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A method for asymptotic integration of almost diagonal systems

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Abstract. A theorem of asymptotic integration is proven for linear systems of differential equations. The theorem is designed to fit a specialized family of differential systems which occur frequently in quantum mechanics. It is shown to be best possible in a certain sense. The method provided differs from an established trend that transforms the differential system, via a preparation theorem, to a differential system, where the coefficient matrix is the sum of a diagonal matrix and a remainder matrix that must be absolutely integrable at infinity. In this work the fundamental matrix solution is given as a product of a diagonal matrix and a perturbation of the identity matrix. The perturbation of the identity matrix being on the right in the product rather than on the left as is common in the literature.

1. Introduction

The asymptotic theory of n-dimensional linear differential systems

$$Y' = A(t)Y, \tag{1.1}$$

where A, Y are $n \times n$ matrix functions asks for representation of fundamental solutions $Y(t)$ of (1.1) in the vicinity of $t = \infty$. If $A = A(t)$ is a constant matrix, then a fundamental solution $Y(t)$ of (1.1) is given by

$$Y(t) = Q\exp(Jt), \tag{1.2}$$

where $Q^{-1}AQ = J$, J is a Jordan matrix and Q is a constant invertible matrix. The representation (1.2) provides all the "local" (namely, as $t \to \infty$) essential asymptotic information of the fundamental matrix $Y(t)$ as well as all the essential global information (namely for all t) for $Y(t)$. Moreover, it also shows in a transparent manner how the properties of A, its eigenvalues and their algebraic and geometric multiplicities affect qualitative properties of $Y(t)$.

However, when $A(t)$ genuinely depends on a real or a complex variable t, a representation of a fundamental matrix $Y(t)$ analogous to (1.2) and which provides all the local and global information of $Y(t)$ is not available. The "theory of asymptotic integration" of (1.1) attempts to remedy this situation. Its importance can hardly be overestimated, as the asymptotic behaviour of solutions of nonlinear problems require quite often asymptotic integration of a linearized problem.

When $A(t)$ is meromorphic at $t = \infty$, the "local theory" is in satisfactory state, although the global theory has many outstanding problems unresolved, notably the "connection problem". See, e.g., [17–20].

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The Hukuhara–Turritin algorithm provides a representation of $Y(t)$ in certain sectors of the complex plane. When $A(t)$ is meromorphic at infinity, it is also possible to carry out the asymptotic integration in sectors of the complex plane by a method of triangularization that can be extracted from [9]. When $A(t)$ ceases to be meromorphic, the theory of asymptotic integration is much more difficult and it requires new techniques.

The asymptotic integration of a perturbed system of the form

$$Y' = \big(D(t) + R(t)\big)Y, \tag{1.3}$$

$$D(t) = \mathrm{diag}\{\lambda_1(t), \dots, \lambda_n(t)\}, \qquad R(t) = \big(r_{jk}(t)\big)_{j,k=1}^{n}, \tag{1.4}$$

is widely discussed in the literature. Since a fundamental solution of the unperturbed equation $Y' = DY$ is

$$\Phi(t) = \exp\biggl(\int_{t_0}^{t} D(s)\,\mathrm{d}s\biggr),$$

one may hope that an asymptotic representation of a fundamental solution of (1.3) be given by

$$Y(t) = \big(I + Q(t)\big)\exp\biggl(\int_{t_0}^{t} D(s)\,\mathrm{d}s\biggr) \tag{1.5}$$

with $Q(t) \to 0$ as $t \to \infty$. Some of the early theorems, like [12], provided asymptotic integration under various assumptions. One of the most simple conditions is $|\mathrm{Re}(\lambda_j(t) - \lambda_k(t))| \geqslant c$ for all $j \neq k$ and some positive constant c.

The ability to integrate (1.3) was substantially enhanced by Levinson [14] who assumed that $R \in L^1(a, \infty)$ plus "dichotomy conditions": *for each pair of integers $j \neq k$ and for all s and t such that $a \leqslant s < t < \infty$, either*

$$\int_s^t \mathrm{Re}(\lambda_j - \lambda_k)\,\mathrm{d}\tau \leqslant K_1 \tag{1.6}$$

or

$$\int_s^t \mathrm{Re}(\lambda_j - \lambda_k)\,\mathrm{d}\tau \geqslant K_2, \tag{1.7}$$

where K_1 and K_2 are some constants.

Levinson utilized an extra similarity transformation which rediagonalizes $D + R$. This was further enhanced by Harris and Lutz [10,11] who showed how to transform (1.3) into a system

$$Y_N' = \big(D_N(t) + R_N(t)\big)Y_N,$$

via repeated diagonalizations $Y_0 \equiv Y$, $Y_{j-1} = (I + Q_j)Y_j$, $j = 1, \dots, N$, so that $Q_j(\infty) = 0$ and so that

$$Y(t) = \prod_{j=1}^{N}\big(I + Q_j(t)\big)\exp\biggl(\int^{t} D_N(s)\,\mathrm{d}s\biggr).$$

However, $D_N(t)$ does not coincide necessarily with $D(t)$ which consists of the eigenvalues of the unperturbed system.

It turns out that quotients made of certain elements of $R(t)$ and certain differences $\lambda_j - \lambda_k$ play a crucial role in the asymptotic integration. If

$$\frac{1}{\lambda_j(t) - \lambda_k(t)} R(t) \to 0 \quad \text{as } t \to \infty, \tag{1.8}$$

$$\left(\frac{1}{\lambda_j(t) - \lambda_k(t)} R(t)\right)' \in L^1[a, \infty), \tag{1.9}$$

for all $j, k = 1, \dots, n$, $j \neq k$, then

$$Y(t) = \bigl(I + Q(t)\bigr) \exp\left(\int^t \widetilde{D}(s)\,\mathrm{d}s\right), \tag{1.10}$$

where $\widetilde{D}$ consists of the eigenvalues of $D + R$. See [3]. Note that these generalizations of Levinson's theorem actually use repeated transformations to reduce the given equation to Levinson's form and then apply Levinson's theorem. See also [1,13,16].

