2\int_s^t D^{1/2}\big)\,d\tau \\ D^{-1}D'\exp\big(-2\int_s^t D^{1/2}\big)\,d\tau & 0 \end{pmatrix} ds.$$

In each of the four nonzero terms of J we define lower limits as the following:

$$\begin{aligned}
4J_{13}(t) &= \int_\infty^t \frac{(\omega-\eta)'}{\omega-\eta} \exp\bigl(2i\int_s^t (\omega-\eta)^{1/2}\,d\tau\bigr)\,ds, \quad J_{31}(t) = \overline{J_{13}(t)} \\
4J_{24}(t) &= \int_\infty^t \frac{(\omega+\eta)'}{\omega+\eta} \exp\bigl(2\int_s^t (\omega+\eta)^{1/2}\,d\tau\bigr)\,ds, \\
4J_{42}(t) &= \int_b^t \frac{(\omega+\eta)'}{\omega+\eta} \exp\bigl(-2\int_s^t (\omega+\eta)^{1/2}\,d\tau\bigr)\,ds, \qquad b \le t < \infty.
\end{aligned}$$

It follows directly by integration by parts or by estimates (23)–(24), that $J(t) = \mathcal{O}(t^{-1})$ as $t \to \infty$. By additional substitution of $\mathcal{L}\bigl[I\bigr] = J$ into (25) we get that $P = \sum \mathcal{L}^m\bigl[I\bigr]$ has the same off diagonal 2×2 block structure as J; i.e., $P_{11} = P_{22} = 0$. It follows thus from (13) that the $2n$ independent vector solutions $\vec{y}$ of (1) are the $2n$ columns of

$$U^* D^{-1/4}(I + P_{21}) \exp\bigl(\int D^{1/2}\bigr), \quad U^* D^{-1/4}(I + P_{21}) \exp\bigl(-\int D^{1/2}\bigr).$$

5. Real-valued representation. The purpose of this section is to derive from (13) $2n$ linearly independent, real-valued solutions of (2) (and (1)) for the case that $A(t)$ is a real-valued symmetric matrix. There are two reasons for doing so. First, it is useful to have a fundamental set of real-valued solutions. Secondly, in the next section we shall need to find the zeros of a determinant that is related to the question of oscillation. Working with complex-valued solutions, we would have to show that both the real and the imaginary parts of that determinant vanish simultaneously. In contrast, working with real-valued solutions (and determinants), this problem is resolved by elementary means.

Without loss of generality let

$$\begin{aligned}
\lambda_i(t) < 0 \quad &\text{on } [b,\infty) \quad \text{for} \quad i = 1,\dots,k, \\
\lambda_i(t) > 0 \quad &\text{on } [b,\infty) \quad \text{for} \quad i = k+1,\dots,n,
\end{aligned}$$

and let D be decomposed into the blocks $D = \begin{pmatrix} D_- & 0 \\ 0 & D_+ \end{pmatrix}$, where

$$D_- = \text{ diag } \{\lambda_1,\dots,\lambda_k\}, \quad D_+ = \text{ diag } \{\lambda_{k+1},\dots,\lambda_n\}.$$

Analogously $I = \begin{pmatrix} I_- & 0 \\ 0 & I_+ \end{pmatrix}$, where $I_-,\ I_+$ denote the $k \times k$ and $(n-k)\times(n-k)$ identity matrices, respectively. Let $|D_-|$ be a shorthand notation for diag $\{|\lambda_1|,\dots,|\lambda_k|\}$. According to this notation

$$\exp\int D^{1/2} = \begin{pmatrix} \exp i \int |D_-|^{1/2} & 0 \\ 0 & \exp\int D_+^{1/2} \end{pmatrix}.$$

The block structure of D induces a similar structure also on the $n\times n$ blocks of the matrix

$$\begin{pmatrix} I+P_{11}+P_{21} & I+P_{12}+P_{22} \\ I+P_{11}-P_{21} & -I-P_{22}+P_{12} \end{pmatrix}$$

of (13), and each P_{ij} splits into

$$\begin{pmatrix} \overbrace{P_{ij}^{11}}^{k} & \overbrace{P_{ij}^{12}}^{n-k} \\ P_{ij}^{21} & P_{ij}^{22} \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}.$$

