DISCONCUGACY - An nth order homogeneous linear differential equation (operator)

$$Ly \stackrel{\text{def}}{\equiv} y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0, \quad (1)$$

is called disconjugate on an interval I if no nontrivial solution has n zeros on I, multiple zeros being counted according to their multiplicity. (In russian literature this is called non-oscillation on I.) If (1) has a solution with n zeros on an interval, then the infimum of all values c, $c > a$, such that some solution has n zeros on $[a, c]$ is called the conjugate point of a and is denoted by $\eta(a)$. This infimum is achieved by a solution which has at least n zeros at a and $\eta(a)$ and is positive on $(a, \eta(a))$. If the equation has continuous coefficients, the conjugate point $\eta(a)$ is a strictly increasing, continuous function of a. The adjoint equation has the same conjugate point as (1). For general properties, see [C], [L].

There are numerous explicit sufficient criteria for the equation (1) to be disconjugate. Many of them are of the form

$$\sum_{k=1}^{n} c_k (b-a)^k ||p_k|| < 1,$$

where $||p_k||$ is some norm of p_k, $I = [a, b]$ and c_k are suitable constants. These are "smallness conditions" which express the proximity of (1) to the disconjugate equation $y^{(n)} = 0$. See [W].

L is disconjugate on $[a, b]$ if and only if it has there the Pólya factorization

$$Ly \equiv \rho_n \frac{d}{dx}(\rho_{n-1} \ldots \frac{d}{dx}(\rho_1 \frac{d}{dx}(\rho_0 y)) \ldots), \quad \rho_i > 0,$$

or the equivalent Mammana factorization $Ly = (\frac{d}{dx} + r_n) \ldots (\frac{d}{dx} + r_1)y$. Among the various Pólya factorizations the most important is Trench's canonical form [T]: If L is disconjugate on (a, b), $b \leq \infty$, there is essentialy one factorization such that $\int^b \rho_i^{-1} = \infty$, $i = 1, \ldots, n-1$.

Disconjugacy is closely related to solvability of the **de la Vallée Poussin multiple point problem** $Ly = g$, $y^{(i)}(x_j) = a_{ij}$, $i = 0, \ldots, r_j - 1$, $\sum_1^m r_j = n$. Green's function of a disconjugate operator L and the related homogeneous bvp satisfies

$$G(x,t)/(x - x_1)^{r_1} \ldots (x - x_m)^{r_m} > 0$$

for $x_1 \leq x \leq x_m$, $x_1 < t < x_m$ [L]. Other interesting boundary value problem is the focal bvp $y^{(i)}(x_j) = 0$, $i = r_{j-1}, \ldots, r_j - 1$, $0 = r_0 < r_1 < \ldots < r_m = n - 1$.

For a second order equation nonoscillation (i.e., no solution has a sequence of zeros converging to $+\infty$) implies by the *Sturm seperation theorem* that there exists a point a such that (1) is disconjugate on $[a, \infty)$. For equations of order $n > 2$ this conclusion holds for a class of equations [E] but not for all equations [Gu].

Particular results about disconjugacy exist for various special types of differential equations:

(1) The **Sturm-Liouville operator**

$$(py')' + qy = 0, \quad p > 0, \quad (2)$$

is studied by the **Sturm** (and Sturm-Picone) **comparison theorem**, the Prüfer transformation and the **Riccati equation** $z' + q + z^2/p = 0$. It is also closely related to the positive definiteness of the quadratic functional $\int_a^b (py'^2 - qy^2)$. See [S], [C], [H]. For example, (2) is disconjugate on $[a, b]$ if $\int_a^b p^{-1} \times \int_a^b |q| < 4$.

(2) Third order equations are studied in [G].

(3) For self adjoint equation $\sum_{i=0}^m (p_{m-i} y^{(i)})^{(i)} = 0$ the existence of a solution with two zeros of multiplicity m is studied. Their absence is called (m, m)-disconjugacy.

(4) Disconjugacy of the analytic equation $w'' + p(z)w = 0$ in a complex domain is connected to the theory of univalent functions. See [N], [H].

(5) Many particularly elegant result are available for two-term equations $y^{(n)} + p(x)y = 0$ and their generalizations $Ly + p(x)y = 0$ [K],[E].

Disconjugacy is also studied for certain second order linear differential systems of higher dimension [C], [R]. Note also the historical prologue of [R] which explains the connection to the calculus of variations. The concepts of disconjugacy and oscillation are also generalized to nonlinear and functional differential equations.

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AMS Subject Classification: 34C10