While (1.5) and (1.10) seem similar, there is an essential difference between them. In (1.5) there appear the eigenvalues of the original diagonal D and so they preserve the original physical meaning, which is in contrast to (1.10). In the setting of quantum mechanics, the "physical meaning" of the eigenvalues of the coefficient matrix $A(t)$ are of great importance. They are proportional to the energy levels of a quantum mechanical system. Whether the representation (1.5) holds or not makes then a substantial difference. Another drawback of (1.10) is that the calculation of the eigenvalues of $\widetilde{D} = D + R$ may be a difficult task.

One way to overcome this difficulty is suggested by Eastham in [3]. It is shown that if in addition to (1.8) and (1.9) also

$$\frac{1}{\lambda_j(t) - \lambda_k(t)} R^2(t) \in L^1[a, \infty), \quad j, k = 1, \dots, n,\ j \neq k, \tag{1.11}$$

is satisfied then the solution is given as in (1.5). It was observed also in [5] that under appropriate circumstances $\widetilde{D}(t)$ can be chosen to be as $D(t)$.

It is also observed that the diagonal elements and the off-diagonal elements of $R(t)$ play different roles. Hence it makes sense to place all diagonal elements of Eq. (1.3) in $D(t)$ while the perturbation term $R(t)$ consists only of off-diagonal terms. This convention will be assumed throughout the rest of our work.

The literature is abundant of representation of fundamental solutions of Eq. (1.3) as in (1.5), namely, a product of two matrices such that the diagonal matrix $\Phi = \exp(\int_{t_0}^t D(s)\,\mathrm{d}s)$ is on the right and the matrix $I + Q(t)$, which is a perturbation of the identity matrix, is on the left. In this work we look, on the contrary, for a solution $Y(t)$ of (1.3) that is represented as $Y = \Phi(I + P)$, i.e.,

$$Y(t) = \exp\left(\int^t D(s)\,\mathrm{d}s\right)\bigl(I + P(t)\bigr) \tag{1.12}$$

with a suitable, still unknown perturbation P such that $P = \mathrm{o}(1)$ as $t \to \infty$.

The distinction between (1.5) and (1.10) makes it worth while to formulate in analogy to [5], the following definition, both for representation (1.5) and representation (1.12):

Definition. Let $D(t) \in C[a,\infty)$ be a diagonal matrix. Let $R(t) \in C[a,\infty)$ be such that its diagonal elements are all zero. We say that the system (1.3) is "right almost diagonal" if it possesses an asymptotic representation (1.12) with $P(t) \in C[a,\infty]$ and $\lim P(t) = 0$ as $t \to \infty$. Similarly, if representation (1.5) holds, the system (1.3) will be called "left almost diagonal".

It has been shown in several problems of mathematical physics, see, e.g., [6–8], that a representation of a fundamental solution as a product $\Phi(I+P)$, where the diagonal matrix Φ is on the left rather than on the right in the given product, has a merit of its own. Moreover, it is also evident from, e.g., [7], that the resulting equation for $I+P$ is simpler in form than the equation for $I+Q$ that would have resulted seeking a representation $(I+Q)\Phi$. Usually, a representation $\Phi(I+P)$ is possible when a special "dichotomy condition" is satisfied. This situation occurs in the case where $A(t)$ is anti-Hermitian, which is frequently encountered in quantum mechanics, see [15]. A main motivation is to gain a better understanding of the adiabatic approximation theorem in quantum mechanics ("If the Hamiltonian is changed slowly from H_0 to H, the system in a given eigenstate of H_0 goes over into the corresponding eigenstate of H but does not make any transitions"), which was originated by [4] and discussed by [2]. See also [6].

The purpose of this study is to present conditions on a special family of systems of differential equations (1.1), which will lead to an asymptotic approximation of a fundamental solution of the form $\Phi(I+P)$. While doing so, a self contained proof of a theorem of asymptotic approximation, guaranteeing the existence of a perturbation matrix P, having certain desired properties, will be given.

The method provided differs from an established trend that transforms the differential system, via a preparation theorem, to a differential system, where the coefficient matrix is the sum of a diagonal matrix and a remainder matrix that must be absolutely integrable at infinity. Our method of proof involves the conversion of a differential equation for $P(t)$ into an integral equation in a manner to be detailed in Section 2. It will be shown that our method will lead to a theorem of asymptotic integration which is best possible in a certain sense. Conditions will be given in Section 3, which will guarantee that the system (1.3) is right almost diagonal. Examples will be given that are amenable to our theorem to which the results of [3,5,10,11,14] do not apply.

2. Some formal calculations

Substitution of $Y = \Phi(I+P)$ into (1.3) with $\Phi' = D\Phi$ leads to

$$D\Phi(I+P) + \Phi(I+P)' = (D+R)\Phi(I+P),$$

i.e.,

$$P' = \Phi^{-1}R\Phi(I+P). \tag{2.1}$$

Let

$$K(t) = \Phi^{-1}R\Phi = \exp\left(-\int_{t_0}^{t} D(s)\,\mathrm{d}s\right)R(t)\exp\left(\int_{t_0}^{t} D(s)\,\mathrm{d}s\right). \tag{2.2}$$

Here

$$R(t)=\big(r_{jk}(t)\big)_{j,k=1}^{n},\qquad K(t)=\left(r_{jk}(t)\exp\int_{t_0}^{t}-(\lambda_j-\lambda_k)\,\mathrm{d}\tau\right)_{j,k=1}^{n}.$$

Then

$$P'=K+KP. \tag{2.3}$$

Assuming that $P(\infty)=0$, let us begin a formal integration of (2.3) as

$$p_{jk}(t)=\int_{\infty}^{t} r_{jk}(s)\,\mathrm{e}^{-\int_{t_0}^{s}(\lambda_j-\lambda_k)\,\mathrm{d}\tau}\,\mathrm{d}s+\int_{\infty}^{t}\sum_{h=1}^{n} r_{jh}(s)\,\mathrm{e}^{-\int_{t_0}^{s}(\lambda_j-\lambda_h)\,\mathrm{d}\tau}\,p_{hk}(s)\,\mathrm{d}s,$$

which may be written as

$$P(t)=-\int_t^{\infty}K(t_1)\,\mathrm{d}t_1-\int_t^{\infty}K(t_1)P(t_1)\,\mathrm{d}t_1. \tag{2.4}$$

