Proceedings of the Royal Society of Edinburgh, **132A**, 1333–1359, *2002*

Nonlinear eigenvalue problems for a class of ordinary differential equations

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(MS received 12 July 2001; accepted 17 January 2002)

We consider the class of nonlinear eigenvalue problems

$$(a_{n-1}(x)(\cdots(a_1(x)((a_0(x)u^{p_0*})')^{p_1*})'\cdots)^{p_{n-1}*})' = \lambda b(x)u^{r*},$$

where $y^{p*} = |y|^p \operatorname{sgn} y$, $p_i > 0$ and $p_0 p_1 \dots p_{n-1} = r$, with various boundary conditions. We prove the existence of eigenvalues and study the zero properties and structure of the corresponding eigenfunctions.

1. Introduction

The study of the oscillation properties of nonlinear differential equations

$$(a(x)(y')^{p*})' + b(x)y^{p*} = 0$$

with the signed power

$$y^{p*} = |y|^p \operatorname{sgn} y$$

originates in the works of Mirzov [17] and Elbert [5], who named these equations 'half linear'. This theory extends various aspects of oscillation theory, such as the Picone identity [10], Sturm comparison theorem [16, 17], oscillation and non-oscillation criteria of Kneser type [16] and Hille type [14], and other oscillation criteria [3].

One branch of this research is the study of the eigenvalue problem

$$\begin{aligned}(a(x)(u')^{p*})' + \lambda b(x)u^{p*} &= 0,\\ u(a) = u(b) &= 0.\end{aligned}$$

The existence of the eigenvalues and a description of the associated eigenfunctions was proved in [4, 15] through the use of a generalized Prüfer transformation. Similar eigenvalue problems had been studied earlier in the context of n-width problems in Sobolev spaces. Tikhomirov and Babadzhanov [19] considered an eigenvalue problem of the type

$$\begin{aligned}((u')^{p*})' + \lambda u^{p*} &= 0,\\ u(a) = u(b) &= 0,\end{aligned}$$

and explicitly calculated its eigenvalues and eigenfunctions. A higher-order eigenvalue problem

$$((u^{(r)})^{p*})^{(r)} = (-1)^r \lambda u^{p*},$$
$$u^{(i)}(a) = u^{(i)}(b) = 0, \quad i = 0, \dots, r-1,$$

was studied by Pinkus [18], who proved an analogue of our main theorem in this particular case. Buslaev and Tikhomirov [2] considered equations of the form $((u^{(r)})^{p*})^{(r)} = (-1)^r \lambda u^{q*}$, with $p \neq q$ and various homogeneous boundary values also in connection with n-width problems. Their results are different in character.

The aim of this work is the study of eigenvalue problems for high-order nonlinear homogeneous equations of the form

$$(a_{n-1}(x)(\cdots(a_1(x)((a_0(x)u^{p_0*})')^{p_1*})'\cdots)^{p_{n-1}*})' = \lambda b(x)u^{r*}, \tag{1.1}$$

where

$$p_i > 0, \quad p_0 p_1 \cdots p_{n-1} = r,$$
$$a_i(x), b(x) > 0, \quad a_i \in C^{(n-i)}, \quad b \in C,$$

with various boundary conditions and under suitable assumptions on the powers $p_0, \dots, p_{n-1}$. For the particular case $p_0 = p_1 = \cdots = p_{n-1} = 1$, equation (1.1) reduces to the well-known linear differential equation

$$(a_{n-1}(x)(\cdots(a_1(x)(a_0(x)u)')'\cdots)')' = \lambda b(x)u. \tag{1.2}$$

We shall mention here some results concerning the eigenvalue problem associated with this linear equation.

The left-hand side of (1.2) consists of the general disconjugate operator written in a Pólya factorization form and, as such, equation (1.2) is traditionally written as

$$L_n[u] = \lambda b(x)u,$$

with the notation

$$\left.\begin{aligned} L_0[u] &= a_0(x)u, \\ L_i[u] &= a_i(x)(L_{i-1}[u])', \quad i = 1, \dots, n, \end{aligned}\right\} \tag{1.3}$$

$a_n(x) \equiv 1$. Reasonable boundary conditions associated with (1.2) are, for example,

$$\left.\begin{aligned} L_i[u](a) &= 0, \quad i = 0, \dots, k-1, \\ L_i[u](b) &= 0, \quad i = 0, \dots, n-k-1, \end{aligned}\right\} \tag{1.4}$$

$1 \leqslant k \leqslant n-1$, which are equivalent to

$$u^{(i)}(a) = 0, \quad i = 0, \dots, k-1,$$
$$u^{(i)}(b) = 0, \quad i = 0, \dots, n-k-1.$$

A description of the eigenvalues and eigenfunctions of (1.2) and (1.4) was announced by Krein [13]. More general boundary conditions of the form

$$\left.\begin{aligned} &\sum_{j=0}^{n-1} \alpha_{ij} L_j[u](a) = 0, \quad i = 0, \dots, k-1, \\ &\sum_{j=0}^{n-1} \beta_{ij} L_j[u](b) = 0, \quad i = 0, \dots, n-k-1, \end{aligned}\right\} \tag{1.5}$$

were studied by Karlin [11] and Karon [12]. In these works, the positivity properties of the associated Green functions were proved and results concerning the differential eigenvalue problem were then obtained from known results concerning integral equations. A totally different approach, which lies within the area of the theory of differential equations, may be found in [6] and [7, ch. 10]. It deals mainly with boundary conditions of the type

$$\left.\begin{aligned} L_i[u](a) = 0, \quad i = i_1, \dots, i_k, \\ L_j[u](b) = 0, \quad j = j_1, \dots, j_{n-k}. \end{aligned}\right\} \tag{1.6}$$

A typical result for these eigenvalue problems is as follows.

The eigenvalue problem (1.2) and (1.4) has an infinite sequence of real eigenvalues

$$0 < (-1)^{n-k}\lambda_1 < (-1)^{n-k}\lambda_2 < \cdots, \quad \lambda_i \to \infty \quad \textit{as } i \to \infty.$$

To each λ_i there corresponds an essentially unique eigenfunction u_i that has precisely $i-1$ zeros, all simple, in (a,b), and there are no real eigenvalues except these. The zeros of u_i in (a,b) interlace with those of u_{i+1}.

The same results hold for the boundary-value problem (1.2) and (1.5) under suitable sign-consistency assumptions on the matrices (α_{ij}) and (β_{ij}). Similar results also hold for boundary conditions (1.6). However, it is necessary to explicitly ensure that $\lambda = 0$ is not an eigenvalue. (If 0 is an eigenvalue, its multiplicity may be bigger than 1, which adds further complications and will be avoided here.) This is done by the following Pólya condition, which will be assumed throughout our work.

Condition 1.1. A necessary and sufficient condition that $\lambda = 0$ is not an eigenvalue of (1.2) and (1.6) is that, for all $\ell = 1, \dots, n$, at least ℓ boundary conditions are imposed in (1.6) on $L_0[u], \dots, L_{\ell-1}[u]$.

Analogously with the factorization (1.3), we define for (1.1) the nonlinear operators

$$\left.\begin{aligned} N_0[u] &= a_0(x) u^{p_0 *}, \\ N_i[u] &= a_i(x)((N_{i-1}[u])')^{p_i *}, \quad i = 1, \dots, n, \end{aligned}\right\} \tag{1.7}$$

with $a_n(x) \equiv 1$, $p_n = 1$ and rewrite (1.1) as

$$N_n[u] = \lambda b(x) u^{r*}.$$

Equation (1.1) will be studied together with various two-point nonlinear boundary conditions. To make the complicated calculations tractable, the boundary conditions will be restricted to those of the type analogous to (1.6), namely,

$$\left.\begin{aligned} N_i[u](a) &= 0, \quad i = i_1, \dots, i_k, \\ N_j[u](b) &= 0, \quad j = j_1, \dots, j_{n-k}, \end{aligned}\right\} \tag{1.8}$$

with $1 \leqslant k \leqslant n-1$. The above-mentioned condition 1.1 will always be assumed to hold where the N_i replace the L_i therein.

By (1.7),

$$\left.\begin{aligned} (N_{i-1}[u])' &= (a_i^{-1}(x)N_i[u])^{1/p_i*}, \quad i = 1, \dots, n-1, \\ (N_{n-1}[u])' &= \lambda b(x)(a_0^{-1}(x)N_0[u])^{r/p_0*}. \end{aligned}\right\} \tag{1.9}$$

With the notation $v_i = N_i[u]$, equation (1.1) is seen to be equivalent to the nonlinear system

$$\left.\begin{aligned} v_{i-1}' &= (a_i^{-1}(x)v_i)^{1/p_i*}, \quad i = 1, \dots, n-1, \\ v_{n-1}' &= \lambda b(x)(a_0^{-1}(x)v_0)^{r/p_0*}. \end{aligned}\right\} \tag{1.10}$$

Since $p_0 \cdots p_{n-1} = r$, the product of all powers on the right-hand side of (1.10) is 1. Thus some of these powers are larger than 1, while others are smaller than 1. The superlinear terms make the global existence of solutions suspect, while the sublinear ones may make this system non-Lipschitzian, from which arise questions of uniqueness. This makes (1.10) intriguing and will deserve a detailed analysis.