Note that (26) enables us to estimate each element in every block P_{ij}^{lm} by the corresponding element of $J=\mathcal{L}\big[I\big]$. For example, each element of P_{11}^{11} corresponds to one of $\mathcal{L}\big[I\big]_{11}^{11}$, which is determined by $\Delta(t)=i|\lambda_j|^{1/2}-i|\lambda_l|^{1/2},\ \lambda_j,\lambda_l<0$. Thus, in the proof of the lemma we take $l_{jl}=\infty$ and by (22), $P_{11}^{11}(t)=\mathcal{O}(t^{-1})$. The elements of P_{11}^{12} correspond to $\Delta=i|\lambda_j|^{1/2}-\lambda_l^{1/2},\ \lambda_j<0<\lambda_l$, Thus by (23) $P_{11}^{12}(t)=\mathcal{O}(t^{-2})$. On the other hand the elements of $\mathcal{L}\big[I\big]_{11}^{22}$ and $\mathcal{L}\big[I\big]_{22}^{22}$ depend on $\Delta=\lambda_j^{1/2}-\lambda_l^{1/2},\ \lambda_j,\lambda_l>0$, hence it is estimated by (23) or (24). Consequently $P_{11}^{22}(t),P_{22}^{22}(t)=\mathcal{O}(t^{-\mu})$ with μ determined by (24), and these are the only blocks with such an estimate. We shall not elaborate on this any more, but use rather an estimate $P_{ij}^{lm}=\mathcal{O}(t^{-\nu})$ for some positive ν.

To simplify (13), we replace $P_{11}+P_{21}$, $P_{12}+P_{22}$, $P_{11}-P_{21}$, $-P_{22}+P_{12}$ by Q_{11}, Q_{12}, Q_{21}, Q_{22}, respectively, and put

$$D^{1/4}=\begin{pmatrix} |D_-|^{1/4} & 0 \\ 0 & D_+^{1/4} \end{pmatrix}\begin{pmatrix} \exp(\frac{i\pi}{4}I_-) & 0 \\ 0 & I_+ \end{pmatrix}.$$

Since $|D_-|^{1/4}$, $D_+^{1/4}$ and the orthogonal U are real valued, it is sufficient first to simplify three of the factors of (13), namely

$$\left[\begin{array}{cc:cc} e^{\frac{-i\pi}{4}I_-} & & & \\ & I_+ & & \\ \hdashline & & e^{\frac{i\pi}{4}I_-} & \\ & & & I_+ \end{array}\right]$$
$$\times\left[\begin{array}{cc:cc} I_-+Q_{11}^{11} & Q_{11}^{12} & I_-+Q_{12}^{11} & Q_{12}^{12} \\ Q_{11}^{21} & I_++Q_{11}^{22} & Q_{12}^{21} & I_++Q_{12}^{22} \\ \hdashline I_-+Q_{21}^{11} & Q_{21}^{12} & I_-+Q_{22}^{11} & Q_{22}^{12} \\ Q_{21}^{21} & I_++Q_{21}^{22} & Q_{22}^{21} & I_++Q_{22}^{22} \end{array}\right] \qquad (27)$$
$$\times\left[\begin{array}{cc:cc} \exp\big(i\int|D_-|^{1/2}\big) & & & \\ & \exp\big(\int D_+^{1/2}\big) & & \\ \hdashline & & \exp\big(-i\int|D_-|^{1/2}\big) & \\ & & & \exp\big(-\int D_+^{1/2}\big) \end{array}\right].$$

To reduce the volume of calculations, we treat first in detail the upper left $n \times n$ block of (27) which simplifies to

$$\begin{pmatrix} \big(I_- + Q_{11}^{11}\big)\exp\big(i\int_a^t |D_-|^{1/2} - \frac{\pi}{4}I_-\big) & Q_{11}^{12}\exp\int_a^t D_+^{1/2} \\ Q_{11}^{21}\exp\big(i\int_a^t |D_-|^{1/2}\big) & \big(I_+ + Q_{11}^{22}\big)\exp\int_a^t D_+^{1/2} \end{pmatrix}.$$

Each diagonal term of the $k \times k$ block $\big(I_- + Q_{11}^{11}\big)\exp\big(i\int_a^t |D_-|^{1/2} - \frac{\pi}{4}I_-\big)$ is

$$(1 + q_j e^{i\theta_j})\exp i\Big(\int |\lambda_j|^{1/2} - \frac{\pi}{4}\Big), \qquad 1 \le j \le k,$$

where $q_j = \mathcal{O}(t^{-\nu})$. Its polar decomposition is of the form

$$(1 + \rho_{11,jj})\exp\Big(i\int |\lambda_j|^{1/2} - \frac{\pi}{4} + \gamma_{11,jj}\Big), \quad j = 1, \dots, k,$$

with real $\rho_{11,jj}, \gamma_{11,jj} = \mathcal{O}(t^{-\nu})$. The *off-diagonal* terms of the same block are