Next we integrate by part the last term of (2.4):

$$\begin{aligned}
P(t)&=-\int_t^{\infty}K(t_1)\,\mathrm{d}t_1-\int_t^{\infty}K(t_1)P(t_1)\,\mathrm{d}t_1\\
&=-\int_t^{\infty}K(t_1)\,\mathrm{d}t_1-\left[-\int_{t_2}^{\infty}K(t_1)\,\mathrm{d}t_1\right]P(t_2)\bigg|_{t_2=t}^{\infty}+\int_t^{\infty}\left[-\int_{t_2}^{\infty}K(t_1)\,\mathrm{d}t_1\right]P'(t_2)\,\mathrm{d}t_2\\
&=-\int_t^{\infty}K(t_1)\,\mathrm{d}t_1-\left[\int_t^{\infty}K(t_1)\,\mathrm{d}t_1\right]P(t)\\
&\quad-\int_t^{\infty}\left[\int_{t_2}^{\infty}K(t_1)\,\mathrm{d}t_1\right]\big[K(t_2)+K(t_2)P(t_2)\big]\,\mathrm{d}t_2.
\end{aligned}\tag{2.5}$$

Denote

$$\begin{aligned}
M_1(t)&=\int_t^{\infty}K(t_1)\,\mathrm{d}t_1,\\
M_2(t)&=\int_t^{\infty}M_1(t_2)K(t_2)\,\mathrm{d}t_2=\int_t^{\infty}\left[\int_{t_2}^{\infty}K(t_1)\,\mathrm{d}t_1\right]K(t_2)\,\mathrm{d}t_2.
\end{aligned}$$

Then (2.4) becomes $P(t)=-M_1(t)-\int_t^{\infty}K(t_1)P(t_1)\,\mathrm{d}t_1$ while (2.5) may be rewritten as

$$\big(I+M_1(t)\big)P(t)=-M_1(t)-M_2(t)-\int_t^{\infty}\left[\int_{t_2}^{\infty}K(t_1)\,\mathrm{d}t_1\right]K(t_2)P(t_2)\,\mathrm{d}t_2. \tag{2.6}$$

This scheme may be formally repeated. Let $M_3(t) = \int_t^\infty M_2(t_3)K(t_3)\,\mathrm{d}t_3$. When the last term of (2.6) is integrated by parts, one gets

$$\begin{aligned}\big(I + M_1(t)\big)P(t) &= -M_1(t) - M_2(t) - \big[-M_2(t_3)P(t_3)\big]_{t_3=t}^{\infty} + \int_t^\infty -M_2(t_3)P'(t_3)\,\mathrm{d}t_3 \\ &= -M_1(t) - M_2(t) - M_2(t)P(t) - \int_t^\infty M_2(t_3)\big[K(t_3) + K(t_3)P(t_3)\big]\,\mathrm{d}t_3 \\ &= -M_1(t) - M_2(t) - M_3(t) - M_2(t)P(t) - \int_t^\infty M_2(t_3)K(t_3)P(t_3)\,\mathrm{d}t_3,\end{aligned}$$

that is,

$$\big(I + M_1(t) + M_2(t)\big)P(t) = -M_1(t) - M_2(t) - M_3(t) - \int_t^\infty M_2(t_3)K(t_3)P(t_3)\,\mathrm{d}t_3.$$

m iterations of this scheme lead formally to the integral equation

$$\left(I + \sum_{\ell=1}^{m-1} M_\ell(t)\right)P(t) = -\sum_{\ell=1}^{m} M_\ell(t) - \int_t^\infty M_{m-1}(t_m)K(t_m)P(t_m)\,\mathrm{d}t_m \tag{2.7}$$

with

$$M_\ell(t) = \int_t^\infty M_{\ell-1}(t_\ell)K(t_\ell)\,\mathrm{d}t_\ell, \quad \ell = 1, \ldots, m, \tag{2.8}$$

with $M_0(t) \equiv I$. If the $M_\ell(t)$'s are small for large values of t, $I + \sum_{\ell=1}^{m-1} M_\ell(t)$ is eventually invertible. In this case we denote for short

$$S(t) = \left(I + \sum_{\ell=1}^{m-1} M_\ell(t)\right)^{-1}, \qquad V(t) = S(t)\sum_{\ell=1}^{m} M_\ell(t).$$

Then, the integral equation (2.7) can be written as

$$P(t) = -V(t) - S(t)\int_t^\infty M_{m-1}(t_m)K(t_m)P(t_m)\,\mathrm{d}t_m, \tag{2.9}$$

or symbolically as

$$P = -V - L[P] \tag{2.10}$$

with the integral operator

$$L[P](t) = S(t)\int_t^\infty M_{m-1}(t_m)K(t_m)P(t_m)\,\mathrm{d}t_m.$$

Moreover, note that if P is a solution of (2.10) it also satisfies

$$P = -V - L\bigl[-V - L[P]\bigr] = -V + L[V] - L^2[P],$$

and by repeated iterations it is also a solution of

$$P = \sum_{\nu=0}^{r-1} (-1)^{\nu-1} L^{\nu}[V] + (-1)^{r-1} L^{r}[P], \quad L^0[V] = V.$$

3. Asymptotic integration

In this section we show that our formal scheme indeed leads to an actual asymptotic solution of Eq. (1.3) under suitable assumptions. First we formulate an abstract criterion and in the next step we extract some explicit conditions on the two parameters in our scheme,

$$R = \bigl(r_{jk}(t)\bigr)_{j,k=1}^{n} \quad \text{and} \quad K(t) = \left(r_{jk}(t) \exp \int_{t_0}^{t} (\lambda_k - \lambda_j)\right)_{j,k=1}^{n}.$$

Proposition. *Let* $D(t), R(t) \in C[a,\infty)$, $K(t)$ *defined by* (2.2) *and* $M_\ell(t)$ *defined by* (2.8). *If*

$$M_\ell(t) \to 0 \quad \textit{as } t \to \infty,\ \ell = 1, 2, \ldots, m-1, \tag{3.1}$$

$$M_{m-1}(t)K(t) \in L^1, \tag{3.2}$$

for some integer m *then the integral equation* (2.7) *possesses a unique solution* $P(t) \in C^1[a,\infty)$ *for some* a, *such that* $P(t) \to 0$ *as* $t \to \infty$.