To achieve differentiability with respect to the parameters, we must impose some restrictions on the p_i. It transpires that the following assumptions will be needed.

Assumption 1.2. Let each of the sets of indices $\{i_1, \dots, i_k\}$, $\{j_1, \dots, j_{n-k}\}$ be decomposed into blocks of consecutive integers and assume that at least one of the following four assumptions holds. For each block of consecutive integers $\{\mu, \mu+1, \dots, \mu+\ell\}$, which are sub-blocks of $\{i_1, \dots, i_k\}$, either

$$p_\mu, \dots, p_{\mu+\ell} \leqslant 1$$

or

$$1 \leqslant p_\mu \leqslant \cdots \leqslant p_{\mu+\ell},$$

or the same holds for sub-blocks of $\{j_1, \dots, j_{n-k}\}$.

What should be done if (1.8) contains boundary conditions of the type

$$N_i[u](a) = 0, \quad i = 0, \dots, h,$$

and also

$$N_i[u](a) = 0, \quad i = \ell, \dots, n-1,$$

where $h < \ell$? Since the equations in (1.10) are arranged in a cyclic order, it is natural not to consider the sub-blocks $\{0, \dots, h\}$ and $\{\ell, \dots, n-1\}$, but rather one sub-block $\{\ell, \ell+1, \dots, n-1, 0, \dots, h\}$, which 'wraps around' in a cyclic order. The powers associated with this block in (1.10) will be $\{p_\ell, \dots, p_{n-1}, p_0/r, p_1, \dots, p_h\}$, and assumption 1.2 should hold for this block.

For example, consider (1.1) with the boundary-value conditions

$$N_i[u](a) = 0, \quad i = 0, \dots, k-1,$$
$$N_j[u](b) = 0, \quad j = k, \dots, n-1.$$

In this case, assumption 1.2 holds if one of the following conditions is satisfied:

(i) $p_0, p_1, \dots, p_{k-1} \leqslant 1$;

(ii) $1 \leqslant p_0 \leqslant p_1 \leqslant \dots \leqslant p_{k-1}$;

(iii) $p_k, p_{k+1}, \dots, p_{n-1} \leqslant 1$;

(iv) $1 \leqslant p_k \leqslant p_{k+1} \leqslant \dots \leqslant p_{n-1}$.

An extreme opposite example is the equation $((y^{(r)})^{p*})^{(n-r)} = \lambda b(x) y^{p*}$, with only one power p_i different from 1. For this equation, every set of boundary conditions (1.8) satisfies assumption 1.2. Note that the examples of second- and $2r$th-order at the beginning of this paper are of this type.

The main result of this work is the following.

THEOREM 1.3. *If (1.1), together with the boundary conditions (1.8), satisfy condition 1.1 and assumption 1.2, then (1.1), (1.8) have an infinite sequence of real eigenvalues*

$$0 < (-1)^{n-k}\lambda_1 < (-1)^{n-k}\lambda_2 < \cdots, \quad \lambda_i \to \infty \quad \text{as } i \to \infty.$$

To each λ_i there corresponds an essentially unique eigenfunction u_i that has precisely $i-1$ zeros in (a,b), all of them simple, and there are no real eigenvalues except these. The zeros of u_i in (a,b) interlace with those of u_{i+1}.

Why do we only consider the nonlinear operator y^{p*}? Could these signed powers be replaced by other nonlinear operators in (1.1), say, $N_i[u] = \varphi((N_{i-1}[u])')$? If we want our equation to be homogeneous, namely, to have the property that if u is an eigenfunction, then cu is an eigenfunction for all c, then we are led to the functional equation $\varphi(c_1 u) = c_2 \varphi(u)$, hence the Cauchy functional equation $\varphi(x)\varphi(y) = \varphi(xy)\varphi(1)$. Thus $\varphi(u)$ must be either $|u|^p$ or u^{p*} (see [1]). It turns out that this choice is mandatory also at some steps of our method of proof. As sign changes of solutions play an important role, it is natural to demand that $\operatorname{sgn}\varphi(u) = \operatorname{sgn} u$.

2. Notation and tools

The linear eigenvalue problem (1.2), (1.4) is equivalent to an integral equation whose kernel is the associated Green function. This integral equation can be approximated by a matrix equation. Hence our linear (and nonlinear) eigenvalue problems are closely related to linear (and nonlinear) matrix eigenvalue problems with strictly total positive matrices. For results in this direction, plus numerous references, see [8].

Let $\sigma(f)$ denote the number of sign changes of the function f in the interval (a,b). If $x_1, x_2, \ldots$ are the totality of the zeros of $N_0[f], \ldots, N_{n-1}[f]$ in (a,b), let $n(x_i)$ denote the maximum number of consecutive $N_j[f]$, $j = 0, \ldots, n-1$, which vanish at x_i,

$$N_j[f](x_i) = 0, \quad j = m, \ldots, m + n(x_i) - 1.$$

Note that, in the linear case ($p_i = 1$), the number of consecutive vanishing $L_j[u](x_i)$ starting at $j = m$ simply count the multiplicity of the zero x_i of $L_m[f]$. However, in the case of the nonlinear operators $N_j[u]$, $n(x_i)$ is not necessarily the multiplicity of the zero x_i of a solution u of (1.1), due to the lack of smoothness. This is because $v_j' = (N_j[u])'$ of (1.10) may be just continuous at x_i and only its signed power $((N_j[u])')^{p_{j+1}*}$ is again differentiable. However, $n(x_i) = 1$ does imply that if $N_j[u](x_i) = 0$, then $N_{j+1}[u](x_i) \neq 0$, so x_i is a simple zero in the usual sense.

Proposition 2.1. *Let $\sigma(f)$ denote the number of sign changes of a function f in (a,b). Then*

$$\sigma(N_n[f]) \geqslant \sigma(f) + \sum \langle n(x_i)\rangle + \#\{i \mid N_i[f](a) = 0\} + \#\{j \mid N_j[f](b) = 0\} - n, \quad (2.1)$$

where $\langle n(x_i)\rangle$ denotes the largest even integer that is not larger than $n(x_i)$ and $\#\{\ldots\}$ counts the number of zero boundary values.

Outline of proof. The proposition is adapted from [7, ch. 1.2], where a more detailed claim is proved in the linear case. The proof of [7] applies almost verbatim, with each $L_i[f]$ replaced by $N_i[f]$. It is a consequence of Rolle's theorem applied to $N_i[u] = a_i((N_{i-1}[u])')^{p_i*}$, a careful count of zeros and sign changes, and $\operatorname{sgn} u^{p*} = \operatorname{sgn} u$. For full details, see [7, lemma 1.11].

By Rolle's theorem, $\sigma(g') \geqslant \sigma(g) - 1$, but if g vanishes at either endpoint, we, in fact, have

$$\sigma(g') \geqslant \sigma(g) + \#\{g(a) = 0\} + \#\{g(b) = 0\} - 1.$$

Here, the counting $\#\{\ldots\}$ means that $\#\{g(c) = 0\} = 1$ if $g(c) = 0$, and 0 otherwise. Apply this to the function $g = N_0[f] = a_0 f^{p_0*}$ and note that

$$\operatorname{sgn} N_0[f] = \operatorname{sgn} f^{p_0*} = \operatorname{sgn} f \quad \text{and} \quad \operatorname{sgn}(N_0[u])' = \operatorname{sgn}((N_0[u])')^{p_1*} = \operatorname{sgn} N_1[u].$$

Thus

$$\sigma(N_1[f]) \geqslant \sigma(f) + \#\{N_0[f](a) = 0\} + \#\{N_0[f](b) = 0\} - 1.$$

Next we repeat the same argument for the functions $N_1[f], N_2[f], \ldots$, and obtain, analogously,

$$\sigma(N_{i+1}[f]) \geqslant \sigma(N_i[f]) + \#\{N_i[f](a) = 0\} + \#\{N_i[f](b) = 0\} - 1, \quad i = 0, \ldots, n-1.$$

Summing these inequalities leads us to (2.1), except for the term $\sum \langle n(x_i)\rangle$.

If $\ell = n(x_i)$, then ℓ applications of Rolle's theorem show that $N_n[f]$ has $\ell - 1$ additional sign changes. If ℓ is odd, then f changes its sign at x_i and the net contribution of this zero to the right-hand side of (2.1) is $\ell - 1$. For even ℓ, f does not change its sign there. Thus, in each case, there is added $\langle \ell \rangle$ to (2.1). ∎

If u is a solution of (1.1) and $a_0(x), b(x) > 0$, then the functions $u, N_0[u]$ and $N_n[u] = \lambda b a_0^{-r/p_0}(N_0[u])^{r/p_0*}$ have the same sign changes, and hence, by (2.1), we have the following result.