$$|q_{11,jl}|\exp\Big(i\int |\lambda_l|^{1/2} - \frac{\pi}{4} + \gamma_{11,jl}\Big), \quad 1 \le j \ne l \le k,$$

with $q_{11,jl} = \mathcal{O}(t^{-\nu})$. Thus the decomposition of the block into diagonal and off-diagonal matrices is

$$\big(I_- + R_-^{11}\big)\exp\Big(i\int_a^t |D_-|^{1/2} - \frac{\pi}{4}I_- + \Gamma_-\Big) + Q_{11}^{OFF},$$

where R_-^{11}, Γ_- are diagonal $k \times k$ matrices, both $\mathcal{O}(t^{-\nu})$, and $Q_{11}^{OFF} = \mathcal{O}(t^{-\nu})$ is off-diagonal.

Similarly, $(I_+ + Q_{11}^{22})\exp\int D_+^{1/2}$ may be decomposed as

$$\big(I_+ + R_+^{22}\big)\exp\Big(\int_a^t D_+^{1/2} + i\Gamma_+\Big) + Q_{22}^{OFF}\exp\Big(i\int_a^t D_+^{1/2}\Big)$$

with real diagonal R_+^{22}, $\Gamma_+ = \mathcal{O}(t^{-\nu})$ and $Q_{22}^{OFF} = \mathcal{O}(t^{-\nu})$. Any element of the $k \times (n-k)$ block

$$Q_{11}^{21}\exp\Big(i\int_a^t |D_-|^{1/2}\Big)$$

is

$$|q_{11,jl}|\exp\Big(i\int |\lambda_l|^{1/2} + \gamma_{11,jl}\Big), \quad l = 1, \dots, k,\ j = k+1, \dots, n,$$

with $|q_{11,jl}| = \mathcal{O}(t^{-\nu})$ and Q_{11}^{12} is $\mathcal{O}(t^{-\nu})\exp\int_a^t D_+^{1/2}$. Thus we get a sum of a diagonal and an off-diagonal matrix

$$\begin{pmatrix} \big(I_- + R_-^{11}\big)\exp\big(i\int_a^t |D_-|^{1/2} - \frac{\pi}{4}I_- + \Gamma_-\big) & 0 \\ 0 & \big(I_+ + R_+^{11}\big)\exp\big(\int_a^t D_+^{1/2} + i\Gamma_+\big) \end{pmatrix}$$
$$+ \begin{pmatrix} \mathcal{O}(t^{-\nu}) & \mathcal{O}(t^{-\nu})\exp\big(\int_a^t D_+^{1/2}\big) \\ \mathcal{O}(t^{-\nu}) & \mathcal{O}(t^{-\nu})\exp\big(\int_a^t D_+^{1/2}\big) \end{pmatrix}^{OFF}.$$

The upper right $n \times n$ block of (27) becomes similarly

$$\begin{pmatrix} \big(I_- + R_-^{12}\big)\exp\big(-i\int_a^t |D_-|^{1/2} - \frac{\pi}{4}I_- + \Gamma_-\big) & 0 \\ 0 & \big(I_+ + R_+^{12}\big)\exp\big(-\int_a^t D_+^{1/2} + i\Gamma_+\big) \end{pmatrix}$$
$$+ \begin{pmatrix} \mathcal{O}(t^{-\nu}) & \mathcal{O}(t^{-\nu})\exp\big(-\int_a^t D_+^{1/2}\big) \\ \mathcal{O}(t^{-\nu}) & \mathcal{O}(t^{-\nu})\exp\big(-\int_a^t D_+^{1/2}\big) \end{pmatrix}^{OFF}.$$

$2n$ real-valued independent vector solutions of (1) are obtained by taking real and imaginary parts of the columns of the above matrices and multiplying them from the left-hand side by $U^*|D|^{-1/4}$. The real parts may be written as follows: $2k$ oscillatory solutions result from the columns of

$$M_1 = \begin{pmatrix} \overbrace{(I_- + R_-^{11})\cos\big(\int_a^t |D_-|^{1/2} - \frac{\pi}{4}I_- + \Gamma_-\big) + \mathcal{O}(t^{-\nu})}^{k} \\ \mathcal{O}(t^{-\nu}) \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}$$

and

$$M_2 = \begin{pmatrix} \overbrace{(I_- + R_-^{11})\sin\big(\int_a^t |D_-|^{1/2} - \frac{\pi}{4}I_- + \Gamma_-\big) + \mathcal{O}(t^{-\nu})}^{k} \\ \mathcal{O}(t^{-\nu}) \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix},$$

