PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 2, Pages 475–484
S 0002-9939(99)05140-0
Article electronically published on May 27, 1999

BOUNDS FOR SOLUTIONS OF A DIFFERENTIAL INEQUALITY

URI ELIAS

(Communicated by Hal L. Smith)

Abstract. This work compares the solutions of an nth order differential inequality plus n boundary conditions with the solution of the related differential equation with $n-1$ boundary conditions. The differential operator is assumed to be disconjugate. It is proved that under suitable conditions the ratio of these solutions is monotone. The solution of the inequality can be replaced by the corresponding Green's function.

1

A series of papers by Erbe and Wang [ErW], Erbe, Hu and Wong [EHW], Eloe and Henderson [EH3], [EH4] (see more references in [EH4]) studies the existence of positive solutions for nonlinear differential equations of the type

$$y^{(n)} + a(t)f(y) = 0$$

with certain boundary conditions, by utilizing lower bounds for solutions of differential inequalities and Green's functions. A typical example for these bounds is the following result [EH1, Theorem 1]: *If*

$$\begin{aligned} &(-1)^{n-k}y^{(n)}(t) \geq 0, \\ &y^{(i)}(0) = 0, \qquad i = 0, \dots, k-1, \\ &y^{(j)}(1) = 0, \qquad j = 0, \dots, n-k-1, \end{aligned} \tag{1}$$

then

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} y(t) \geq 4^{-m} \max_{t\in[0,1]} y(t), \qquad m = \max\{k, n-k\}. \tag{2}$$

A similar inequality holds for $(-1)^{n-k}G(t,s)$ for each s, $0 < s < 1$, when G is Green's function of the operator $\frac{d^n}{dx^n}$ and the bv conditions of (1). A generalization of (2) for multipoint boundary value problems is proved in [EH2].

The proof of inequality (2) is based on the following [EH1, Lemma 2]:

Proposition. *Let $y(t)$ satisfy (1) and let t_1 denote its unique local maximum point in $[0,1]$. Then*

$$y(t) \geq \begin{cases} \left(\frac{t}{t_1}\right)^k \max_{[0,1]} |y|, & 0 \leq t \leq t_1, \\ \left(\frac{1-t}{1-t_1}\right)^{n-k} \max_{[0,1]} |y|, & t_1 \leq t \leq 1. \end{cases} \tag{3}$$

Received by the editors March 25, 1998.
1991 *Mathematics Subject Classification.* Primary 34C10.
Key words and phrases. Differential inequality, disconjugate differential operator.

The aim of this work is to present some variations on the theme of inequality (3). It will be shown that the proof of [EH1, Lemma 2] in fact contains more than is claimed in (3), and it will be generalized in various directions. For example, not only a lower bound for y/t^k is available but even its monotonicity. It extends also to other boundary conditions and is compared with other solutions of the corresponding homogeneous differential equation.

We prefer to formulate the results in terms of *quasi-derivatives* $L_0 y = \rho_0 y$, $L_i y = \rho_i (L_{i-1} y)'$, $\rho_i > 0$, $i = 1, \dots, n$, instead of derivatives $y^{(i)}$, and a general disconjugate operator

$$L_n y = \rho_n(\rho_{n-1} \dots (\rho_1(\rho_0 y)')' \dots)'$$

which replaces $y^{(n)}$, since these exchanges have no effect on most of the proofs.

Theorem 1. *Let y satisfy*

$$(-1)^{n-k} L_n y(t) \ge 0 \qquad \text{on } [a,b], \tag{4}$$

$$\begin{aligned} L_i y(a) &= 0, \qquad i = 0, \dots, k-1, \\ L_j y(b) &= 0, \qquad j = 0, \dots, n-k-1, \end{aligned} \tag{5}$$

and let φ_k be the solution of the initial value problem

$$\begin{aligned} &L_n u = 0, \\ &L_i u(a) = \delta_{i,k}, \qquad i = 0, \dots, n-1, \end{aligned}$$

i.e., $\varphi_k(t) = \rho_0^{-1}(t) \int_a^t \rho_1^{-1}(t_1) \int_a^{t_1} \dots \int_a^{t_{k-1}} \rho_k^{-1}(t_k)\, dt_k \dots dt_1$. Then y/φ_k is nonincreasing on $[a,b]$. If $(-1)^{n-k} L_n y(t) > 0$ on (a,b), then y/φ_k is strictly decreasing there.