Proof. Suppose that $P(t)$ is indeed a bounded solution of (2.7) in $C[a,\infty)$. According to assumption (3.1), $M_\ell(t) \to 0$ for $\ell = 1, \ldots, m-1$ and by (3.2), also $M_m(t) = \int_t^\infty M_{m-1}(t_m)K(t_m)\,\mathrm{d}t_m \to 0$ as $t \to \infty$. Note that for $l = 1, \ldots, m-1$ no absolute integrability is involved. It follows that $I + \sum_{\ell=1}^{m-1} M_\ell(t)$ is an invertible matrix on $[a,\infty)$ for some sufficiently large a, the integral equation (2.7) may indeed be written as (2.9) with $V(t) \to 0$, $S(t) \to I$. With a suitable norm, say $\|P\| = \sum\sum |p_{jk}|$,

$$\|P(t)\| \leqslant \|V(t)\| + \|S(t)\| \int_t^\infty \|M_{m-1}(t_m)K(t_m)\| \|P(t_m)\|\,\mathrm{d}t_m. \tag{3.3}$$

For any bounded matrix valued function $A(t)$ let $|||A||| = \sup_{[a,\infty)} \|A(t)\|$. (3.3) holds for every t, $a \leqslant t < \infty$, therefore

$$|||P||| \leqslant |||V||| + |||S||| \left(\int_a^\infty \|M_{m-1}(t_m)K(t_m)\|\,\mathrm{d}t_m\right) |||P|||. \tag{3.4}$$

Since $M_{m-1}(t)K(t) \in L^1$, then for a given s, $0 \leqslant s < 1$, we can choose a large enough such that

$$|||S||| \int_a^\infty \|M_{m-1}(t_m)K(t_m)\|\,\mathrm{d}t_m \leqslant s < 1. \tag{3.5}$$

Let us fix this value of a for the rest of this work. Then by (3.4)

$$|||P||| \leqslant \frac{|||V|||}{1-s}.$$

The proof follows now by a standard fixed point argument. Take $0 \leqslant s < 1$ and let a be chosen so that (3.5) holds. Define the sequence

$$P_0 = V, \qquad P_j = V + L[P_{j-1}], \quad j = 1, 2, \ldots.$$

Then

$$\|P_{j+1}(t) - P_j(t)\| \leqslant \|L[P_j - P_{j-1}]\| \leqslant s|||P_j - P_{j-1}|||,$$

and also

$$|||P_{j+1} - P_j||| \leqslant s|||P_j - P_{j-1}|||.$$

Hence, the sequence $P_j(t)$ converges uniformly on $C[a,\infty)$ to a limit function $P(t)$. It is evident that $P \in C[a,\infty)$ and it is the unique solution of the integral equation (2.7). Consequently, also $P \in C^1$. Since $V(t) \to 0$, $S(t) \to I$, it follows from inequality (3.3) that

$$\lim_{t\to\infty} P(t) = 0.$$

Note that once the existence of $P(t)$ is established, a more specific estimate may be obtained. By Gronwall's inequality the integral inequality (3.3) implies

$$\|P(t)\| \leqslant \|V(t)\| + \|S(t)\| \int_t^\infty \|M_{m-1}(\tau)K(\tau)\| \|V(\tau)\| \exp\left(\int_t^\tau \|M_{m-1}K\|\|S\| \,\mathrm{d}\nu\right) \mathrm{d}\tau. \quad \square$$

This criterion has some meaning even for $m = 1$, i.e., if it is applied directly to Eq. (2.4). All one has to assume is $K(t) \in L^1$, namely

$$r_{\alpha\beta}(t)\,\mathrm{e}^{-\int_{t_0}^t (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}\tau} \in L^1 \quad \text{for } \alpha, \beta = 1, \ldots, n,\ \alpha \neq \beta.$$

In comparison, Levinson's theorem requires $r_{\alpha\beta} \in L^1$ for all $\alpha, \beta = 1, \ldots, n$.

The essence of this paper is to give reasonable conditions under which assumptions (3.1) and (3.2) hold and so the solutions of Eq. (1.3) have nice asymptotic representations. Of course, we wish that our conditions will extend other ones which are available in the literature. We discuss a relatively simple situation, when in Eq. (2.7) we take $m = 2$, namely Eq. (2.6):

$$[I + M_1(t)]P = -M_1(t) - M_2(t) - \int_t^\infty M_1(t_2)K(t_2)P(t_2)\,\mathrm{d}t_2. \tag{3.6}$$

The asymptotic solution of type (1.12) works well when the differences of the eigenvalues of $D(t)$ are imaginary or close to the imaginary axis.

