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INTEGRAL MEANS AND PÓLYA FACTORIZATIONS

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ABSTRACT. Integral means of functions and their derivatives are studied. We find a relation between integral means and the Pólya factorization of ordinary linear differential operators.

The purpose of this note is to present relations between integral means of functions and the corresponding integral means of their derivatives. Also, we point out that these relations are consequences of the Pólya factorization of differential operators.

We begin with two identities:

Theorem 1. *Let there be given the integral mean with weight function* $w_0 > 0$*,*

$$F(t) = \frac{\int_a^t w_0(s) f(s)\, ds}{\int_a^t w_0(s)\, ds}. \tag{1}$$

(a) If $w_1 = \int_a^t w_0$*, then the generalized* k*-th derivatives* $L_k f = \left(\frac{w_1}{w_0}\frac{d}{dt}\right)^k f$ *have the integral means*

$$L_k F(t) = \frac{\int_a^t w_0(s) L_k f(s)\, ds}{\int_a^t w_0(s)\, ds}, \qquad k = 1, 2, \dots\ . \tag{2}$$

(b) Let $w_k = \int_a^t w_{k-1}$*,* $r_k = w_k^2 / w_{k-1} w_{k+1}$*,* $k = 1, 2, \dots$ *. Then the integral mean with weight* w_k *satisfies*

$$r_k \frac{d}{dt} r_{k-1} \frac{d}{dt} \dots r_1 \frac{d}{dt} F(t) = \frac{\int_a^t w_k(s) f^{(k)}(s)\, ds}{\int_a^t w_k(s)\, ds}, \qquad k = 1, 2, \dots . \tag{3}$$

Proof. By (1), $F(t) = \int_a^t w_0(s) f(s)\, ds / w_1(t)$, with $w_1(t) = \int_a^t w_0$. By differentiation and then integration by parts (with $w_1(a) = 0$),

$$\begin{aligned} F' &= w_1^{-2}\Big[w_0 f w_1 - w_0 \int_a^t w_0 f\Big] \\ &= w_1^{-2}\Big[w_0 f w_1 - w_0\Big(w_1 f - \int_a^t w_1 f'\Big)\Big] \\ &= w_1^{-2} w_0 \int_a^t w_1 f'. \end{aligned}$$

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This may be written as

$$\frac{w_1}{w_0}F' = \frac{1}{w_1}\int_a^t w_1 f' = \frac{1}{w_1}\int_a^t w_0\Big(\frac{w_1}{w_0}f'\Big), \tag{4}$$

which is precisely (2) with $k = 1$, $L_1 = \frac{w_1}{w_0}\frac{d}{dt}$. Induction on k completes the proof of (2).

Let us multiply (4) by w_1/w_2, where $w_2 = \int_a^t w_1$, and rewrite it as

$$\frac{w_1^2}{w_0 w_2}F'(t) = \frac{\int_a^t w_1(s) f'(s)\,ds}{\int_a^t w_1(s)\,ds}.$$

When this argument is repeated k times with $w_{\ell+1} = \int_0^t w_\ell$, $\ell = 1, 2, \dots, k$, (3) follows. □

Take for example the weight function $w_0(t) = t^p$, $p > -1$, and $a = 0$. Then one has $w_k = t^{p+k}/(p+1)\dots(p+k)$, $w_1/w_0 = t/(p+1)$, $r_k = w_k^2/w_{k-1}w_{k+1} = (p+k+1)/(p+k)$. If

$$F(t) = \frac{\int_0^t s^p f(s)\,ds}{\int_0^t s^p\,ds}, \tag{5}$$

then (2) becomes

$$(t\frac{d}{dt})^k F(t) = \frac{\int_0^t s^p (s\frac{d}{ds})^k f(s)\,ds}{\int_0^t s^p\,ds}, \qquad k = 1, 2, \dots, \tag{6}$$

and (3) becomes

$$\frac{p+k+1}{p+1}F^{(k)}(t) = \frac{\int_0^t s^{p+k} f^{(k)}(s)\,ds}{\int_0^t s^{p+k}\,ds}, \qquad k = 1, 2, \dots . \tag{7}$$

(7) yields easily

$$t^k F^{(k)}(t) = \frac{\int_0^t s^{p+k} f^{(k)}(s)\,ds}{\int_0^t s^p\,ds}, \qquad k = 1, 2, \dots . \tag{8}$$

Since $\big(t\frac{d}{dt}\big)^n = t^n\frac{d^n}{dt^n} + \sum_1^{n-1} c_i t^i \frac{d^i}{dt^i}$ with certain constants c_i, one can deduce (8) also from (6).