PROPOSITION 2.2. *For every non-trivial solution u of (1.1),*

$$\sum \langle n(x_i)\rangle + \#\{i \mid N_i[u](a) = 0\} + \#\{j \mid N_j[u](b) = 0\} \leqslant n. \tag{2.2}$$

If a solution of (1.1) satisfies the boundary conditions (1.8), then

$$\#\{i \mid N_i[u](a) = 0\} \geqslant k \quad \text{and} \quad \#\{j \mid N_j[u](b) = 0\} \geqslant n - k.$$

It thus follows from (2.2) that, in fact,

$$\#\{i \mid N_i[u](a) = 0\} = k, \qquad \#\{j \mid N_j[u](b) = 0\} = n - k$$

and $n(x_i) = 1$ for every x_i in (a, b). So, if eigenfunctions u exist, they necessarily have only simple zeros in (a, b), as do the $N_j[u]$, $j = 1, \dots, n-1$.

Another conclusion is that any non-trivial solution of (1.1) may have only a finite number of zeros in $[a, b]$. Suppose, to the contrary, that u has a sequence of zeros $x_i \to c$ as $i \to \infty$. By Rolle's theorem, each $N_j[u]$ has a sequence of zeros $x_i^j \to c$ and consequently $N_j[u](c) = 0$ for all $j = 0, \dots, n-1$. But then $\#\{i \mid N_i[u](c) = 0\} = n$, $\#\{j \mid N_j[u](x_i^j) = 0\} \geqslant 1$, contradicting (2.2) on the interval with endpoints x_i^j and c.

3. Proof of theorem 1.3

The strategy of our proof of theorem 1.3 is to connect the nonlinear (1.1), (1.8) to the linear (1.2), (1.6) through a continuum of intermediate problems, and show by means of the implicit function theorem that the appropriate information about the linear problem can be extended to the nonlinear problem.

For all t, $0 \leqslant t \leqslant 1$, let

$$\begin{aligned} p_i(t) &= t p_i + (1 - t), \quad i = 1, \dots, n-1, \\ r(t) &= p_0(t) \cdots p_{n-1}(t). \end{aligned}$$

Then

$$(p_0(1), \dots, p_{n-1}(1), r(1)) = (p_1, \dots, p_{n-1}, r),$$

while

$$(p_0(0), \dots, p_{n-1}(0), r(0)) = (1, \dots, 1, 1).$$

We now consider (1.1) together with the boundary conditions (1.8), with p_i replaced by $p_i(t)$, and verify the existence of the desired eigenvalues and eigenfunctions for all $0 \leqslant t \leqslant 1$. Note that if assumption 1.2 holds for the given powers p_i, it also holds for $p_i(t)$ for all $t \in [0, 1]$. For ease of notation, the dependence on t will be explicitly stated only when needed.

Solutions of (1.10) may be parametrized by an initial-value problem

$$(v_0(a), \dots, v_{n-1}(a)) = (N_0[u](a), \dots, N_{n-1}[u](a)) = (\alpha_0, \dots, \alpha_{n-1}). \tag{3.1}$$

This can be done only if the initial-value problem for the nonlinear system (1.10) has a unique solution $v(x, \alpha_0, \dots, \alpha_{n-1})$ and this solution is extendable to the whole interval $[a, b]$. This will be verified in theorems A.1 and A.2 of the appendix. In this section we will assume this fact.

An eigenfunction is a solution of the system (1.10) that satisfies the boundary conditions (1.8). To satisfy the k boundary conditions of (1.8) at $x = a$, we require that

$$\alpha_i = 0 \quad \text{for } i = i_1, \dots, i_k.$$

Our aim is to verify that during the change of t we are permitted to determine the other $n-k$ parameters α_i, $i \neq i_1, \dots, i_k$, plus λ, so that the boundary conditions at $x = b$ are also satisfied.

As we already mentioned, if a solution $u \not\equiv 0$ of (1.1) satisfies the boundary conditions (1.8), then $\#\{i \mid N_i[u](a) = 0\} = k$, $\#\{j \mid N_j[u](b) = 0\} = n - k$, and consequently,

$$\begin{aligned} N_i[u](a) &\neq 0, \quad i \neq i_1, \dots, i_k, \\ N_j[u](b) &\neq 0, \quad j \neq j_1, \dots, j_{n-k}. \end{aligned}$$

In particular, $\alpha_i = N_i[u](a) \neq 0$ for $i \neq i_1, \dots, i_k$. If u is a solution of (1.1), then from the condition $p_0 \cdots p_{n-1} = r$, cu is also a solution for every constant c. Therefore, without loss of generality, we may assume one $\alpha_i \neq 0$ to be 1. Furthermore, at present, the precise choice of the $i_1, \dots, i_k$ makes no difference in the analysis, so, for simplicity of notation, we will assume that the $n-k$ non-vanishing initial values $\{\alpha_i \mid i \neq i_1, \dots, i_k\}$ are $\alpha_k, \dots, \alpha_{n-2}, \alpha_{n-1}$ and $\alpha_{n-1} = 1$.

To summarize, we shall search for eigenfunctions of the form

$$u(x, \alpha_k, \dots, \alpha_{n-2}, \lambda, t)$$

that satisfy the $n-k$ boundary conditions of (1.8) at $x = b$,

$$N_j[u(b, \alpha_k, \dots, \alpha_{n-2}, \lambda, t)] = 0, \quad j = j_1, \dots, j_{n-k}. \tag{3.2}$$

These are $n-k$ equations in the $n-k$ variables $\alpha_k, \dots, \alpha_{n-2}, \lambda$. Our aim is to solve them and obtain the continuous functions $\alpha_k(t), \dots, \alpha_{n-2}(t), \lambda(t)$ for $0 \leqslant t \leqslant 1$. For $t = 0$, it is known from the linear theory that, for every integer m, there exists an eigenvalue λ_m and an essentially unique eigenfunction u_m with $m-1$ simple zeros in (a, b). Hence (3.2) has, for $t = 0$, a solution $\alpha_k(0), \dots, \alpha_{n-2}(0), \lambda(0)$ (and $u_m(x) = u(x, \alpha_k(0), \dots, \alpha_{n-2}(0), \lambda(0), 0)$). This solution of (3.2) will be continued by the implicit function theorem on $t > 0$ and generate an eigenfunction of the nonlinear (1.1), (1.8) also with $m-1$ simple zeros. Technically, we prove the following result.

Proposition 3.1. *Let $\tilde{t} \in [0, 1]$ and suppose that, for $t = \tilde{t}$, the system (3.2) has a solution $\alpha_k(\tilde{t}), \dots, \alpha_{n-1}(\tilde{t}), \lambda(\tilde{t})$. Then, for $t = \tilde{t}$,*

$$\left.\frac{\partial(N_{j_1}[u], \dots, N_{j_{n-k}}[u])}{\partial(\alpha_k, \dots, \alpha_{n-2}, \lambda)}\right|_{x=b} \neq 0. \tag{3.3}$$

Proof. We need the differentiability of the solution of the initial-value problem (1.10), (3.1) with respect to each initial value α_i and λ. If $1/p_i < 1$, then at the points where v_i vanishes, the right-hand side of (1.10) does not have continuous derivatives with respect to v_i. So the differentiability with respect to initial values does not follow from the standard Peano theorem [9] and it is a quite delicate problem. It will be verified in theorem A.3 of the appendix that if assumption 1.2 about the powers p_i holds, then all $\partial N_j[u]/\partial\alpha_\ell$ and $\partial N_j[u]/\partial\lambda$ indeed exist and are continuous in $[a,b]$. This will be postulated in the present section. The differentiabilty of the $N_j[u]$ with respect to t is easily proved.

In the following calculations we omit $\tilde{t}$ altogether, so keep in mind that $p_i = p_i(\tilde{t})$, u is the corresponding eigenfunction, etc.

Suppose that the Jacobian determinant (3.3) does vanish. Then there exist $n-k$ values $(c_k,\ldots,c_{n-2},c_\lambda)\neq(0,\ldots,0)$ such that

$$c_k\frac{\partial N_j[u]}{\partial\alpha_k}(b)+\cdots+c_{n-2}\frac{\partial N_j[u]}{\partial\alpha_{n-2}}(b)+c_\lambda\frac{\partial N_j[u]}{\partial\lambda}(b)=0,\quad j=j_1,\ldots,j_{n-k}. \tag{3.4}$$

The next step is to characterize the derivatives

$$\frac{\partial}{\partial\alpha}N_j[u]\quad\text{and}\quad\frac{\partial}{\partial\lambda}N_j[u],$$

when α is one of the $\alpha_k,\ldots,\alpha_{n-2}$. Once it is known, by theorem A.3, that these derivatives exist, we may straightforwardly differentiate (1.10),

$$\begin{aligned} v_{i-1}' &= (a_i^{-1}(x)v_i)^{1/p_i*},\quad i=1,\ldots,n-1,\\ v_{n-1}' &= \lambda b(x)(a_0^{-1}(x)v_0)^{r/p_0*} \end{aligned}$$

with respect to $\alpha=\alpha_\ell$, using $(x^{p*})'=p|x|^{p-1}$. It follows that the

$$z_i=\frac{\partial v_i}{\partial\alpha}=\frac{\partial N_i[u]}{\partial\alpha}$$

satisfy

$$\left.\begin{aligned} z_{i-1}' &= (a_i^{-1/p_i}p_i^{-1}|v_i|^{1/p_i-1})z_i,\quad i=1,\ldots,n-1,\\ z_{n-1}' &= \lambda b\left(a_0^{-r/p_0}\frac{r}{p_0}|v_0|^{r/p_0-1}\right)z_0. \end{aligned}\right\} \tag{3.5}$$