Analogously, if ψ_{n-k} is the solution of $L_n u = 0$, $L_i u(b) = (-1)^{n-k}\delta_{i,n-k}$, $i = 0, \dots, n-1$, then y/ψ_{n-k} is nondecreasing on $[a,b]$ and it is strictly increasing if inequality (4) is strict.

For $L_n y = y^{(n)}$, $\varphi_k = t^k/k!$, we get that the quotient y/t^k is decreasing on $(0,1]$. In particular (3) follows. However, much more is available:

Theorem 2. *If y satisfies*

$$\begin{aligned} (-1)^{n-k} y^{(n)}(t) &\ge 0, \qquad \text{on } [0,1], \\ y^{(i)}(0) &\ge 0, \qquad i = 0, \dots, k-1, \\ (-1)^j y^{(j)}(1) &\ge 0, \qquad j = 0, \dots, n-k-1, \end{aligned}$$

then

$$(-1)^\ell \frac{d^\ell}{dt^\ell}\Big(\frac{y}{t^k}\Big) \ge 0, \qquad \text{on } (0,1], \quad \ell = 0, \dots, n-k. \tag{6}$$

If an inequality in one of the n boundary conditions at $t = 0$ or $t = 1$ is strict or if $(-1)^{n-k} y^{(n)}(t) > 0$ on $[0,1]$, then also inequality (6) is strict on $(0,1)$.

This applies, in particular, to the solutions of (1).

The solutions $\varphi_k(t), \psi_{n-k}$ are not the only possible comparison functions. For example,

Theorem 3. *Let y satisfy the differential inequality (4) and boundary conditions (5) and let $w_k(x)$ be the solution of the differential equation*

$$L_n w = 0 \tag{7}$$

and the $n-1$ boundary conditions

$$\begin{aligned} L_i w(a) &= 0, \qquad i = 0, \dots, k-1, \\ L_j w(b) &= 0, \qquad j = 0, \dots, n-k-2, \end{aligned} \tag{8}$$

normalized, say, by $L_k w(a) = 1$. Then y/w_k is nonincreasing on $[a,b]$. If inequality (4) is strict, then y/w_k is strictly decreasing.

If $w_{k-1}(x)$ is the solution of the same equation and the boundary conditions

$$\begin{aligned} L_i w(a) &= 0, \qquad i = 0, \dots, k-2, \\ L_j w(b) &= 0, \qquad j = 0, \dots, n-k-1, \end{aligned}$$

then y/w_{k-1} is nondecreasing on $[a,b]$; it is strictly increasing if $(-1)^{n-k} L_n y > 0$ there.

In the next section we prove Theorems 1-3. In the third section they are generalized for more boundary value conditions. We shall also show that Green's functions are analogous to solutions of differential inequalities.

2

Our arguments rely on some facts which are well known in the theory of disconjugacy and which we recall here. Namely, the n boundary conditions (5) enforce through Rolle's theorem a minimal number of zeros for each of the quasi-derivatives $L_0 y, \dots, L_n y$, while inequality (4) sets an upper bound to their number. Putting these two opposing trends together, we have

Lemma. *If y satisfies the differential inequality $(-1)^{n-k} L_n y > 0$ on (a,b) and the boundary conditions (5), then*

(a) Between any two zeros of $L_t y$, $t = 0, 1, \dots, n-1$, there is precisely one simple zero of $L_{t+1} y$ and, in addition to the n zeros which are prescribed by (5), these are the only zeros of $L_{t+1} y$ in $[a,b]$.

(b) $y > 0$ on (a,b).

Proof. First we observe that since $(-1)^{n-k} L_n y > 0$ on (a,b), no $L_t y$ may vanish identically on any subinterval of $[a,b]$ and all their zeros are isolated. Since y has $k + (n-k) = n$ zeros at $t = a$ and $t = b$, it follows by Rolle's theorem that each of $L_0 y, \dots, L_{n-1} y$ has at least one zero in $[a,b]$. Suppose that one of them also has some zero which is not deduced by Rolle's theorem, i.e., some $L_i y$ has in (a,b) a zero which is not located between two zeros of $L_{i-1} y$, or $L_i y$ has more than one simple zero between two zeros of $L_{i-1} y$. Then a repeated application of Rolle's theorem will imply that $L_n y$ changes its sign in (a,b), which is impossible. This verifies (a). The same argument shows, in particular, that $y \neq 0$ in (a,b).