Many other identities about integral means are available. For example, if we transform (5) and (6) by $s = u^{-1}$, $s\frac{d}{ds} = -u\frac{d}{du}$, we get that for $q > 1$, $v > 0$,

$$F(v) = \frac{\int_v^\infty u^{-q} f(u)\,du}{\int_v^\infty u^{-q}\,du} \quad \text{implies} \quad (v\frac{d}{dv})^k F(v) = \frac{\int_v^\infty u^{-q}(u\frac{d}{du})^k f(u)\,du}{\int_v^\infty u^{-q}\,du},$$

$k = 1, 2, \dots$. If we choose $a = \infty$, $w_0 = e^{-pt}$, $w_k = \int_t^\infty w_{k-1} = e^{-pt}/p^k$, then by (3), $r_k = 1$ and

$$F(t) = \frac{\int_t^\infty e^{-ps} f(s)\,ds}{\int_t^\infty e^{-ps}\,ds} \quad \text{implies} \quad F^{(k)}(t) = \frac{\int_t^\infty e^{-ps} f^{(k)}(s)\,ds}{\int_t^\infty e^{-ps}\,ds}.$$

Corollary 1. *Let $w_0 = t^p$ as in (5).*

(a) If $f^{(k)}$ has a fixed sign for $t \geq 0$, then $F^{(k)}$ has the same fixed sign as well.

(b) If $f, f', \ldots f^{(\ell)} \geq 0$ for $t \geq 0$, then

$$F^{(k)}(t) \leq \frac{p+1}{p+k+1} f^{(k)}(t), \qquad t \geq 0, \quad k = 0, \ldots, \ell - 1. \tag{9}$$

(c) If $f, f', \ldots f^{(\ell)} \geq 0$, $f^{(\ell+1)} \leq 0$, then

$$0 \leq \frac{p+1}{p+\ell+1} f^{(k)}(t) \leq F^{(k)}(t), \qquad t \geq 0, \quad k = 0, \ldots, \ell. \tag{10}$$

Proof. (a) is a consequence of (7). By (7) also

$$\begin{aligned} \frac{t^{p+k+1}}{p+1} F^{(k)}(t) &= \int_0^t s^{p+k} f^{(k)}(s)\, ds \\ &= \frac{t^{p+k+1}}{p+k+1} f^{(k)}(t) - \int_0^t \frac{s^{p+k+1}}{p+k+1} f^{(k+1)}(s)\, ds, \end{aligned} \tag{11}$$

and the upper bound (9) follows for $k + 1 \leq \ell$.

If $f^{(\ell+1)} \leq 0$, then by (11) (with $k = \ell$)

$$F^{(\ell)}(t) \geq \frac{p+1}{p+\ell+1} f^{(\ell)}(t), \qquad t \geq 0,$$

which is (10) for $k = \ell$. Integration yields

$$F^{(\ell-1)}(t) - \frac{p+1}{p+\ell+1} f^{(\ell-1)}(t) \geq F^{(\ell-1)}(0) - \frac{p+1}{p+\ell+1} f^{(\ell-1)}(0), \quad t \geq 0. \tag{12}$$

But by (7), $F^{(k)}(0) = \big((p+1)/(p+k+1)\big) f^{(k)}(0)$ for $k = 1, 2, \ldots$. Using this with $k = \ell - 1$, (12) becomes

$$F^{(\ell-1)}(t) - \frac{p+1}{p+\ell+1} f^{(\ell-1)}(t) \geq \Big(\frac{p+1}{p+\ell} - \frac{p+1}{p+\ell+1}\Big) f^{(\ell-1)}(0) \geq 0$$

for $t \geq 0$. Thus we get the lower bound (10) for $k = \ell - 1$. Repeated integrations of the last inequality prove (10) for $k = \ell - 2, \ldots, 1, 0$. □