To simplify this linear system, let us define

$$\rho_i=a_i^{1/p_i}p_i|v_i|^{-1/p_i+1},\quad i=0,\ldots,n-1.$$

Then, for $i=1,\ldots,n-1$, system (3.5) becomes $z_{i-1}'=\rho_i^{-1}z_i$. Substituting $v_i=N_i[u]=a_i((N_{i-1}[u])')^{p_i*}$, we get

$$\rho_i=a_i^{1/p_i}p_i|a_i((N_{i-1}[u])')^{p_i}|^{-1/p_i+1}=a_ip_i|(N_{i-1}[u])'|^{p_i-1}.$$

For $i=0$, we substitute $v_0=N_0[u]=a_0u^{p_0*}$ and obtain

$$\rho_0=a_0^{1/p_0}p_0|a_0u^{p_0}|^{-1/p_0+1}=a_0p_0|u|^{p_0-1},$$

and the last equation of (3.5) becomes

$$z_{n-1}'=\lambda ba_0^{-r/p_0}\frac{r}{p_0}|a_0u^{p_0}|^{r/p_0-1}z_0=\lambda b\frac{r|u|^{r-1}}{a_0p_0|u|^{p_0-1}}z_0=\frac{\lambda b}{\rho_0}r|u|^{r-1}z_0.$$

Thus

$$(z_0, \ldots, z_{n-1}) = \frac{\partial}{\partial \alpha}(N_0[u], \ldots, N_{n-1}[u])$$

solves the linear system

$$\left.\begin{aligned} z'_{i-1} &= \rho_i^{-1} z_i, \quad i = 1, \ldots, n-1, \\ z'_{n-1} &= \lambda b r |u|^{r-1} \rho_0^{-1} z_0, \end{aligned}\right\} \tag{3.6}$$

with

$$\left.\begin{aligned} \rho_i &= a_i p_i |(N_{i-1}[u])'|^{p_i - 1}, \quad i = 1, \ldots, n-1, \\ \rho_0 &= a_0 p_0 |u|^{p_0 - 1}, \end{aligned}\right\} \tag{3.7}$$

and $\rho_i \geqslant 0$.

We are also interested in $u_\alpha = \partial u / \partial \alpha$. Note that

$$z_0 = \frac{\partial}{\partial \alpha} N_0[u] = \frac{\partial}{\partial \alpha}(a_0 u^{p_0 *}) = a_0 p_0 |u|^{p_0 - 1} \frac{\partial u}{\partial \alpha} = \rho_0 \frac{\partial u}{\partial \alpha}.$$

Thus

$$u_\alpha = \frac{\partial u}{\partial \alpha} = \rho_0^{-1} z_0$$

and (3.6) is equivalent to the nth-order linear differential equation

$$(\rho_{n-1} \cdots (\rho_1 (\rho_0 u_\alpha)')' \cdots)' = \lambda b r |u|^{r-1} u_\alpha, \tag{3.8}$$

or more explicitly,

$$(a_{n-1} p_{n-1} |(N_{n-2}[u])'|^{p_{n-1} - 1} (\cdots (a_1 p_1 |(N_0[u])'|^{p_1 - 1} (a_0 p_0 |u|^{p_0 - 1} u_\alpha)')' \cdots)')' = \lambda b(x) r |u|^{r-1} u_\alpha.$$

This is the *variational equation of (1.1) with respect to the solution* u. Of course, it can be derived by direct differentiation of (1.1), but we also need (3.6). As in (1.3), we define the linear differential operators

$$\begin{aligned} M_0[z] &= \rho_0 z, \\ M_i[z] &= \rho_i (M_{i-1}[z])', \quad i = 1, \ldots, n, \end{aligned}$$

$\rho_n \equiv 1$. With this notation, equation (3.8) is written as

$$M_n[u_\alpha] = \lambda b r |u|^{r-1} u_\alpha. \tag{3.9}$$

The $(\partial / \partial \alpha) N_i[u] = z_i$ are conveniently characterized in terms of (3.8) by

$$\begin{aligned} z_i &= \rho_i z'_{i-1} \\ &= \rho_i (\rho_{i-1} z'_{i-2})' \\ &= \cdots = \rho_i (\cdots (\rho_1 z'_0) \cdots)' \\ &= \rho_i \left(\cdots \left(\rho_1 \left(\rho_0 \frac{\partial u}{\partial \alpha} \right)' \right) \cdots \right)' \\ &= M_i \left[\frac{\partial u}{\partial \alpha} \right], \end{aligned}$$

i.e.

$$\frac{\partial N_i[u]}{\partial\alpha} = M_i\left[\frac{\partial u}{\partial\alpha}\right], \quad i = 0,\ldots,n-1. \tag{3.10}$$

This relation between the nonlinear operator N_i and its linearization M_i will be used later. Note that all calculations above are independent of the choice of the parameter $\alpha = \alpha_\ell$.

We observe that proposition 2.1 holds for the operator M_n and proposition 2.2 holds for (3.9). This is not obvious, since if $p_i < 1$, then $\rho_i = a_i^{1/p_i} p_i |v_i|^{-1/p_i+1}$ is singular at the points where $v_i = N_i[u]$ vanishes. However, in theorem A.3, we prove that under assumption 1.2, the derivatives

$$z_i = \frac{\partial N_i[u]}{\partial\alpha} = M_i[u_\alpha]$$

exist and are continuous. Moreover, it follows from (A 14) that even if $z_i' = (M_i[u_\alpha])'$ is unbounded at a point $x = c$, $a < c < b$, it does not change its sign there, say, $z_i'(c^+) = z_i'(c^-) = +\infty$. Therefore, a Rolle's-theorem-type argument may be applied to $M_{i+1}[u_\alpha] = \rho_{i+1}(M_i[u_\alpha])'$ and propositions 2.1 and 2.2 follow.

The $(\partial/\partial\lambda)N_i[u]$ are calculated similarly. Let

$$(\zeta_0,\ldots,\zeta_{n-1}) = \frac{\partial}{\partial\lambda}(N_0[u],\ldots,N_{n-1}[u]).$$

The differentiation of (1.10) with respect to λ is similar to (3.5), except that the last equation is now

$$\zeta_{n-1}' = \lambda b\left(a_0^{-r/p_0}\frac{r}{p_0}|v_0|^{r/p_0-1}\right)\zeta_0 + b(a_0^{-1}v_0)^{r/p_0*}.$$

Since

$$(a_0^{-1}v_0)^{r/p_0*} = (u^{p_0*})^{r/p_0*} = u^{r*},$$

$(\zeta_0,\ldots,\zeta_{n-1})$ is a solution of the linear system

$$\begin{aligned}\zeta_{i-1}' &= \rho_i^{-1}\zeta_i, \quad i = 1,\ldots,n-1,\\ \zeta_{n-1}' &= \lambda br|u|^{r-1}\rho_0^{-1}\zeta_0 + bu^{r*}.\end{aligned}$$

We also have, as previously,

$$\zeta_0 = \frac{\partial}{\partial\lambda}N_0[u] = \rho_0\frac{\partial u}{\partial\lambda}.$$

Thus, as in (3.10),

$$\frac{\partial N_i[u]}{\partial\lambda} = M_i\left[\frac{\partial u}{\partial\lambda}\right], \quad i = 0,\ldots,n-1, \tag{3.11}$$

and $u_\lambda = \partial u/\partial\lambda$ satisfies

$$M_n[u_\lambda] = \lambda br|u|^{r-1}u_\lambda + bu^{r*}. \tag{3.12}$$

Finally, we show how the eigenfunction u of equation (1.1) is related to the linearization of the same equation. Namely, what is $M_i[u]$?

$$\begin{aligned} M_0[u] &= \rho_0 u \\ &= a_0 p_0 |u|^{p_0-1} u \\ &= p_0 a_0 u^{p_0 *} \\ &= p_0 N_0[u], \\ M_1[u] &= \rho_1(\rho_0 u)' \\ &= (a_1 p_1 |(N_0[u])'|^{p_1-1})(p_0 N_0[u])' \\ &= p_0 p_1 a_1((N_0[u])')^{p_1 *} \\ &= p_0 p_1 N_1[u], \end{aligned}$$

and generally,

$$M_i[u] = p_0 p_1 \cdots p_i N_i[u], \quad i = 0, \ldots, n-1. \tag{3.13}$$

Recall that $N_n[u] = (N_{n-1}[u])'$ and $p_0 p_1 \cdots p_{n-1} = r$. Thus

$$M_n[u] = (M_{n-1}[u])' = p_0 \cdots p_{n-1}(N_{n-1}[u])' = rN_n[u]$$

and, consequently, u satisfies

$$M_n[u] = r\lambda b u^{r*}. \tag{3.14}$$

After these preparations, we can now return to (3.4). From the identities (3.10) and (3.11) and due to the linearity of the operators M_j, equations (3.4) may be rewritten as

$$M_j[c_k u_{\alpha_k} + \cdots + c_{n-2} u_{\alpha_{n-2}} + c_\lambda u_\lambda](b) = 0, \quad j = j_1, \ldots, j_{n-k}, \tag{3.15}$$

where

$$u_{\alpha_\ell} = \frac{\partial u}{\partial \alpha_\ell}, \qquad u_\lambda = \frac{\partial u}{\partial \lambda}.$$

Our aim is to show that this is impossible.