Theorem 1. *Let for all $\alpha \neq \beta$, $\alpha, \beta = 1, \ldots, n$,*

$$\left|\operatorname{Re}\int_{t_0}^{t} \bigl(\lambda_\alpha(s) - \lambda_\beta(s)\bigr)\,\mathrm{d}s\right| \leqslant B, \quad a \leqslant t_0,\ t < \infty, \tag{3.7}$$

where B is a nonnegative constant,

$$\frac{r_{\alpha\beta}(t)}{\lambda_\beta(t) - \lambda_\alpha(t)} \to 0 \quad \text{as } t \to \infty, \tag{3.8}$$

$$\left[\frac{r_{\alpha\beta}(t)}{\lambda_\alpha(t) - \lambda_\beta(t)}\right]' \in L^1[a, \infty), \tag{3.9}$$

and for all $k \neq \beta$,

$$\left(\int_t^\infty \left|\left[\frac{r_{\alpha\beta}(t_1)}{\lambda_\alpha(t_1) - \lambda_\beta(t_1)}\right]'\right| \mathrm{d}t_1\right) \bigl|r_{\beta k}(t)\bigr| \in L^1[a, \infty). \tag{3.10}$$

Then a fundamental solution of Eq. (1.3) *is given by* (1.12),

$$Y = \exp\left(\int_{t_0}^{t} D(s)\,\mathrm{d}s\right)\bigl(I + P(t)\bigr)$$

with $P \in C^1[a, \infty)$, $P(t) \to 0$ as $t \to \infty$ and $\det(I + P)(t) \equiv 1$ *on* $[a, \infty)$.

Proof. Our aim is to show that assumptions (3.1) and (3.2) of the proposition, namely $M_1(t) \to 0$, $M_1(t)K(t) \in L[a, \infty)$, hold for Eq. (3.6). The relevant matrices and their respective elements are

$$\begin{aligned}
K(t) &= \Phi^{-1}(t)R(t)\Phi(t) = \left(r_{\alpha\beta}(t)\exp\left[\int_{t_0}^{t} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\right]\right)_{\alpha,\beta=1}^{n}, \\
M_1(t) &= \int_t^\infty K(t_1)\,\mathrm{d}t_1 = \left(\int_t^\infty r_{\alpha\beta}(t_1)\exp\left[\int_{t_0}^{t_1} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\right]\mathrm{d}t_1\right)_{\alpha,\beta=1}^{n}, \\
M_2(t) &= \int_t^\infty M_1(t_2)K(t_2)\,\mathrm{d}t_2.
\end{aligned}$$

Recall that K, M_1 are off-diagonal matrices since R is such.

Now we integrate by parts the elements of $M_1(t)$ which will be denoted, for short, $M_{1,\alpha\beta}$:

$$\begin{aligned}
M_{1,\alpha\beta}(t) &= \int_t^\infty r_{\alpha\beta}(t_1)\exp\left[\int_{t_0}^{t_1} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\right]\mathrm{d}t_1 \\
&= \int_t^\infty \frac{r_{\alpha\beta}(t_1)}{\lambda_\beta(t_1) - \lambda_\alpha(t_1)}\bigl(\lambda_\beta(t_1) - \lambda_\alpha(t_1)\bigr)\exp\left[\int_{t_0}^{t_1} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\right]\mathrm{d}t_1 \\
&= \frac{r_{\alpha\beta}(t_1)}{\lambda_\beta(t_1) - \lambda_\alpha(t_1)}\exp\left(\int_{t_0}^{t_1} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\Bigg|_{t_1=t}^{\infty}\right) \\
&\quad - \int_t^\infty \left[\frac{r_{\alpha\beta}(t_1)}{\lambda_\beta(t_1) - \lambda_\alpha(t_1)}\right]'\exp\left[\int_{t_0}^{t_1} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\right]\mathrm{d}t_1.
\end{aligned}$$

By (3.7) and (3.8) the integrated term vanishes at the upper limit $t_1 = \infty$. So

$$M_{1,\alpha\beta}(t) = \frac{r_{\alpha\beta}(t)}{\lambda_\alpha(t) - \lambda_\beta(t)} \exp \int_{t_0}^{t} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\, \mathrm{d}s - \int_t^\infty \left[\frac{r_{\alpha\beta}(t_1)}{\lambda_\beta(t_1) - \lambda_\alpha(t_1)}\right]' \exp\left[\int_{t_0}^{t_1} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\, \mathrm{d}s\right] \mathrm{d}t_1. \tag{3.11}$$

By another application of (3.7) the elements of M_1 are bounded by

$$|M_{1,\alpha\beta}(t)| \leqslant \left(\left|\frac{r_{\alpha\beta}(t)}{\lambda_\beta(t) - \lambda_\alpha(t)}\right| + \int_t^\infty \left| \left[\frac{r_{\alpha\beta}(t_1)}{\lambda_\beta(t_1) - \lambda_\alpha(t_1)}\right]' \right| \mathrm{d}t_1 \right) \exp B.$$

Therefore assumptions (3.8), (3.9) imply that $M_1(t) \to 0$ as $t \to \infty$.

Next we turn to the kernel matrix of (3.6), $M_1(t)K(t)$. Its elements are estimated by

$$\begin{aligned} \bigl|\bigl(M_1(t)K(t)\bigr)_{\alpha\beta}\bigr| &\leqslant \sum_{j=1}^{n} |M_{1,\alpha j}(t)| |K_{j\beta}(t)| \\ &\leqslant \sum_{j \neq \alpha,\beta} \left(\left|\frac{r_{\alpha j}(t)}{\lambda_j(t) - \lambda_\alpha(t)}\right| + \int_t^\infty \left| \left[\frac{r_{\alpha j}(t_1)}{\lambda_j(t_1) - \lambda_\alpha(t_1)}\right]' \right| \mathrm{d}t_1 \right) |r_{j\beta}(t)| (\exp B)^2. \end{aligned}$$

Due to (3.8) the first term in the sum is

$$\left|\frac{r_{\alpha j}(t)}{\lambda_j(t) - \lambda_\alpha(t)}\right| \leqslant \int_t^\infty \left| \left[\frac{r_{\alpha j}(t_1)}{\lambda_j(t_1) - \lambda_\alpha(t_1)}\right]' \right| \mathrm{d}t_1,$$

so $M_1 K \in L^1[a, \infty)$ follows from (3.10). So condition (3.2) of the proposition is verified and the existence of the appropriate P is proved.

According to (2.1), $(I + P)' = P' = \Phi^{-1} R \Phi (I + P)$, so by Abel's formula

$$\det(I + P)(t) = \det(I + P)(\infty) \exp\left(\int_\infty^t \operatorname{trace}\bigl(\Phi^{-1} R \Phi\bigr) \right) \equiv 1,$$

since $\operatorname{trace}(\Phi^{-1} R \Phi) = \operatorname{trace}(R) \equiv 0$.