Corollary 1 shows how convexity properties are inherited by averaging. About the convexity of $F(t) = t^{-1} \int_0^t f(s)\, ds$, see [Mit, §1.4.7] and references there. Part (a) of Corollary 1 also implies, for example,

Corollary 2. *Let f be completely monotone, that is, $(-1)^n f^{(n)} \geq 0$, $n = 0, 1, \ldots$. Then the average F in (5) is completely monotone, too. If f is completely convex, i.e. $(-1)^n f^{(2n)} \geq 0$, $n = 0, 1, \ldots$, then F of (5) is also completely convex.*

The integral identities (2),(3) may also be written as differential identities. For example, (1) and (2) are equivalent, respectively, to

$$\frac{d}{dt}(w_1 F) = w_0 f \qquad \text{and} \qquad \frac{d}{dt}(w_1 L_k F) = w_0 L_k f.$$

If we substitute $f = w_0^{-1} \frac{d}{dt}(w_1 F)$ from the first equation into the second one, we get

$$\frac{d}{dt}(w_1 L_k F) = w_0 L_k (w_0^{-1} \frac{d}{dt} w_1 F).$$

However, L_k is the differential operator $\left(\frac{w_1}{w_0}\frac{d}{dt}\right)^k$, so each side of the last equation may be presented as a product of differential operators

$$\frac{1}{w_0}\frac{d}{dt}\left(w_1\Big(\frac{w_1}{w_0}\frac{d}{dt}\Big)^k F\right) \equiv \Big(\frac{w_1}{w_0}\frac{d}{dt}\Big)^k\Big(\frac{1}{w_0}\frac{d}{dt}\,w_1 F\Big). \tag{13}$$

Similarly, (1) and (3) are equivalent, respectively, to

$$\frac{d}{dt}(w_1 F) = w_0 f \quad\text{ and }\quad \frac{d}{dt}w_{k+1}\frac{w_k^2}{w_{k-1}w_{k+1}}\frac{d}{dt}\cdots\frac{w_1^2}{w_0 w_2}\frac{d}{dt}F = w_k\frac{d^k}{dt^k}f,$$

which imply, as above, the differential operator identity

$$\frac{1}{w_k}\frac{d}{dt}\frac{w_k^2}{w_{k-1}}\frac{d}{dt}\frac{w_{k-1}^2}{w_{k-2}w_k}\cdots\frac{w_1^2}{w_0 w_2}\frac{d}{dt}F \equiv \frac{d^k}{dt^k}\Big(\frac{1}{w_0}\frac{d}{dt}w_1 F\Big),\qquad w_i = \int w_{i-1}. \tag{14}$$

(13) represents two different Pólya factorizations of the same differential operator, say

$$\alpha_{k+1}\frac{d}{dt}\alpha_k\ldots\alpha_1\frac{d}{dt}\alpha_0 F \equiv \beta_{k+1}\frac{d}{dt}\beta_k\ldots\beta_1\frac{d}{dt}\beta_0 F,\qquad \alpha_i,\beta_i > 0. \tag{15}$$

The same holds for (14). For more details about such factorizations, see [Pol], [Tre], [Cop]. One way to verify such identities is to show that the $(k+1)$st order linear differential operators on both sides have the same leading coefficient $\alpha_{k+1}\ldots\alpha_0 = \beta_{k+1}\ldots\beta_0$ and the same null space. For example, for (13) it follows by direct calculation (and using repeatedly $w_0 = w_1{}'$) that the null spaces of both sides are spanned by the $k+1$ functions $\{w_1^{-1}, 1, \log w_1, (\log w_1)^2, \ldots, (\log w_1)^{k-1}\}$ and $\alpha_{k+1}\ldots\alpha_0 = \beta_{k+1}\ldots\beta_0 = (w_1/w_0)^{k+1}$.