As a consequence it follows that if t_i denotes the first zero of $L_i y$ in $(a,b]$, then

$$a < t_k < t_{k-1} < \dots < t_1 < t_0 = b, \tag{9}$$

$$a < t_k < t_{k+1} < \dots < t_{n-1} < b. \tag{10}$$

(9) is obvious since by the boundary values at $t = a$ Rolle's theorem implies $t_i \in (a, t_{i-1})$, $i = 1, \dots, k$. (10) is a more delicate consequence, which follows from (a),

since for $i = k+1, \dots, n$, $L_i y$ cannot have a zero in (a, t_{i-1}). Another formulation is that on a right neighborhood of a,

$$L_i y(a+\epsilon) L_{i+1} y(a+\epsilon) > 0, \qquad i = 0, \dots, k-1, \tag{11}$$

$$L_i y(a+\epsilon) L_{i+1} y(a+\epsilon) < 0, \qquad i = k, \dots, n-1. \tag{12}$$

(11) is obvious and (12) is a consequence of (10). For, if $L_i y$ and $L_{i+1} y = \rho_{i+1}(L_i y)'$ have the same sign on the right hand side of a, then $|L_i y|$ increases near a and the first zero of $L_{i+1} y$ must precede that of $L_i y$, contradicting (10).

It follows by (11),(12) that $\operatorname{sgn}(L_n y) = (-1)^{n-k} \operatorname{sgn}(L_0 y)$, and since $y \neq 0$ and $\operatorname{sgn}(L_n y) = (-1)^{n-k}$ by (4), we have $y > 0$.

Note that the lemma may be formulated even if $(-1)^{n-k} L_n y \geq 0$ and it vanishes identically on some subinterval. However, in this situation, one has to replace 'zero points' by 'zero components' — maximal subintervals throughout which a function vanishes. See [Cop, p. 107]. □

Proof of Theorem 1. Let us discuss first the case $(-1)^{n-k} L_n y > 0$. Suppose that y/φ_k is not monotone in $[a, b]$. Then for some $c > 0$, y/φ_k attains the value c at least at two different points and the corresponding function

$$h = y - c\varphi_k$$

has two distinct zeros in (a, b). Note that since $(-1)^{n-k} L_n h > 0$, $h \not\equiv 0$ on any subinterval and y/φ_k cannot be constant on any subinterval. After these elementary observations one can follow verbatim the arguments of [EH1, Lemma 2]. For the sake of convenience we summarize it here.

Let us denote the first two zeros of $L_i h$, $i = 0, \dots, k$, in $(a, b]$ by $c_{i,1}, c_{i,2}$. We claim that they exist and that

$$a < c_{i,1}, c_{i,2} < t_i, \qquad i = 0, \dots, k. \tag{13}$$

For $i = 0$ this is self-evident since $t_0 = b$ and $h(t)$ has two zeros in (a, b). Suppose that for some $j < k$, (13) is proved, i.e., $L_j h$ has two zeros $c_{j,1}, c_{j,2}$ such that

$$a < c_{j,1} < c_{j,2} < t_j.$$

Then the next quasi-derivative $L_{j+1} h$ has two zeros $c_{j+1,1} \in (a, c_{j,1})$, $c_{j+1,2} \in (c_{j,1}, c_{j,2})$, i.e., both in (a, t_j). We want to show that they are in fact in the smaller interval $(a, t_{j+1}) \subset (a, t_j)$. On the right hand side of the k-tuple zero of y at $t = a$, we have $L_0 y > 0, \dots, L_k y > 0$. Hence $L_{j+1} y > 0$ in (a, t_{j+1}) and $L_{j+1} y < 0$ in (t_{j+1}, t_j), while $L_{j+1} \varphi_k > 0$ on the whole $(a, b]$. Consequently

$$L_{j+1} h = L_{j+1} y - c L_{j+1} \varphi_k \neq 0 \quad \text{ in } [t_{j+1}, t_j],$$

and $c_{j+1,1}, c_{j+1,2}$ must be located in (a, t_{j+1}). This proves (13) for all $j = 0, \dots, k$.

Thus, we conclude finally that $L_k h$ has two zeros in (a, t_k) and consequently $L_{k+1} h$ has at least one zero in (a, t_k). But this is impossible since $L_{k+1} h \equiv L_{k+1} y$, and the first zero of $L_{k+1} y$ is t_{k+1}, which satisfies $t_{k+1} > t_k$ according to (10).