Note that, if in addition $R(t) \to 0$ as $t \to \infty$, then also $P' = \Phi^{-1} R \Phi (I + P) = K(I + P) \to 0$ since $|K_{jk}| \leqslant |r_{jk}| \exp B \to 0$. □

Remarks. On the face of it, it looks as if Theorem 1 fails if the difference of two eigenvalues is zero at a point of the interval $[t_0, \infty)$. However a close examination of the conditions that guarantee the validity of Theorem 1 reveals that a proper interpretation of the relations (3.8), (3.9) and (3.10) leads to a pleasant surprise. If a difference $\lambda_\beta - \lambda_\alpha$ becomes zero at a point t_1, we have to worry only about the behaviour of $r_{\alpha\beta}$, $r_{\beta\alpha}$ near t_1. The theorem holds if we interpret the quotients $r_{\alpha\beta}/(\lambda_\beta - \lambda_\alpha)$, $r_{\beta\alpha}/(\lambda_\beta - \lambda_\alpha)$ as limits as $t \to t_1$ and these quotients satisfy the assumptions (3.8), (3.9) and (3.10). By doing so we of course allow the quotients to possess removable singularities or some mild singularities at t_1. Note that Theorem 1 holds even if $\lambda_\alpha \equiv \lambda_\beta$ for some $\alpha \neq \beta$ provided that $r_{\alpha\beta} \equiv r_{\beta\alpha} \equiv 0$.

When $r_{\alpha\beta}/(\lambda_\alpha - \lambda_\beta)$ is monotone, (3.10) becomes

$$\frac{r_{\alpha\beta}(t) r_{\beta k}(t)}{\lambda_\alpha(t) - \lambda_\beta(t)} \in L^1[a, \infty), \quad \alpha \neq \beta,\ \beta \neq k. \tag{3.12}$$

Naturally, our assumptions (3.8), (3.9) and (3.10) (or (3.12)) should be compared with the assumptions (1.8), (1.9) and (1.11) of [3]. The advantage of Theorem 1 over [3] is our reduced number of assumptions. It is essential in the proof of [3] that

$$\frac{r_{\alpha\beta}}{\lambda_j - \lambda_k} \in L^1 \quad \text{for all } \alpha \neq \beta,\ j \neq k,$$

i.e., $(n^2 - n)^2$ conditions; in contrast, our analog (3.8) consists of only $n^2 - n$ conditions. Similarly, (1.11) contains $n^4(n-1)$ conditions while (3.10) has only $n(n-1)^2$.

4. Some examples

We start with a simple example that demonstrates the difference between our result and the theory in [3]. Consider the equation

$$Y' = \left[\begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & 2\mathrm{i} & 0 \\ 0 & 0 & \mathrm{i}t^\gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 & c_{13} \\ 0 & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix} t^\delta \right] Y, \quad t_0 \leqslant t < \infty, \tag{4.1}$$

for some large t_0, where c_{jk} are constants, $-1/2 < \delta < 0$ and $\gamma > 2\delta + 1$. Theorem 1.7.1 of [3] cannot be applied naively to this example since it requires condition (1.11) to hold. However (1.11) fails since $((\lambda_1 - \lambda_2)^{-1} R^2)_{11} \approx t^{2\delta} \notin L^1$. On the other hand (3.10) (or (3.12)) do hold: for large values of t the eigenvalues are distinct and

$$\frac{r_{12} r_{2k}}{\lambda_1 - \lambda_2} \equiv 0, \qquad \frac{r_{13} r_{3k}}{\lambda_1 - \lambda_3} \approx t^{2\delta - \gamma} \in L^1,$$

since $2\delta - \gamma < -1$. (3.8) also holds since

$$\frac{r_{12}}{\lambda_1 - \lambda_2} \equiv 0, \qquad \frac{r_{13}}{\lambda_1 - \lambda_3} \approx t^{\delta - \gamma}$$

and $\delta - \gamma < -\delta - 1 < 0$. Of course this simple example can be treated also by other methods.

Here are two examples that demonstrate the range of applicability of Theorem 1. Given the system

$$Y' = \left[\begin{pmatrix} \mathrm{i}t^p & 0 & 0 \\ 0 & \mathrm{i}t^q & 0 \\ 0 & 0 & \mathrm{i}t^r \end{pmatrix} + \begin{pmatrix} 0 & c_{12} t^{w_{12}} & c_{13} t^{w_{13}} \\ c_{21} t^{w_{21}} & 0 & c_{23} t^{w_{23}} \\ c_{31} t^{w_{31}} & c_{32} t^{w_{32}} & 0 \end{pmatrix} \right] Y, \quad t_0 \leqslant t < \infty, \tag{4.2}$$

where c_{jk} are constants, $p < q < r < 0$,

$$\begin{aligned} w_{12}, w_{21} &< q < 0, \\ w_{13}, w_{31}, w_{23}, w_{32} &< r < 0. \end{aligned} \tag{4.3}$$

While (4.1) has completely distinct diagonal elements, the diagonal elements of (4.2) coalesce as $t \to \infty$. Theorem 1 applies to (4.2). Condition (3.7) is obviously satisfied. Observe that $\lambda_1 - \lambda_2 = \mathcal{O}(t^q)$ while $\lambda_1 - \lambda_3$, $\lambda_2 - \lambda_3 = \mathcal{O}(t^r)$. Conditions (3.8) and (3.9) are satisfied by virtue of (4.3). (3.10) (or (3.12)) is satisfied if

$$w_{12} + w_{21}, w_{12} + w_{23}, w_{21} + w_{31} < q - 1,$$

and $w_{jk} + w_{kl} < r - 1$ for the other indices $j \neq k \neq l$, namely

$$w_{13} + w_{31}, w_{13} + w_{32}, w_{31} + w_{12}, w_{23} + w_{31}, w_{23} + w_{32}, w_{32} + w_{21} < r - 1.$$

On the other hand (1.8), (1.9) and (1.11) guarantee by [3] that the system (4.2) is almost diagonal only if $w_{jk} < r < 0$ and $w_{jk} + w_{kl} < r - 1$ holds for *all* $j \neq k \neq l$.