Now we turn to the inverse question, namely are any two factorizations like (15) of a differential operator a source of integral mean identities like (2) and (3)? If (15) is factored into products of differential operators

$$\frac{1}{v_0}\frac{d}{dt}\big(v_1 N_k[F]\big) \equiv M_k\Big(\frac{1}{w_0}\frac{d}{dt}(w_1[F])\Big), \tag{16}$$

where M_k, N_k are kth order differential operators and $v_0^{-1}\frac{d}{dt}v_1$, $w_0^{-1}\frac{d}{dt}w_1$ are first order differential operators, $v_1 = \int v_0$, $w_1 = \int w_0$, then

$$F(t) = \frac{\int w_0 f}{\int w_0} \quad\text{ implies }\quad N_k[F](t) = \frac{\int v_0\, M_k[f]}{\int v_0}.$$

(15) can be always written as (16) in a trivial way. All one has to do is to multiply (15) from the right and the left hand sides by suitable factors and split α_k, β_1 into products. This is, of course, useless, since multiplying, say, v_0 by a positive function leads to multiplication of M_k by the reciprocal of the same function. So, the real question is to obtain interesting factorizations (16) with "nice" operators M_k, N_k. We exhibit such a result, based on the identity

$$t^{-n+1}\Big(\frac{d}{dt}\Big)t^n\Big(\frac{d}{dt}\Big)^{2n-1}F \equiv \Big(\frac{d}{dt}\Big)^{2n-1}t^n\Big(\frac{d}{dt}\Big)t^{-n+1}F. \tag{17}$$

(17) is easily established, since both sides are equal to $tF^{(2n)} + nF^{(2n-1)}$.

Theorem 2. *Let $f \in C^{2n-1}[0,\infty)$, $t^n f \in L(t_0,\infty)$ and $f^{(n-1)}(0) = 0$. If*

$$F(t) = \frac{\int_t^\infty s^{-n} f(s)\,ds}{\int_t^\infty s^{-n}\,ds}, \qquad 0 \le t < \infty, \tag{18}$$

then

$$\frac{n}{n-1}\,F^{(2n-1)}(t) = \frac{\int_0^t s^{n-1} f^{(2n-1)}(s)\,ds}{\int_0^t s^{n-1}\,ds}. \tag{19}$$

Proof. Let $F(t)$ be defined by (18), that is, $t^{-n+1}F(t)/(n-1) = \int_t^\infty s^{-n} f(s)\,ds$, or

$$(n-1)f(t) = t^n \frac{d}{dt}\big(t^{-n+1}F(t)\big). \tag{20}$$

We differentiate (20) $2n-1$ times and get by the identity (17) that

$$(n-1)f^{(2n-1)} = \frac{d^{2n-1}}{dt^{2n-1}} t^n \frac{d}{dt}\big(t^{-n+1}F\big) = t^{-n+1}\frac{d}{dt}\big(t^n F^{(2n-1)}\big),$$

or

$$t^n F^{(2n-1)}\Big|_0^t = (n-1)\int_0^t s^{n-1} f^{(2n-1)}\,ds. \tag{21}$$

The lower limit has to be calculated carefully due to the definition of F at $t=0$. To determine the behaviour of F and its derivatives near $t=0$, we differentiate (20) $n-1$ times,

$$(n-1)f^{(n-1)} = \big(t^n(t^{-n+1}F)'\big)^{(n-1)} = \big(tF' + (-n+1)F\big)^{(n-1)} = tF^{(n)},$$

i.e., $F^{(n)} = (n-1)f^{(n-1)}(t)t^{-1}$. By $n-1$ more differentiations and passing to limit as $t \to 0$, it follows that $t^n F^{(2n-1)}\big|_0 = (-1)^n(n-1)(n-1)!f^{(n-1)}(0)$, and this equals 0 due to our assumption. Thus (21) becomes

$$t^n F^{(2n-1)}(t) = (n-1)\int_0^t s^{n-1} f^{(2n-1)}\,ds,$$

which is precisely (19). □

Beginning with the identity

$$t^{-n+1}\frac{d}{dt}t^{n+1}\Big(\frac{d}{dt}\Big)^n \equiv \Big(\frac{d}{dt}\Big)^n t^{n+1}\frac{d}{dt}t^{-n+1},$$

it can be shown that (18) implies

$$\frac{n}{n-1}\,tF^{(n)}(t) = \frac{\int_0^t s^{n-1}(sf(s))^{(n)}\,ds}{\int_0^t s^{n-1}\,ds}.$$

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