while $2(n-k)$ nonoscillatory solutions are extracted from

$$M_3 = \begin{pmatrix} \overbrace{\mathcal{O}(t^{-\nu})\exp\big(\int_a^t D_+^{1/2}\big)}^{n-k} \\ \big(I_+ + R_+^{11}\big)\exp\big(\int_a^t D_+^{1/2}\big) + \mathcal{O}(t^{-\nu})\exp\big(\int_a^t D_+^{1/2}\big) \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix},$$

and

$$M_4 = \begin{pmatrix} \overbrace{\mathcal{O}(t^{-\nu})\exp\big(-\int_a^t D_+^{1/2}\big)}^{n-k} \\ \big(I_+ + R_+^{12}\big)\exp\big(-\int_a^t D_+^{1/2}\big) + \mathcal{O}(t^{-\nu})\exp\big(-\int_a^t D_+^{1/2}\big) \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}.$$

Here $I_\pm + R_\pm$ denote the diagonal parts of their blocks and they determine the dimensions of the block structure. Phase shifts $\Gamma_\pm$ appear explicitly in M_1, M_2 but are absorbed in the $\mathcal{O}$ terms in M_3, M_4. The $2n$-dimensional space of solutions of (1) consists of

$$U^*|D|^{-1/4}\big(M_1(t)\ M_3(t)\ M_2(t)\ M_4(t)\big)\vec{c}, \qquad \vec{c} \in \mathbf{R}^{2n}.$$

6. Applications to oscillation theory. There exists a solution of (1) which satisfies the two-point boundary conditions

$$\vec{y}(a) = \vec{y}(t) = 0 \tag{28}$$

if and only if the determinant of the $2n \times 2n$ matrix

$$\omega = \begin{pmatrix} M_1(a) & M_3(a) & M_2(a) & M_4(a) \\ M_1(t) & M_3(t) & M_2(t) & M_4(t) \end{pmatrix} \tag{29}$$

vanishes. Due to the complexity of the formulas which appear in each of its blocks, it is technically difficult to write them explicitly all at once, so we need some compact notation and verbal description of the required calculations.

As remarked in the lemma, the perturbation term $P(t)$ not only tends to zero as $t \to \infty$, but by a suitable choice of the point a it can be made uniformly small on the whole $[a, \infty)$. Consequently, this holds also for all terms of the form $\mathcal{O}(t^{-\nu})$ in (29). So first we concentrate on the role of the other explicitly given elements.

Let

$$\Psi_- = \int_a^t |D_-|^{1/2}, \qquad \Psi_+ = \int_a^t D_+^{1/2}.$$

To neutralize the effect of the exponentially growing terms, let us multiply (29) from the right by the diagonal matrix $\mathrm{diag}\{I_-,\, \exp(-\Psi_+),\, I_-,\, I_+\}$. This multiplies $\det(\omega)$ by $\prod_{j=k+1}^n \exp\big(-\int \lambda_j^{1/2}\big)$ and has no effect on its vanishing. The so-obtained matrix is

$$\left[\begin{array}{cc|cc} \cos\big(-\frac{\pi}{4}I_-\big) & & \sin\big(-\frac{\pi}{4}I_-\big) & \\ & I_+ & & I_+ \\ \hline \cos\big(\Psi_- - \frac{\pi}{4}I_-\big) & & \sin\big(\Psi_- - \frac{\pi}{4}I_-\big) & \\ & I_+ & & \exp\big(-\Psi_+\big) \end{array}\right]$$

$$+ \text{ uniformly small perturbation terms.}$$

The explicitly given matrix consists of four $n \times n$ diagonal matrices (Γ_-, Γ_+ are included in the perturbation terms). Since

$$\cos\big(-\frac{\pi}{4}I_-\big) = \sin\big(-\frac{\pi}{4}I_-\big) = \frac{\sqrt{2}}{2}I_-, \quad \cos(\psi - \pi/4) + \sin(\psi - \pi/4) = \frac{\sqrt{2}}{2}\sin\psi,$$

it is column-equivalent to

$$\left[\begin{array}{cc:cc} \frac{\sqrt{2}}{2} I_- & & & \\ & I_+ & & \\ \hdashline \cos\bigl(\Psi_- - \frac{\pi}{4} I_-\bigr) & & \frac{\sqrt{2}}{2}\sin\bigl(\Psi_-\bigr) & \\ & I_+ & & \exp\bigl(-\Psi_+\bigr) - I_+ \end{array}\right]$$

and its determinant is, up to a constant factor,

$$\det\bigl(\sin\Psi_-\bigr)\det\bigl(I_+ - \exp(-\Psi_+)\bigr) = \prod_{j=1}^{k} \sin\Bigl(\int_a^t |\lambda_j|^{1/2}\Bigr) \prod_{j=k+1}^{n} \bigl(1 - \exp\bigl(-\int_a^t \lambda_j^{1/2}\bigr)\bigr). \tag{30}$$

In the absence of the perturbation terms, that is when $A(t)$ itself is a diagonal matrix, $\det(\omega)$ vanishes at the points $t_{m,j}$ where

$$m\pi = \int_a^{t_{m,j}} |\lambda_j|^{1/2}, \qquad j = 1, \ldots, k, \quad m = 1, \ldots \tag{31}$$

and these are all the conjugate points of a.