This contradiction shows that y/φ_k must be monotone. Since y/φ_k is positive in (a, b) and vanishes at b, it is strictly decreasing. Note that a separate treatment of the cases $k = 1$, $k = n-1$, as in [EH1], is avoided.

If it is only assumed that $(-1)^{n-k} L_n y \geq 0$, we replace y by $y_\epsilon = y + \epsilon \int_a^b G(t, s)\, ds$, $\epsilon > 0$, where G is Green's function of the operator $(-1)^{n-k} L_n$ and bc (5). Then $(-1)^{n-k} L_n y_\epsilon = (-1)^{n-k} L_n y + \epsilon > 0$ and y_ϵ / φ_k is strictly increasing. As $\epsilon \searrow 0$, we conclude that y/φ_k is nondecreasing. □

Proof of Theorem 2. We need the following generalization of the inequality of Kiguradze: *If for some* $k < n$

$$\begin{aligned} y^{(i)}(0) &\geq 0, \qquad i = 0, \dots, k-1, \\ (-1)^{n-k} y^{(n)}(t) &\geq 0 \quad \text{on } [0, L], \end{aligned} \tag{14}$$

then

$$(-1)^{n-k}\Big(\frac{y}{t^k}\Big)^{(n-k)} \geq 0 \quad \text{on} \quad (0, L].$$

(See [Eli, Theorem 6.12].) Our y satisfies (14) according to (1), therefore

$$z(t) = y/t^k$$

satisfies $(-1)^{n-k} z^{(n-k)} \geq 0$ on $(0, 1]$. By the identity

$$\begin{aligned} z^{(\ell)} = (y/t^k)^{(\ell)} &= \sum_{j=0}^{\ell} \binom{\ell}{j} y^{(j)} (t^{-k})^{(\ell-j)} \\ &= \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \frac{(k+\ell-j-1)!}{(k-1)!} y^{(j)} t^{-k-\ell+j}, \end{aligned}$$

and the boundary conditions $(-1)^j y^{(j)}(1) \geq 0$, $j = 0, \dots, n-k-1$, it follows that $(-1)^\ell z^{(\ell)}(1) \geq 0$, $\ell = 0, \dots, n-k-1$. Now the inequalities

$$\begin{aligned} (-1)^{n-k} z^{(n-k)}(t) &\geq 0 \qquad \text{on } (0, 1], \\ (-1)^\ell z^{(\ell)}(1) &\geq 0, \qquad \ell = 0, \dots, n-k-1, \end{aligned} \tag{15}$$

imply (6) by $n-k$ repeated integrations of $z^{(n-k)}$ on $[t, 1]$.

Next we turn to the cases of strict inequalities. If, say, $y^{(i_0)}(0) > 0$ for some i_0, $0 \leq i_0 \leq k-1$, then take $v = y - y^{(i_0)}(0) t^{i_0}/i_0!$. v satisfies (14) and it is concluded as above that $(-1)^{n-k}(v/t^k)^{(n-k)} \geq 0$ on $(0, 1]$. Consequently

$$(-1)^{n-k} z^{(n-k)} = (-1)^{n-k}(v/t^k)^{(n-k)} + y^{(i_0)}(0)(-1)^{n-k}(t^{i_0-k}/i_0!)^{(n-k)} > 0$$

on $(0, 1]$. If, on the other hand, $(-1)^{j_0} y^{(j_0)}(1) > 0$ for some j_0, $0 \leq j_0 \leq n-k-1$, we get $(-1)^\ell z^{(\ell)}(1) > 0$ for all $j_0 \leq \ell \leq n-k-1$. In both cases it follows by repeated integrations of (15) that $(-1)^\ell z^{(\ell)}(t) > 0$ on $(0, 1)$.

Finally, if $(-1)^{n-k} y^{(n)} > 0$ on $[0, 1]$, take $v = y - \epsilon t^k (1-t)^{n-k}$, with $\epsilon > 0$ small enough such that $(-1)^{n-k} v^{(n)} = (-1)^{n-k} y^{(n)} - n!\epsilon \geq 0$. Then

$$(-1)^\ell (y/t^k)^{(\ell)} = (-1)^\ell (v/t^k)^{(\ell)} + \epsilon (-1)^\ell ((1-t)^{n-k})^{(\ell)} > 0$$

for $\ell = 1, \dots, n-k$ on $(0, 1)$. □

Proof of Theorem 3. The proof is even more simple than that of Theorem 1, and we shall outline only some points. If y/w_k is not monotone in $[a, b]$, then for some $c > 0$, y/w_k attains the value c at least at two different points and the function $h = y - cw_k$ has two zeros in (a, b). According to the boundary conditions $y - cw_k$ also has $n-1$ zeros at the endpoints a, b, i.e., altogether $n+1$ zeros in $[a, b]$. But then the nth quasi-derivative $L_n h$ must change its sign in (a, b), in contradiction with $(-1)^{n-k} L_n h \equiv (-1)^{n-k} L_n y(t) > 0$. Therefore y/w_k is monotone.