Recall that $\sum_k^{n-2} c_i u_{\alpha_i}$ is a solution of (3.9), while $c_\lambda u_\lambda$ is a solution of (3.12). For brevity, we write $w_\alpha = \sum_k^{n-2} c_i u_{\alpha_i}$ and consider the function $w_\alpha + c_\lambda u_\lambda$. Equation (3.15) implies that the function $w_\alpha + c_\lambda u_\lambda$ satisfies at b the $n-k$ conditions

$$M_j[w_\alpha + c_\lambda u_\lambda](b) = 0, \quad j = j_1, \ldots, j_{n-k}.$$

$u = u(x, \alpha_k, \ldots, \alpha_{n-2}, \lambda)$ was defined by the initial value (3.1), so that it satisfies at a

$$N_i[u](a) = 0, \quad i = i_1, \ldots, i_k,$$

plus the normalization

$$N_{n-1}[u](a) = 1$$

for all $\alpha_k, \ldots, \alpha_{n-2}, \lambda$. Differentiation with respect to some α_ℓ or λ, together with (3.10) and (3.11) evidently yields that u_{α_ℓ}, u_λ, as well as any of their linear combinations, satisfy at a the $k+1$ conditions

$$M_i[w_\alpha + c_\lambda u_\lambda](a) = 0, \quad i = i_1, \ldots, i_k, n-1.$$

We remark that this implies that $c_\lambda \neq 0$. Otherwise, w_α, which is a solution of the linear equation (3.9), satisfies $(k+1)+(n-k)=n+1$ homogeneous boundary conditions at the endpoints a and b, which is impossible by proposition 2.2.

The solution u of (1.1) also satisfies, by (3.13), the first k of these conditions at a and the same $n-k$ conditions at b. Thus u, $w_\alpha + c_\lambda u_\lambda$ and any of their linear combinations satisfy the n linear boundary conditions

$$\left.\begin{aligned} M_i[y](a) = 0, &\quad i = i_1, \dots, i_k, \\ M_j[y](b) = 0, &\quad j = j_1, \dots, j_{n-k}. \end{aligned}\right\} \tag{3.16}$$

Let us choose a linear combination $f_0 = w_\alpha + c_\lambda u_\lambda + cu$ that satisfies one additional boundary condition of the same type at one of the endpoints, say,

$$\left.\begin{aligned} M_i[f_0](a) = 0, &\quad i = i_1, \dots, i_k, i_{k+1}, \\ M_j[f_0](b) = 0, &\quad j = j_1, \dots, j_{n-k}. \end{aligned}\right\} \tag{3.17}$$

This implies that $\#\{i \mid M_i[f_0](a) = 0\} \geqslant k+1$ and $\#\{j \mid M_j[f_0](b) = 0\} \geqslant n-k$, so, by proposition 2.1,

$$\sigma(M_n[w_\alpha + c_\lambda u_\lambda + cu]) \geqslant \sigma(w_\alpha + c_\lambda u_\lambda + cu) + 1.$$

But, by (3.9), (3.12) and (3.14),

$$\begin{aligned} M_n[w_\alpha + c_\lambda u_\lambda + cu] &= \lambda br|u|^{r-1} w_\alpha + c_\lambda(\lambda br|u|^{r-1} u_\lambda + bu^{r*}) + c(\lambda bru^{r*}) \\ &= \lambda br|u|^{r-1}\left[w_\alpha + c_\lambda\left(u_\lambda + \frac{1}{\lambda r}u\right) + cu\right], \end{aligned}$$

and therefore

$$\sigma\left(w_\alpha + c_\lambda u_\lambda + \left(\frac{c_\lambda}{\lambda r} + c\right)u\right) \geqslant \sigma(w_\alpha + c_\lambda u_\lambda + cu) + 1. \tag{3.18}$$

The new combination

$$f_1 = w_\alpha + c_\lambda u_\lambda + \left(\frac{c_\lambda}{\lambda r} + c\right)u$$

satisfies again the n boundary conditions (3.16) (but not the $n+1$ boundary conditions (3.17)!). So $\#\{i \mid N_i[f_1](a) = 0\} \geqslant k$, $\#\{j \mid N_j[f_1](b) = 0\} \geqslant n-k$ and another application of the operator M_n, together with proposition 2.1, gives $\sigma(M_n[f_1]) \geqslant \sigma(f_1)$. Explicitly,

$$\sigma\left(w_\alpha + c_\lambda u_\lambda + \left(\frac{2c_\lambda}{\lambda r} + c\right)u\right) \geqslant \sigma\left(w_\alpha + c_\lambda u_\lambda + \left(\frac{c_\lambda}{\lambda r} + c\right)u\right). \tag{3.19}$$

After m iterations of this process, we get, together with (3.18),

$$\sigma\left(w_\alpha + c_\lambda u_\lambda + \left(\frac{mc_\lambda}{\lambda r} + c\right)u\right) \geqslant \sigma(w_\alpha + c_\lambda u_\lambda + cu) + 1. \tag{3.20}$$

As $m \to \infty$,

$$\frac{1}{m}\left(w_\alpha + c_\lambda u_\lambda + \left(\frac{mc_\lambda}{\lambda r} + c\right)u\right) \to \frac{c_\lambda}{\lambda r}u.$$

What happens to the zeros of

$$f_m = w_\alpha + c_\lambda u_\lambda + \left(\frac{mc_\lambda}{\lambda r} + c\right)u$$

as $m \to \infty$? No zero of f_m in (a, b) can tend to either endpoint a or b, since this would add a vanishing $M_i[u]$ there and we would obtain

$$\#\{i \mid M_i[u](a) = 0\} + \#\{j \mid M_j[u](b) = 0\} \geqslant n + 1,$$

contradicting proposition 2.2 applied to (3.14). Nor can two zeros of f_m coalesce at an internal point x_i of (a, b), since this would imply that $M_0[u](x_i) = M_1[u](x_i) = 0$. But we had already concluded after proposition 2.2 that $n(x_i) \geqslant 2$ is impossible for an eigenfunction u. It therefore follows that as $m \to \infty$, the number of sign changes of f_m in (a, b) remains fixed. So, for all sufficiently large integers m,

$$\sigma\left(w_\alpha + c_\lambda u_\lambda + \left(\frac{mc_\lambda}{\lambda r} + c\right)u\right) = \sigma(u).$$

Together with (3.20), this means that

$$\sigma(u) \geqslant \sigma(w_\alpha + c_\lambda u_\lambda + cu) + 1. \tag{3.21}$$

On the other hand, consider the function

$$f_{-m} = w_\alpha + c_\lambda u_\lambda + \left(\frac{-mc_\lambda}{\lambda r} + c\right)u.$$

As above,

$$\frac{1}{m} f_{-m} \to \left(\frac{c_\lambda}{\lambda r}\right)u$$

when $m \to \infty$ and $\sigma(f_{-m}) = \sigma(u)$ for sufficiently large m. Now we apply the operator M_n to f_{-m} m times and obtain

$$\begin{aligned}\sigma(w_\alpha + c_\lambda u_\lambda + cu) &\geqslant \sigma\left(w_\alpha + c_\lambda u_\lambda + \left(\frac{-c_\lambda}{\lambda r} + c\right)u\right)\\ &\geqslant \cdots \geqslant \sigma\left(w_\alpha + c_\lambda u_\lambda + \left(\frac{-mc_\lambda}{\lambda r} + c\right)u\right)\\ &= \sigma(u).\end{aligned}$$

This contradicts (3.21). This contradiction shows that the Jacobian (3.3) indeed cannot be 0 and the proof of proposition 3.1 is completed. $\square$

We now return to the proof of the main theorem. Eigenvalues and eigenfunctions of (1.1), (1.8) are obtainable by solving the unknown $\alpha_k(t), \dots, \alpha_{n-2}(t), \lambda(t)$ from the system (3.2) for every $0 \leqslant t \leqslant 1$. Take a fixed integer m. It is known that the system (3.2) has a solution for $t = 0$ that generates the mth eigenvalue λ_m and corresponding eigenfunction u_m of the (linear) system. Thanks to proposition 3.1, the implicit function theorem ensures that the solution of (3.2) can be continued for some $t > 0$. Let $\hat{t}$ be the supremum of the values such that (3.2) is solvable for all $0 \leqslant t < \hat{t}$ and assume that $\hat{t}$ is finite.

During the process, we normalized the solution u of (1.1), (3.1) by

$$N_{n-1}[u](a) = \alpha_{n-1} = 1.$$

However, since (1.1) is homogeneous, i.e. u is a solution if and only if cu is a solution, every other normalization is equivalent. Let us renormalize u by multiplication with a suitable constant, so that

$$\sum_{i \neq i_1, \ldots, i_k} (N_i[u](a))^2 = 1,$$

and, in addition, of course, $N_i[u](a) = 0$, $i = i_1, \ldots, i_k$.