The other example is

$$Y' = \left[\begin{pmatrix} \mathrm{i}t^p & 0 & 0 \\ 0 & \mathrm{i}t^p & 0 \\ 0 & 0 & \mathrm{i} \end{pmatrix} + \begin{pmatrix} 0 & 0 & c_{13}t^{w_{13}} \\ 0 & 0 & c_{23}t^{w_{23}} \\ c_{31}t^{w_{31}} & c_{32}t^{w_{32}} & 0 \end{pmatrix} \right] Y, \quad t_0 \leqslant t < \infty, \tag{4.4}$$

with two eigenvalues of the diagonal matrix being identical, $p < 0$, w_{jk} are negative constants such that $-1 < w_{jk} < 0$ and

$$w_{jk} + w_{kl} < -1 \quad \text{for } j \neq k \neq l. \tag{4.5}$$

Conditions (3.8) and (3.9) are satisfied by virtue of $p < 0$ and $w_{jk} < 0$. Moreover, (3.10) is satisfied thanks to condition (4.5). Hence Theorem 1 applies. It is noteworthy that our theorem is valid no matter how close to zero is one or some of the values w_{jk} are, as long as (4.5) is satisfied.

Levinson's methods cannot be applied to system (4.4). This is so on two counts. The first count is that the off diagonal terms are not in L^1. The second count being that two elements of the diagonal matrix are identical and therefore a continuously differentiable invertible transformation that diagonalizes the system cannot be guaranteed. Because of same reasons the results in [5] cannot be applied. The methods in [3,10,11] cannot conclude either that the system is almost diagonal. Moreover, techniques of block-diagonalization for nonanalytic systems of differential systems, e.g., [9], could lead to the asymptotic integration of our example, but they cannot guarantee that our system is almost diagonal.

Finally we show that Theorem 1 is best possible in the following sense. We demonstrate that (3.10) (or (3.12)) is necessary for system (1.3) to be almost diagonal, i.e., for approximation (1.12) to hold. Consider

$$Y' = \left[\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} + \begin{pmatrix} 0 & t^{-p} \\ t^{-q} & 0 \end{pmatrix} \right] Y \tag{4.6}$$

with $p, q > 0$. (3.12) requires that $r_{12}r_{21}/(\lambda_1 - \lambda_2) = t^{-p-q}/2\mathrm{i} \in L^1$, i.e., $\alpha \equiv p + q > 1$. We show that if on the contrary, $\alpha \leqslant 1$ then representation (1.12) of a fundamental solution, namely

$$Y(t) = \exp \int^t \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \mathrm{d}s \, (I + P(t)) \tag{4.7}$$

is impossible.

Levinson's theorem guarantees an asymptotic approximation

$$Y(t) = (I + Q(t)) \exp\biggl(\int^t \begin{pmatrix} \mu_1(s) & 0 \\ 0 & \mu_2(s) \end{pmatrix} \mathrm{d}s\biggr), \tag{4.8}$$

where $\mu_{1,2}(t) = \pm \mathrm{i}\sqrt{1 - t^{-\alpha}}$ are the eigenvalues of $D + R$. If both (4.7) and (4.8) hold for a fundamental solution, then

$$\begin{aligned} &\exp \int_{t_0}^t \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \mathrm{d}s \,(I + P(t))\bigl(I + P(t_0)\bigr)^{-1} \\ &\quad \equiv (I + Q(t)) \exp \int_{t_0}^t \begin{pmatrix} \mu_1(s) & 0 \\ 0 & \mu_2(s) \end{pmatrix} \mathrm{d}s \,\bigl(I + Q(t_0)\bigr)^{-1} \end{aligned}$$

with some $P(t)$, $Q(t) \to 0$ as $t \to \infty$. Comparing the $(1,1)$-term on each side of the matrix identity above yields

$$\bigl(1 + p_{11}(t)\bigr) = A\bigl(1 + q_{11}(t)\bigr)\,\mathrm{e}^{\mathrm{i}\int_{t_0}^t (-1+\sqrt{1-s^{-\alpha}})\,\mathrm{d}s} + Bq_{12}(t)\,\mathrm{e}^{-\mathrm{i}\int_{t_0}^t (1+\sqrt{1-s^{-\alpha}})\,\mathrm{d}s} \tag{4.9}$$

with some constants A, B. Here $p_{ij}, q_{ij} \to 0$ and

$$\int_{t_0}^t \bigl(-1 + \sqrt{1 - s^{-\alpha}}\bigr)\,\mathrm{d}s = -\int_{t_0}^t \frac{s^{-\alpha}}{1 + \sqrt{1 - s^{-\alpha}}}\,\mathrm{d}s \to -\infty \quad \text{as } t \to \infty$$

if $\alpha \leqslant 1$. The left-hand side of (4.9) satisfies $\lim(1 + p_{11}(t)) = 1$ as $t \to \infty$. If $A = 0$ then evidently we have a contradiction as the right-hand side of (4.9) tends to 0. If $A \neq 0$ then the real part of the right-hand side would have extreme values arbitrary close to $\pm|A|$, evidently a contradiction. Thus (4.6) is not almost diagonal without assumption (3.12).