Since w_k is a solution of a disconjugate equation, it cannot have more than the $k+(n-k-1) = n-1$ zeros which are prescribed at $t = a, b$, so their multiplicities are precisely k and $n-k-1$, respectively. Since $y/w_k > 0$ on (a, b) while $\lim_{t \to b^-} y/w_k =$

0, y/w_k must be decreasing. The nonstrict inequality is treated as in the proof of Theorem 1.

The arguments for y/w_{k-1} are analogous. □

3

As one observes the proofs of the Lemma and Theorem 3, it is evident that they depend on nothing but a repeated application of Rolle's theorem to ensure existence of zeros of higher quasi-derivatives. Therefore the results can be extended to boundary conditions other than (5).

For example, the proof of (9) in the Lemma relies on the k-tuple zero of y at a. However, even if the quasi-derivatives of y do not vanish at a but rather $L_{i-1}y$, and $L_i y = \rho_i(L_{i-1}y)'$ have the same sign on the right hand side of a, $L_i y$ must have a zero in $(a, b]$ before the first zero of $L_{i-1}y$ (recall the graph of $L_{i-1}y(t)$!). Similarly, if $L_{j-1}y$ and $L_j y$ have opposite signs on the left hand side of b, $L_j y$ has a zero after the last zero of $L_{j-1}y$ in $[a, b)$. Therefore one can replace the boundary conditions (5) by

$$\begin{aligned}
&(L_i y - \alpha_i L_{i+1}y)(a) = 0, \qquad i = 0, \dots, k-1,\\
&(L_j y + \beta_j L_{j+1}y)(b) = 0, \qquad j = 0, \dots, n-k-1,\\
&\qquad \alpha_i \ge 0,\ \beta_j \ge 0.
\end{aligned}$$

It is easily checked that (9)-(10), (11)-(12) and Theorem 1 hold true for these boundary conditions. We skip the technical details. Once it is known that the quotient y/φ_k is monotone, it is possible to show that it decreases by considering $\alpha_i,\ \beta_j \to 0^+$ and a continuity argument.

Another possibility is to take such n boundary conditions

$$\begin{aligned}
L_i y(a) &= 0, \qquad i = i_1, \dots, i_k,\\
L_j y(b) &= 0, \qquad j = j_1, \dots, j_{n-k},
\end{aligned} \tag{16}$$

$i_1 < i_2 < \dots < i_k$, $j_1 < \dots < j_{n-k}$, which imply that each quasi-derivative $L_0 y, \dots, L_{n-1}y$ has some zero in $[a, b]$. This is guaranteed by Pólya's condition for Hermite-Birkhoff interpolation:

At least ℓ boundary conditions of the boundary conditions are imposed on the first ℓ quasi-derivatives $L_0 y, \dots, L_{\ell-1}y$ for $\ell = 1, \dots, n$.

By this condition for $\ell = 1$, at least one boundary condition is imposed on $L_0 y$. For $\ell = 2$, either one zero is imposed on $L_1 y$ or two zeros on $L_0 y$. In the latter case $L_1 y$ has a zero in (a, b). An inductive argument shows that each $L_0 y, \dots, L_{n-1}y$ has some zero in $[a, b]$.

Pólya's condition is satisfied, for example, by the $(k, n-k)$-focal boundary conditions $L_i y(a) = 0,\ i = 0, \dots, k-1$, $L_j y(b) = 0,\ j = k, \dots, n-1$.

Theorem 4. *Let y satisfy differential inequality (4),*

$$(-1)^{n-k} L_n y(t) \ge 0 \qquad \text{on } [a, b],$$

and the boundary conditions (16),

$$\begin{aligned}
L_i y(a) &= 0, \qquad i = i_1, \dots, i_k,\\
L_j y(b) &= 0, \qquad j = j_1, \dots, j_{n-k},
\end{aligned}$$

which obey Pólya's condition. Let us delete from (16) one boundary condition so that the remaining $n-1$ satisfy Pólya's condition for the $n-1$ quasi-derivatives

$L_0y, \dots, L_{n-2}y$, and let $w(t)$ be a positive solution of the differential equation (7) which satisfies these $n-1$ boundary conditions. Then y/w is monotone on $[a,b]$.