What happens to the solution u as $t \uparrow \hat{t}$? By the above normalization, it is obvious that we can choose a convergent subsequence of initial values. For simplicity, we denote the limit by $\alpha_0(\hat{t}), \ldots, \alpha_{n-1}(\hat{t})$. The corresponding solution u satisfies the given boundary conditions. As argued above for $\lim f_m$, no zero of u can tend to either endpoint a or b, nor can two of its zeros coalesce at an internal point of (a, b). Hence the number of sign changes of u in (a, b) remains unchanged for $0 \leqslant t \leqslant \hat{t}$. By considering (1.1) at $x = a$, it also follows that $\lambda(t)$ tends to a finite limit. Therefore, we can again apply the implicit function theorem, starting at $\hat{t}$ and continue the solution of (3.2), contradicting the definition of $\hat{t}$. This verifies that our process continues, in fact, for all $t > 0$ and, in particular, holds for $t = 1$. This proves the existence of an eigenvalue λ_m of (1.1) such that the corresponding eigenfunction has exactly $m - 1$ changes of sign in (a, b).

Note that the sign of the eigenvalues is determined from the linear problem and it cannot change with t, since we assume condition 1.1. Thus $(-1)^{n-k}\lambda_m > 0$ for all m.

The eigenvalues of the linear problem are arranged according to the number of sign changes of the corresponding eigenfunction. That is, $|\lambda_{\ell+1}| > |\lambda_\ell|$ and $\sigma(u_{\ell+1}) = \sigma(u_\ell) + 1 = \ell$. We had already seen that as the parameter t grows, the number of sign changes remains fixed. Is the strict order of the eigenvalues preserved in the process? The following proposition shows that no eigenvalues meet as t grows and the nonlinear problem preserves the same ordering $0 < (-1)^{n-k}\lambda_\ell < (-1)^{n-k}\lambda_{\ell+1}$ as the linear one.

Proposition 3.2.

(i) *Each eigenvalue of (1.1), (1.8) has an essentially unique eigenfunction.*

(ii) *Let λ, μ be two eigenvalues of (1.1), (1.8), and u, w the corresponding eigenfunctions. If $|\lambda| > |\mu|$, then $\sigma(u) > \sigma(w)$.*

Proof. First we prove that, for any two functions u, w for which $N_n[u]$ and $N_n[w]$ are well defined, and constants α, β,

$$\begin{aligned} \sigma(&\alpha^{p_0 \cdots p_{n-1}*} N_n[u] + \beta^{p_0 \cdots p_{n-1}*} N_n[w]) \\ &\geqslant \sigma(\alpha u + \beta w) \\ &\quad + \#\{i \mid 0 \leqslant i \leqslant n-1,\ \alpha^{p_0 \cdots p_i *} N_i[u](a) + \beta^{p_0 \cdots p_i *} N_i[w](a) = 0\} \\ &\quad + \#\{j \mid 0 \leqslant j \leqslant n-1,\ \alpha^{p_0 \cdots p_j *} N_j[u](b) + \beta^{p_0 \cdots p_j *} N_j[w](b) = 0\} - n. \end{aligned} \tag{3.22}$$

As in the proof of proposition 2.1, we start with

$$\sigma(f') \geqslant \sigma(f) + \#\{f(a)=0\} + \#\{f(b)=0\} - 1, \tag{3.23}$$

and apply this to the function $f_0 = \alpha^{p_0*}N_0[u] + \beta^{p_0*}N_0[w]$. By the easily checked identity

$$\operatorname{sgn}(u^{p*} + v^{p*}) = \operatorname{sgn}(u+v), \tag{3.24}$$

and since $N_0[u] = a_0 u^{p_0*}$, we have

$$\sigma(\alpha^{p_0*}N_0[u] + \beta^{p_0*}N_0[w]) = \sigma(\alpha^{p_0*}u^{p_0*} + \beta^{p_0*}w^{p_0*}) = \sigma(\alpha u + \beta w).$$

So (3.23) becomes

$$\begin{aligned}\sigma(\alpha^{p_0*}(N_0[u])' &+ \beta^{p_0*}(N_0[w])') \\ &\geqslant \sigma(\alpha u + \beta w) + \#\{\alpha^{p_0*}N_0[u](a) + \beta^{p_0*}N_0[w](a) = 0\} \\ &\qquad + \#\{\alpha^{p_0*}N_0[u](b) + \beta^{p_0*}N_0[w](b) = 0\} - 1.\end{aligned}$$

We apply (3.24) with $p = p_1$ to the left-hand side of the last inequality, and substitute $((N_0[u])')^{p_1*} = a_1^{-1}N_1[u]$, so as to obtain

$$\begin{aligned}\sigma(\alpha^{p_0p_1*}N_1[u] &+ \beta^{p_0p_1*}N_1[w]) \\ &\geqslant \sigma(\alpha u + \beta w) + \#\{\alpha^{p_0*}N_0[u](a) + \beta^{p_0*}N_0[w](a) = 0\} \\ &\qquad + \#\{\alpha^{p_0*}N_0[u](b) + \beta^{p_0*}N_0[w](b) = 0\} - 1.\end{aligned}$$

Next we repeat the same step with $f_1 = \alpha^{p_0p_1*}N_1[u] + \beta^{p_0p_1*}N_1[w]$, focusing attention on the vanishing of $N_1[u]$, $N_1[w]$ at the endpoints. This gives

$$\begin{aligned}\sigma(\alpha^{p_0p_1p_2*}N_2[u] &+ \beta^{p_0p_1p_2*}N_2[w]) \\ &\geqslant \sigma(\alpha^{p_0p_1*}N_1[u] + \beta^{p_0p_1*}N_1[w]) \\ &\qquad + \#\{\alpha^{p_0p_1*}N_1[u](a) + \beta^{p_0p_1*}N_1[w](a) = 0\} \\ &\qquad + \#\{\alpha^{p_0p_1*}N_1[u](b) + \beta^{p_0p_1*}N_1[w](b) = 0\} - 1.\end{aligned}$$

Summing up the n repetitions of the above inequalities leads us to (3.22).

Now we turn to the proof of (i). Let u, w be two eigenfunctions corresponding to the same eigenvalue λ and suppose that they are linearly independent. On the right-hand side of (3.22), the n boundary conditions that are satisfied by both u and w are counted. But, if we choose α, β so that one additional intermediate function $\alpha^{p_0\cdots p_i*}N_i[u] + \beta^{p_0\cdots p_i*}N_i[w]$ vanishes at one of the endpoints, then the right-hand side increases by 1 and we get

$$\sigma(\alpha^{p_0\cdots p_{n-1}*}N_n[u] + \beta^{p_0\cdots p_{n-1}*}N_n[w]) \geqslant \sigma(\alpha u + \beta w) + 1.$$

By $p_0\cdots p_{n-1} = r$, the differential equations $N_n[u] = \lambda b u^{r*}, N_n[w] = \lambda b w^{r*}$ and, by (3.24), the left-hand side of (3.22) becomes

$$\sigma(\alpha^{p_0\cdots p_{n-1}*}N_n[u] + \beta^{p_0\cdots p_{n-1}*}N_n[w]) = \sigma((\alpha u)^{r*} + (\beta w)^{r*}) = \sigma(\alpha u + \beta w).$$

Thus we arrive at the contradiction

$$\sigma(\alpha u + \beta w) \geqslant \sigma(\alpha u + \beta w) + 1.$$

This contradiction implies that λ cannot have two linearly independent eigenfunctions.

To prove (ii), let λ, μ be two distinct eigenvalues, $|\lambda| > |\mu|$, and u, w the corresponding eigenfunctions. We choose $f = \alpha u + \beta w$ as above, with an additional zero at either endpoint, and obtain

$$\sigma(\alpha^{p_0 \cdots p_{n-1}*} N_n[u] + \beta^{p_0 \cdots p_{n-1}*} N_n[w]) \geqslant \sigma(\alpha u + \beta w) + 1.$$

Since $N_n[u] = \lambda b u^{r*}$, $N_n[w] = \mu b w^{r*}$, we obtain

$$\sigma(\lambda(\alpha u)^{r*} + \mu(\beta w)^{r*}) \geqslant \sigma(\alpha u + \beta w) + 1$$

and, by (3.24),

$$\sigma\left(\alpha u + \left(\frac{\mu}{\lambda}\right)^{1/r} \beta w\right) \geqslant \sigma(\alpha u + \beta w) + 1.$$

Note that, consequently, we must have $\alpha \neq 0$, $\beta \neq 0$. We repeat the same argument m times, beginning with the function $\alpha u + (\mu/\lambda)^{1/r} \beta w$ (but this time without any extra sign change!), and obtain

$$\sigma\left(\alpha u + \left(\frac{\mu}{\lambda}\right)^{m/r} \beta w\right) \geqslant \sigma(\alpha u + \beta w) + 1.$$