5. More integrations by part

Let us now modify Theorem 1 by additional integrations by parts of (3.11). Define successively a sequence of generalized derivatives

$$r_{\alpha\beta}^{[0]}(t) = r_{\alpha\beta}(t), \qquad r_{\alpha\beta}^{[m+1]}(t) = \left[\frac{r_{\alpha\beta}^{[m]}(t)}{\lambda_\alpha(t) - \lambda_\beta(t)}\right]', \quad m = 0, 1, \ldots.$$

Then (3.11) may be written as

$$\begin{aligned} M_{1,\alpha\beta}(t) &= \int_t^\infty r_{\alpha\beta}^{[0]}(t_1) \exp\biggl[\int_{t_0}^{t_1} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\biggr]\,\mathrm{d}t_1 \\ &= \frac{r_{\alpha\beta}^{[0]}(t)}{\lambda_\alpha(t) - \lambda_\beta(t)} \exp\biggl[\int_{t_0}^{t} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\biggr] \\ &\quad + \int_t^\infty r_{\alpha\beta}^{[1]}(t_1) \exp\biggl[\int_{t_0}^{t_1} \bigl(\lambda_\beta(s) - \lambda_\alpha(s)\bigr)\,\mathrm{d}s\biggr]\,\mathrm{d}t_1. \end{aligned}$$

(Note that the denominator of $r^{[1]}_{\alpha\beta}$ is $\lambda_\alpha - \lambda_\beta$ and not $(\lambda_\beta - \lambda_\alpha)$!) By ℓ repeated integrations by parts and assumptions which ensure the vanishing of the integrated terms at $t_1 = \infty$, we get

$$\begin{aligned} M_{1,\alpha\beta}(t) = &\sum_{m=0}^{\ell} \frac{r^{[m]}_{\alpha\beta}(t)}{\lambda_\alpha(t) - \lambda_\beta(t)} \exp\left[\int_{t_0}^{t} \left(\lambda_\beta(s) - \lambda_\alpha(s)\right) \mathrm{d}s\right] \\ &+ \int_t^\infty r^{[\ell+1]}_{\alpha\beta}(t_1) \exp\left[\int_{t_0}^{t_1} \left(\lambda_\beta(s) - \lambda_\alpha(s)\right) \mathrm{d}s\right] \mathrm{d}t_1. \end{aligned}$$

Analogously with the previous proof, we get

Theorem 2. *Let ℓ be a fixed integer. Assume that* (3.7) *holds and for all $\alpha, \beta = 1, \ldots, n$, $\alpha \neq \beta$,*

$$\frac{r^{[m]}_{\alpha\beta}(t)}{\lambda_\beta(t) - \lambda_\alpha(t)} \to 0 \quad \text{as } t \to \infty,\ m = 0, \ldots, \ell, \tag{5.1}$$

$$r^{[\ell+1]}_{\alpha\beta} = \left[\frac{r^{[\ell]}_{\alpha\beta}(t)}{\lambda_\alpha(t) - \lambda_\beta(t)}\right]' \in L^1[a, \infty), \tag{5.2}$$

and for $k \neq \beta$,

$$\left(\int_t^\infty \left|r^{[\ell+1]}_{\alpha\beta}(t_1)\right| \mathrm{d}t_1\right) r_{\beta k}(t) \in L^1[a, \infty). \tag{5.3}$$

Then a fundamental solution of Eq. (1.3) *is given by* (1.12).

Outline of proof. As in the proof of Theorem 1, we show also here that by (5.1), (5.2) and (5.3), $M_1(t) \to 0$ as $t \to \infty$ and $M_1(t)K(t) \in L[a, \infty)$. Hence the small perturbation $P(t)$ is obtained as the solution of (3.6).

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References

[1] H. Behncke and C. Remling, Asymptotic integration of linear differential equations, *J. Math. Anal. Appl.* **210** (1997), 585–597.
[2] M. Born and V. Fock, Beweis des Adiabatensatzes, *Z. Phys.* **51** (1928), 165–180.
[3] M.S.P. Eastham, *The Asymptotic Solution of Linear Differential Systems*, Oxford University Press, Oxford, 1989.
[4] P. Ehrenfest, On adiabatic changes of a system in connection with the quantum theory, *Proc. Roy. Acad. Amsterdam* **19** (1917), 576–597.
[5] H. Gingold, Almost diagonal systems in asymptotic integration, *Proc. Edinburgh Math. Soc.* **28** (1985), 143–158.
[6] H. Gingold, In general, the less degeneracy the less transition. A principle for time dependent Hamiltonian systems in quantum mechanics, *J. Math. Phys.* **28** (1987), 2400–2406.
[7] H. Gingold and P.F. Hsieh, Global approximation of perturbed Hamiltonian differential equations with several turning points, *SIAM J. Math. Anal.* **18** (1987), 1275–1293.

[8] H. Gingold, J. She and W.E. Zorumski, Local principles of wave propagation in homogeneous media, *J. Acoust. Soc. Amer.* **93** (1993), 599–604.
[9] H. Gingold and A. Tovbis, Solutions of finitely smooth nonlinear singular differential equations and problems of diagonalization and triangularization, *SIAM J. Math. Anal.* **29** (1998), 757–778.
[10] W.A. Harris Jr. and D. Lutz, On the asymptotic integration of linear differential systems, *J. Math. Anal. Appl.* **48** (1974), 1–16.
[11] W.A. Harris Jr. and D. Lutz, A unified theory of asymptotic integration, *J. Math. Anal. Appl.* **57** (1977), 571–586.
[12] P. Hartman and A. Wintner, Asymptotic integrations of linear differential equations, *Amer. J. Math.* **77** (1955), 48–86, 932.
[13] P. Hsieh and F. Xie, Asymptotic diagonalization of a system of linear ordinary differential equations, *Dynam. Contin. Discrete Impuls. Systems* **1** (1996), 51–74.
[14] N. Levinson, The asymptotic nature of solutions of linear differential equations, *Duke Math. J.* **15** (1948), 111–126.
[15] R.L. Liboff, *Introductory Quantum Mechanics*, Holden Day, San Francisco, 1980.
[16] R. Medina and M. Pinto, Linear differential systems with conditionally integrable coefficients, *J. Math. Anal. Appl.* **166** (1992), 52–64.
[17] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.
[18] Y. Sibuya, *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient*, Mathematics Studies, Vol. 18, North-Holland, Amsterdam, 1975.
[19] W.R. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Wiley, New York, 1965.
[20] W.R. Wasow, *Linear Turning Point Theory*, Springer, New York, 1984.