If the bc which was omitted from (16) is at the endpoint b, y/w is nonincreasing, while if it is at $t=a$, y/w is nondecreasing. If $(-1)^{n-k}L_ny(t) > 0$ on (a,b), y/w is strictly monotone.

Theorems 1, 3 and 4 have analogues, where the solution of the differential inequality $(-1)^{n-k}L_ny(t) \geq 0$ is replaced by the Green's function of the operator $(-1)^{n-k}L_n$ plus n suitable boundary conditions. We formulate and prove only one of these results, the analogue of Theorem 4.

Theorem 5. *Let $G(t,s)$ be Green's function of the operator $(-1)^{n-k}L_n$ and the boundary conditions (16) which obey Pólya's condition and let $w(t)$ be the solution of $L_ny = 0$ which was defined in Theorem 4. Then for each fixed s, $a < s < b$, $G(t,s)/w(t)$ is increasing or decreasing according to whether the highest boundary condition which was omitted from (16) was at $t=a$ or at $t=b$, respectively. It is strictly monotone unless all $n-1$ bc which define w are concentrated at one of the endpoints a or b.*

Note that the results of [EH1] are formulated for Green's function of the operator L_n which differs from our result by the factor $(-1)^{n-k}$.

Proof of Theorem 4. This proof, too, is analogous to that of Theorem 1. By Pólya's condition, for every function y which satisfies (16), each L_iy, $i = 1, \dots, n-1$, has at least one zero in $[a,b]$. The same argument as in the proof of the Lemma shows that for y which satisfies inequality (4), no L_iy has in (a,b) a zero which is not located between two zeros of $L_{i-1}y$. Also (9),(10) have analogies:

$$\begin{aligned} t_{i+1} < t_i \quad &\text{if} \quad i = i_1, \dots, i_k, \\ t_{i+1} > t_i \quad &\text{if} \quad i \neq i_1, \dots, i_k. \end{aligned}$$

Similarly,

$$\begin{aligned} L_iy(a+\epsilon)L_{i+1}y(a+\epsilon) > 0, \qquad & i = i_1, \dots, i_k, \\ L_iy(a+\epsilon)L_{i+1}y(a+\epsilon) < 0, \qquad & i \neq i_1, \dots, i_k. \end{aligned}$$

Consequently (4),(16) imply that $y > 0$ on (a,b).

According to the assumption, the $n-1$ boundary conditions which determine w, satisfy Pólya's condition only for the $n-1$ quasi-derivatives $L_0w, \dots, L_{n-2}w$, therefore each of $L_0w, \dots, L_{n-2}w$ has at least one zero in $[a,b]$. If w also has an additional zero in (a,b), the same reasoning implies that even $L_{n-1}w$ has a zero in (a,b), contradicting $L_nw = \rho_n(L_{n-1}w)' = 0$. Therefore $w \neq 0$ in (a,b) and it may be taken that $w > 0$. Note that one obvious way to choose w is to delete the bc from (16) which involves the highest among the appearing quasi-derivatives.

Now we continue along the outlines of the proof of Theorem 3. $h = y - cw$ satisfies the same $n-1$ bc as w does, and if for some $c > 0$, it also has two zeros in (a,b), then each $L_0h, \dots, L_{n-2}h$ has at least 3 zeros in $[a,b]$. But then $L_nh = L_ny$ must change its sign in (a,b), in contradiction with $(-1)^{n-k}L_nh > 0$. It follows that any equation $y/w = c$, $c > 0$, has at most one root in (a,b). Since $w \neq 0$ on (a,b), it follows that the continuous y/w is strictly monotone there.

It remains to decide whether y/w is decreasing or increasing. Consider the case in which the bc which was omitted from (16) is at the endpoint b, say $L_{j_{n-k}}y(b) = 0$.