Since, by assumption, $|\mu/\lambda| < 1$, we obtain as $m \to \infty$ that

$$\sigma(u) \geqslant \sigma(\alpha u + \beta w) + 1.$$

Next we begin with the function $(\mu/\lambda)^{m/r} \alpha u + \beta w$ and carry out the same argument. Now the result is

$$\sigma(\alpha u + \beta w) \geqslant \cdots \geqslant \sigma\left(\left(\frac{\mu}{\lambda}\right)^{m/r} \alpha u + \beta w\right) \to \sigma(w),$$

and together we have

$$\sigma(u) \geqslant \sigma(\alpha u + \beta w) + 1 > \sigma(\alpha u + \beta w) \geqslant \sigma(w). \tag{3.25}$$

Thus $\sigma(u) > \sigma(w)$. □

REMARK 3.3. The inequality (3.25) was proved for a particular combination $\alpha u + \beta w$, with an additional zero at one endpoint. If we repeat this same argument for an arbitrary combination $\gamma u + \delta w$, $(\gamma, \delta) \neq (0, 0)$, we get

$$\sigma(u) \geqslant \sigma(\gamma u + \delta w) \geqslant \sigma(w).$$

In particular, we have

$$\ell \geqslant \sigma(\gamma u_{\ell+1} + \delta u_\ell) \geqslant \ell - 1. \tag{3.26}$$

Proposition 3.2 also implies that there are no other eigenvalue–eigenfunction pairs, except those that we constructed by the implicit function theorem. Suppose, to the contrary, that there exists another pair $\tilde{\lambda}_\ell$, $\tilde{u}_\ell$ such that $\tilde{u}_\ell$ has $\ell - 1$ changes of sign. (Recall that any non-trivial solution of (1.1) has at most a finite number of zeros in $[a,b]$.) Then either $\tilde\lambda_\ell = \lambda_\ell$, $\tilde\lambda_\ell > \lambda_\ell$ or $\tilde\lambda_\ell < \lambda_\ell$, and proposition 3.2 leads to a contradiction.

The last detail of theorem 1.3 is the separation of the zeros of consecutive eigenfunctions. Let λ_ℓ, $\lambda_{\ell+1}$, $|\lambda_\ell| < |\lambda_{\ell+1}|$, be two consecutive eigenvalues and u_ℓ, $u_{\ell+1}$ be the corresponding eigenfunctions, with $\ell-1$ and ℓ simple zeros in (a,b), respectively. Suppose that their zeros do not separate each other in (a,b). Then we check (and reject) two possibilities.

(a) If u_ℓ, $u_{\ell+1}$ have a common zero at $x_1 \in (a,b)$, then there exist α, β such that $\alpha^{p_0p_1*}N_1[u_{\ell+1}] + \beta^{p_0p_1*}N_1[u_\ell]$ vanishes at x_1 (and, of course, every combination of $N_0[u_\ell]$, $N_0[u_{\ell+1}]$ is zero there). Here, $\alpha,\beta \neq 0$, since we had already seen that no eigenfunction u can satisfy $N_0[u](x_1) = N_1[u](x_1) = 0$. Thus we have $n(x_1) \geqslant 2$, which adds two changes of sign to our count and, as in the proof of (3.25), we obtain

$$\sigma\biggl(\alpha u_{\ell+1} + \biggl(\frac{\lambda_\ell}{\lambda_{\ell+1}}\biggr)^{1/r}\beta u_\ell\biggr) \geqslant \sigma(\alpha u_{\ell+1} + \beta u_\ell) + 2. \tag{3.27}$$

But this contradicts (3.26).

(b) Suppose that $u_{\ell+1}$ has consecutive zeros at $x_1, x_2 \in (a,b)$, but $u_\ell \neq 0$ in $[x_1,x_2]$. Then there exist $\alpha,\beta \neq 0$ such that $\alpha u_{\ell+1} + \beta u_\ell$ vanishes at a point of (x_1,x_2) and does not change its sign there ('double' zero). By the definitions of $N_0[u]$, $N_1[u]$ and (3.24), this is also a zero of $\alpha^{p_0*}N_0[u_{\ell+1}] + \beta^{p_0*}N_0[u_\ell]$, and $\alpha^{p_0p_1*}N_1[u_{\ell+1}] + \beta^{p_0p_1*}N_1[u_\ell]$ changes it sign there. This adds two changes of sign to our count and (3.27) again holds, which is impossible by (3.26).

This completes the proof of the separation property, and of theorem 1.3.

Appendix A. Existence, uniqueness and differentiability

THEOREM A.1. *Assume we are given the cyclic system*

$$\left.\begin{aligned} v'_{i-1} &= b_i(x)v_i^{r_i*}, \quad i = 1,\dots,n-1,\\ v'_{n-1} &= b_0(x)v_0^{r_0*}, \end{aligned}\right\} \tag{A 1}$$

where $b_i \in C[a,\infty)$, $b_i > 0$ and $r_i > 0$. If $r_0r_1\cdots r_{n-1} \leqslant 1$, then every solution is extendable to the whole interval $[a,\infty)$. The same holds if $v_i^{r_i*}$ is replaced by $v_i^{r_i}$.

Proof. The proof is inspired by [17, lemma 2.1]. Suppose, to the contrary, that a solution $(v_0,\dots,v_{n-1})$ can be continued only until $x=\omega$, $\omega < \infty$. Choose a point x_0, $x_0 < \omega$, such that

$$\int_{x_0}^{\omega} b_i(s)\,\mathrm{d}s < 2^{-r_i}, \quad i = 0,\dots,n-1.$$

Integration of (A 1) on $[x_0,x]$ yields

$$v_{i-1}(x) = v_{i-1}(x_0) + \int_{x_0}^{x} b_i(s)v_i^{r_i*}(s)\,\mathrm{d}s, \quad x_0 \leqslant x < \omega, \quad i = 1,\dots,n, \tag{A 2}$$

where we use the cyclic notation $v_n \equiv v_0$, $b_n \equiv b_0$.

Let $z_i(x) = \max_{[x_0,x]} |v_i(s)|$. By (A 2),

$$0 \leqslant z_{i-1}(x) \leqslant z_{i-1}(x_0) + 2^{-r_i} z_i(x)^{r_i}, \quad x_0 \leqslant x < \omega, \quad i = 1, \dots, n. \tag{A 3}$$

We prove by induction that, for every $j = 1, \dots, n-1$,

$$0 \leqslant z_0(x) \leqslant \text{const.} + (\tfrac{1}{2} z_j(x))^{r_1 \cdots r_j} \tag{A 4}$$

where 'const.' means various positive constants. For $j = 1$, this is (A 3) with $i = 1$. Suppose (A 4) is valid for a certain integer j. Then, by an additional application of (A 3) with $i = j+1$, we get

$$\begin{aligned} z_0(x) &\leqslant \text{const.} + (\tfrac{1}{2} z_j(x))^{r_1 \cdots r_j} \\ &\leqslant \text{const.} + (\tfrac{1}{2}(z_j(x_0) + 2^{-r_{j+1}} z_{j+1}(x)^{r_{j+1}}))^{r_1 \cdots r_j}. \end{aligned}$$

Since $(\frac{1}{2}(a+b))^m \leqslant a^m + b^m$ for every $a, b, m > 0$, the above inequality continues as

$$\begin{aligned} z_0(x) &\leqslant \text{const.} + z_j(x_0)^{r_1 \cdots r_j} + 2^{-r_1 \cdots r_j r_{j+1}} z_{j+1}(x)^{r_1 \cdots r_j r_{j+1}} \\ &\leqslant \text{const.} + (\tfrac{1}{2} z_{j+1}(x))^{r_1 \cdots r_{j+1}}. \end{aligned}$$

This verifies (A 4). In particular, for $j = n$, we have, with $z_n \equiv z_0$ and $r_n \equiv r_0$,

$$0 \leqslant z_0(x) \leqslant \text{const.} + (\tfrac{1}{2} z_0(x))^{r_0 r_1 \cdots r_{n-1}}.$$

As $r_0 \cdots r_{n-1} \leqslant 1$, it follows that $z_0(x)$ is bounded on $[x_0, \omega)$ and so, of course, is $v_0(x)$. Analogously, as a consequence of the cyclic behaviour, one proves that all the $z_j(x)$, and thus the $v_j(x)$, are bounded as well. From the differential equations (A 1), it follows that the $v_0', \dots, v_{n-1}'$ are bounded on $[x_0, \omega)$, and so $\lim_{x \to \omega^-} v_j(x)$ exists and is finite for each $j = 0, \dots, n-1$. But then the solution can be continued in a right-hand neighbourhood of ω, contradicting the definition of ω. Thus $\omega < \infty$ is impossible. Note that, as we took absolute values, the above holds for $v_i^{r_i *}$ and $v_i^{r_i}$. □

Theorem A.2. *Equation (A 1), with $r_i > 0$, $i = 0, \dots, n-1$, and initial value*

$$(v_0(a), \dots, v_{n-1}(a)) = (\alpha_0, \dots, \alpha_{n-1}) \neq (0, \dots, 0), \tag{A 5}$$

has at most one solution. The same holds if $v_i^{r_i *}$ is replaced by $v_i^{r_i}$.