For the example of Section 4 the formula (30) takes the form

$$\sin\Bigl(\int_a^t |\lambda_j|^{1/2}\Bigr)\bigl(1 - \exp\bigl(-\int_a^t \lambda_j^{1/2}\bigr)\bigr).$$

Therefore the asymptotic location of the conjugate points to a are given as $t_m(a) \approx a + m\pi/\omega_0$ as $a \to \infty$.

Now we turn to the perturbed matrix ω. $\det(\omega)\prod_{k+1}^{n}\exp\bigl(\int \lambda_j^{1/2}\bigr)$ consists of the product (30) and other terms, each of which includes at least one factor which is a perturbation term and which can be made arbitrarily small on $[a, \infty)$. Thus,

$$\prod_{j=k+1}^{n} \exp\Bigl(\int_a^t \lambda_j^{1/2}\Bigr)\det(\omega) = \prod_{j=1}^{k} \sin\Bigl(\int_a^t |\lambda_j|^{1/2}\Bigr) \prod_{j=k+1}^{n} \bigl(1 - \exp\bigl(-\int_a^t \lambda_j^{1/2}\bigr)\bigr) + \text{uniformly small perturbation terms.} \tag{32}$$

This expression may be used to approximate the first conjugate point of a for large values of a. By a suitable rearrangment of the columns of U, let

$$\lambda_1(\infty) \le \lambda_2(\infty) \le \cdots \le \lambda_k(\infty) < 0$$

and define $\alpha = \alpha(a)$ as the unique value such that

$$\int_a^{\alpha} |\lambda_1|^{1/2} = \pi.$$

For a fixed $\delta > 0$, the trigonometric product in (32) is bounded away from 0 on $[a+\delta, \alpha-\delta]$. Consequently equation (1) has no conjugate point of a at least on an interval whose approximate length is $\alpha - a$.

Suppose now that $\lambda_1(\infty)$ is a simple eigenvalue; that is,

$$\lambda_1(\infty) < \lambda_2(\infty) \le \cdots \le \lambda_k(\infty) < 0,$$

The trigonometric product actually changes its sign at $t = \alpha$, so we can conclude that (1) indeed has a conjugate point approximately at α. In the absence of the perturbation terms, the solution which corresponds to this first conjugate point of a is

$$U^*|D|^{-1/4}\big(\sin\big(\int_a^t |\lambda_1|^{1/2}\big), 0, \dots, 0\big). \tag{33}$$

When the perturbation terms are present, the solution of bvp (28) differs from (33) by $\mathcal{O}(a^{-\mu})$. This may be summarized as

Theorem. *Let $\lambda_1(\infty) < 0$ be the minimal eigenvalue of A at $t = \infty$ and*

$$J(t) = \int_a^t |\lambda_1|^{1/2}.$$

Then $J^{-1}(\pi)$ is an asymptotic lower bound for the first conjugate point $\eta(a)$ of a for large values of a. If in addition $\lambda_1(\infty)$ is a simple eigenvalue, then

$$\eta(a) = J^{-1}(\pi) + \mathcal{O}(a^{-\mu}).$$

The above argument clarifies the relation of the most negative eigenvalue λ_1 with the first conjugate point. There remains an interesting question of the role played by the other points t_{mj} which were defined in (31). Now the trigonometric product $\prod \sin\big(\int_a^t |\lambda_j|^{1/2}\big)$ may have arbitrarily close zeros, each of which results from another factor ("resonance"), and a small pertubation may eliminate some of its zeros. Nevertheless, since $|\lambda_j|^{1/2} = |\lambda_j(\infty)|^{1/2} + o(1)$, it follows that $\prod \sin\big(\int |\lambda_j|^{1/2}\big)$ does oscillate infinitely many times between two constant, nonzero values $\pm\delta$. Consequently $\det(\omega)$ has infinitely many zeros and there exist infinitely many conjugate points.

The authors wish to thank the referee for his helpful remarks and for drawing attention to reference [9].

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