Since y/w is monotone, the limits

$$\lim_{t\to a^+} y/w = A \geq 0, \qquad \lim_{t\to b^-} y/w = B \geq 0$$

exist, possibly $+\infty$. Suppose that y/w is increasing, i.e.,

$$0 \leq A \leq y/w \leq B \leq +\infty, \qquad A < B.$$

Consider the function $h = y - Aw \geq 0$. Then $\lim h/w = 0$ and h has at $t = a$ a zero of multiplicity greater than y has there. If, say, the multiplicity of the zero of y at $t = a$ is r, i.e., $i_1 = 0, i_2 = 1, \ldots, i_r = r-1$ but $i_{r+1} > r$, then h has a zero of multiplicity there at least $r+1$: $L_i h(a) = 0$ for $i = 0, \ldots, r, i_{r+1}, \ldots i_k$. But thus $(-h)$ satisfies

$$(-1)^{n-k-1} L_n(-h) \equiv (-1)^{n-k} L_n y > 0 \qquad \text{on } [a,b],$$

and the $(k+1)+(n-k-1) = n$ boundary conditions

$$\begin{aligned} L_i h(a) &= 0, \qquad i = 0, \ldots, r, i_{r+1}, \ldots i_k, \\ L_j h(b) &= 0, \qquad j = j_1, \ldots, j_{n-k-1}, \end{aligned}$$

i.e., $(-h)$ satisfies an assumption similar to (5),(16) with k replaced by $k+1$. But we saw that this implies $-h \geq 0$, a contradiction. Therefore y/w cannot increase.

If $L_n y$ satisfies a nonstrict inequality, the proof is completed as that of Theorem 1. The properties of the required Green's function are verified below. □

Proof of Theorem 5. Green's function of the operator $(-1)^{n-k} L_n y = 0$ plus the boundary conditions (16) exists since the equation $(-1)^{n-k} L_n y = 0$ with the same boundary conditions has only the trivial solution. Otherwise, if such $y \not\equiv 0$ exist, then by Pólya's condition each $L_i y$, $i = 0, \ldots, n-1$, has a zero in $[a,b]$. But $\rho_n (L_{n-1} y)' = L_n y = 0$, so $L_{n-1} y = \text{const}$, and if it vanishes at one point of $[a,b]$, then it is identically zero there. By the same argument one gets recursively that $L_{n-2} y \equiv 0, \ldots, L_0 y \equiv 0$. Consequently Green's function exists.

Green's function of bc (16) and a solution y of (4),(16) have some analogy: they both satisfy the same n bc but on the other hand the $(n-1)$th quasi-derivative $L_{n-1} G$ is discontinuous. Thus, one cannot apply Rolle's theorem to $L_{n-2} G$ even if it has two zeros in $[a,b]$; $L_{n-1} G$ may change its sign at its discontinuity point without vanishing there. Nevertheless we claim that $G(t,s) > 0$ on (a,b) for any fixed s, $a < s < b$.

To verify that $G(t,s) \neq 0$, we distinguish between two cases, namely whether a boundary condition is imposed on the $(n-1)$th quasi-derivative or not.

(i) If all n boundary conditions of (16) are imposed on $L_0 y, \ldots, L_{n-2} y$, then we deduce that $L_{n-2} G(t,s)$ has at least two zeros in $[a,b]$. If, in addition, $G(t,s)$ has a zero in (a,b), then $L_{n-2} G(t,s)$ would have at least 3 zeros. Thus $L_{n-1} G(t,s)$ would change its sign at least twice in (a,b). But this is impossible since $L_n G = \rho_n (L_{n-1} G)' = 0$, and $L_{n-1} G$ is constant on $[a,s)$ and on $(s,b]$.

(ii) Suppose that a boundary condition is imposed by (16) on $L_{n-1} y$, say $L_{n-1} y(b) = \ldots = L_r y(b) = 0$, $L_{r-1} y(b) \neq 0$. Then $L_r G = \ldots = L_{n-1} G \equiv 0$ on $(s,b]$. By Pólya's condition at least r boundary conditions are imposed at the endpoints on $L_0 y, \ldots, L_{r-1} y$. If in addition $G(t,s)$ has a zero in (a,b), then $L_r G$ changes its sign in (a,b). That exchange of sign cannot be located in (s,b) since $L_r G \equiv 0$. So $L_r G$ has a change of sign in (a,s) and, of course, $L_r G(s,s) = \ldots =$

$L_{n-2}G(s,s) = 0$. But this would imply that $L_{n-1}G$ changes its sign in (a,s), contradicting $L_{n-1}G$ =const.

Thus it is shown that $G(t,s) \neq 0$ in (a,b), and it remains only to determine its sign. But $y(t) = \int_a^b G(t,s)\,ds$ is the solution of the equation $(-1)^{n-k}L_n y = 1$ plus bc (16), which is positive according to Theorem 4. Therefore $G(t,s) > 0$ on (a,b) also. The analogous $w > 0$ was proved in Theorem 4.

Finally we turn to the main claim of the theorem. If for some fixed $s \in (a,b)$, $G(t,s)/w(t)$ is not monotone, then a function $h(t) = G(t,s) - cw(t)$ has two zeros in (a,b) and it satisfies the same $n-1$ of the bc (16) as w does. This leads to a contradiction as above: For each $\ell - 1 \leq n-2$, $L_0 h, \ldots, L_{\ell-1}h$ satisfy at least ℓ boundary conditions at the endpoints and by the two additional zeros in (a,b), each of $L_0 h, \ldots, L_{n-2}h$ has at least three zeros in $[a,b]$. So $L_{n-1}h$ has two changes of sign in (a,b), which is impossible since $L_{n-1}h$ is constant on $[a,s]$ and on $[s,b]$. Thus $G(t,s)/w(t)$ is monotone, and it is determined whether it increases or decreases as in the proof of Theorem 4.

If $G(t,s)/w(t)$ is not strictly monotone but rather constant on some subinterval, it must be constant either on $[a,s]$ or on $[s,b]$. Let, for example, $G(t,s)/w(t) = \alpha(s)$ on $[a,s]$ and consider the function $h(t) = G(t,s) - w(t)\alpha(s)$ on $[s,b]$. By the standard definition of Green's functions and its continuity conditions at $t = s$, $h(t)$ is the solution of the initial value problem

$$\begin{aligned} L_n h &= 0 \qquad \text{on } [s,b], \\ L_i h(s) &= (-1)^{n-k}(\rho_0(s)\ldots\rho_n(s))^{-1}\delta_{i,n-1}, \quad i = 0,\ldots,n-1, \end{aligned}$$

i.e., $h(t)$ is a constant multiple of $\rho_0^{-1}(t)\int_s^t \rho_1^{-1}(t_1)\int_s^{t_1}\ldots\int_s^{t_{n-2}}\rho_{n-1}^{-1}(t_{n-1})$ on $[s,b]$. But this $h(t)$ cannot satisfy any bc of (16) at $t = b$, so a contradiction is achieved when both G and w satisfy any common bc at b. The case when G/w is constant on $[s,b]$ is discussed similarly. Consequently G/w is strictly monotone when the $n-1$ bc which define w are distributed at a and at b.

On the other hand, if from the n boundary conditions $L_i y(a) = 0$, $i = 0,\ldots,n-2$, $L_0 y(b) = 0$, we delete the bc at b, then both $G(t,s)$ on $[a,s]$ and $w(t)$ are determined by the same $n-1$ homogeneous initial values at $t = a$ and so $G(t,s)/w(t)$ is constant on $[a,s]$. □

Acknowledgment

The author sincerely thanks the referee for his valuable remarks.

References

[Cop] Coppel, W. A., *Disconjugacy*, Lecture Notes in Math., vol. 220, Springer Verlag, Berlin, 1971. MR **57:**778

[ErW] Erbe, L. H. and Wang, H., *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc. **120** (1994), 743-748. MR **94e:**34025

[EHW] Erbe, L. H., Hu, S., and Wang, H., *Multiple positive solutions of some boundary value problems*, J. Math. Anal. Appl. **184** (1994), 640-648. MR **95f:**34023

[Eli] Elias, U., *Oscillation theory of two-term differential equations*, Kluwer Academic Publishers, Dordrecht, 1997. MR **98e:**34058

[EH1] Eloe, P. W. and Henderson, J., *Inequalities based on a generalization of concavity*, Proc. Amer. Math. Soc. **125** (1997), 2103-2107. MR **97i:**34018

[EH2] Eloe, P. W. and Henderson, J., *Inequalities for solutions of multipoint boundary value problems*, preprint.

[EH3] Eloe, P. W. and Henderson, J., *Positive solutions and nonlinear* $(k, n-k)$ *conjugate eigenvalue problems*, Diff. Equ. Dyn. Sys., in press.

[EH4] Eloe, P. W. and Henderson, J., *Positive solutions and nonlinear multipoint conjugate eigenvalue problems*, Electronic J. of Diff. Equ. **1997** (1997), 1-11. MR **97j:**34023

Department of Mathematics, Technion, Haifa 32000, Israel
E-mail address: elias@tx.technion.ac.il