Proof. If, for each i, either $r_i \geqslant 1$ or $\alpha_i \neq 0$, then there is nothing to prove, since the right-hand side of (A 2) is Lipschitzian and standard uniqueness theorems hold [9]. Therefore, let $r_i < 1$ and $\alpha_i = 0$ simultaneously, at least for some values of i. As the proofs for both $v_i^{r_i *}$ and $v_i^{r_i}$ are identical, we omit the $*$ everywhere.

Let us break the sequence $\alpha_0, \dots, \alpha_{n-1}$ into blocks such that

$$\left.\begin{aligned} \alpha_i = 0, &\quad k_j \leqslant i \leqslant n_j - 1, \\ \alpha_i \neq 0, &\quad n_j \leqslant i \leqslant k_{j+1} - 1, \end{aligned}\right\} \tag{A 6}$$

where $k_j < n_j < k_{j+1}$. Due to the cyclic structure of (A 1), we may renumerate the equations so that $\alpha_0 \neq 0$, $\alpha_{n-1} = 0$. Thus, without loss of generality, we may assume that $0 = n_0 < k_1 < n_1 < \dots < k_q < n_q = n$.

We start with a block $k_j \leqslant i \leqslant n_j - 1$, i.e. $\alpha_{k_j-1} \neq 0$, $\alpha_{k_j} = \cdots = \alpha_{n_j-1} = 0$, $\alpha_{n_j} \neq 0$. Near $x = a$,

$$v_{n_j-1}(x) = \int_a^x b_{n_j} v_{n_j}^{r_{n_j}} \,\mathrm{d}s = \int_a^x b_{n_j}(\alpha_{n_j} + o(1))^{r_{n_j}} \,\mathrm{d}s \approx \text{const.}(x-a),$$

where $\approx$ means 'equal up to multiplication by $1 + o(1)$' and 'const.' is some non-zero constant value. Note that all the b_i are continuous, positive and bounded near $x = a$. Now

$$\begin{aligned} v_{n_j-2}(x) &= \int_a^x b_{n_j-1} v_{n_j-1}^{r_{n_j-1}} \,\mathrm{d}s \\ &\approx \text{const.} \int_a^x b_{n_j-1}(s-a)^{r_{n_j-1}} \,\mathrm{d}s \\ &\approx \text{const.}(x-a)^{r_{n_j-1}+1}. \end{aligned}$$

We prove by decreasing induction that, for all i,

$$\left.\begin{aligned} v_i(x) &\approx \text{const.}(x-a)^{P_i}, \quad i = k_j, \dots, n_j - 1, \\ v_{k_j-1}(x) - \alpha_{k_j-1} &\approx \text{const.}(x-a)^{P_{k_j-1}}, \end{aligned}\right\} \tag{A 7}$$

near a for suitable P_i. Indeed, we have already computed $P_{n_j-1} = 1$, while

$$v_{i-1}(x) = \int_a^x b_i v_i^{r_i} \,\mathrm{d}s \approx \text{const.} \int_a^x b_i (s-a)^{P_i r_i} \,\mathrm{d}s \approx \text{const.}(x-a)^{P_i r_i + 1}$$

for $i = k_j + 1, \dots, n_j - 1$, and similarly for $v_{k_j-1} - \alpha_{k_j-1}$ $(v_{k_j-1}(a) = \alpha_{k_j-1})$. Thus

$$\begin{aligned} P_{i-1} &= P_i r_i + 1 \\ &= (P_{i+1} r_{i+1} + 1) r_i + 1 \\ &= 1 + r_i + r_i r_{i+1} + \cdots + r_i r_{i+1} \cdots r_{n_j-1}, \quad i = k_j, \dots, n_j - 1. \end{aligned} \tag{A 8}$$

Suppose that the initial-value problem has two solutions, $(u_0, \dots, u_{n-1})$ and $(v_0, \dots, v_{n-1})$. Then

$$v_{i-1}(s_{i-1}) - u_{i-1}(s_{i-1}) = \int_a^{s_{i-1}} b_i (v_i^{r_i} - u_i^{r_i}) \,\mathrm{d}s_i = \int_a^{s_{i-1}} b_i r_i \eta_i^{r_i - 1}(v_i - u_i) \,\mathrm{d}s_i,$$

where η_i is between v_i and u_i. For $i \in \{k_j, \dots, n_j - 1\}$, both v_i and u_i satisfy (A 7), and so does $\eta_i(x) \approx \text{const.}(x-a)^{P_i}$. Consequently, near a, the last integral is approximately

$$\text{const.} \int_a^{s_{i-1}} b_i (s_i - a)^{P_i(r_i - 1)} (v_i - u_i) \,\mathrm{d}s_i.$$

According to the definition of P_i,

$$P_i(r_i - 1) = P_i r_i - P_i = (P_{i-1} - 1) - P_i,$$

and since b_i is bounded,

$$|v_{i-1} - u_{i-1}|(s_{i-1}) \leqslant \text{const.} \int_a^{s_{i-1}} (s_i - a)^{P_{i-1} - P_i - 1} |v_i - u_i| \,\mathrm{d}s_i, \quad i = k_j, \dots, n_j - 1, \tag{A 9}$$

near $x = a$. Note that when $r_i < 1$, then $P_{i-1} - P_i - 1 = P_i(r_i - 1) < 0$ and the integral is singular. This is the source of the difficulty, which requires careful estimates.

The blocks of the other type with $\alpha_i \neq 0$ are relatively simple. As $v_i, u_i \approx \alpha_i \neq 0$, $\eta_i \approx \text{const.} \neq 0$, and therefore

$$|v_{i-1} - u_{i-1}|(s_{i-1}) \leqslant \text{const.} \int_a^{s_{i-1}} |v_i - u_i| \, \mathrm{d}s_i, \quad i = n_j, \ldots, k_{j+1} - 1. \tag{A 10}$$

By assumption, $\alpha_0 \neq 0$ and therefore, for $i = n$,

$$|v_{n-1} - u_{n-1}|(s_{n-1}) \leqslant \text{const.} \int_a^{s_{n-1}} |v_0 - u_0| \, \mathrm{d}s_n. \tag{A 11}$$

From the combination of the inequalities (A 9), (A 10) and (A 11) for $i = 0, \ldots, n$, we obtain

$$\begin{aligned}
&|v_0 - u_0|(x) \\
&\quad \leqslant \text{const.} \int_a^x \cdots \int_a^{s_{k_1-2}} \int_a^{s_{k_1-1}} (s_{k_1} - a)^{P_{k_1-1} - P_{k_1} - 1} \ldots \\
&\qquad \int_a^{s_{k_j-1}} (s_{k_j} - a)^{P_{k_j-1} - P_{k_j} - 1} \int_a^{s_{k_j}} \cdots \int_a^{s_{n_j-2}} (s_{n_j-1} - a)^{P_{n_j-2} - P_{n_j-1} - 1} \\
&\qquad \int_a^{s_{n_j-1}} \int_a^{s_{n_j}} \cdots \int_a^{s_{k_{j+1}-2}} \\
&\qquad \int_a^{s_{k_{j+1}-1}} (s_{k_{j+1}} - a)^{P_{k_{j+1}-1} - P_{k_{j+1}} - 1} \ldots \int_a^{s_{n-1}} |v_0 - u_0| \, \mathrm{d}s_n \cdots \mathrm{d}s_1.
\end{aligned}$$

On a small right-hand neighbourhood of a, let $z(x) = \max_{[a,x]} |v_0 - u_0|$. $z(x)$ is a positive non-decreasing function, so that

$$|v_0 - u_0|(x) \leqslant \text{const.} \int_a^x \ldots \int_a^{s_{n-1}} \mathrm{d}s_n \ldots \mathrm{d}s_1 \times z(x),$$

where the iterated integral is similar to the previous one except that the final integrand $|v_0 - u_0|$ is missing. Take a point x where $z(x) = |v_0(x) - u_0(x)|$. Such points exist arbitrarily close to a since $v_0 - u_0$ vanishes at a. If we prove that the iterated integral exists and tends to 0 as $x \to a$, then we get $0 \leqslant z(x) \leqslant o(1) z(x)$. This implies $z(x) = 0$ and thus $v_0 \equiv u_0$.

It remains to verify that the iterated integral

$$\begin{aligned}
&\int_a^x \cdots \int_a^{s_{k_j-1}} (s_{k_j} - a)^{P_{k_j-1} - P_{k_j} - 1} \int_a^{s_{k_j}} \cdots \int_a^{s_{n_j-2}} (s_{n_j-1} - a)^{P_{n_j-2} - P_{n_j-1} - 1} \\
&\int_a^{s_{n_j-1}} \int_a^{s_{n_j}} \cdots \int_a^{s_{k_{j+1}-2}} \int_a^{s_{k_{j+1}-1}} (s_{k_{j+1}} - a)^{P_{k_{j+1}-1} - P_{k_{j+1}} - 1} \ldots \int_a^{s_{n-1}} \mathrm{d}s_n \cdots \mathrm{d}s_1
\end{aligned} \tag{A 12}$$

exists and converges to 0 in spite of the fact that we may have $P_{i-1} - P_i - 1 < 0$ for some i.

Since $\alpha_{n-1} = 0$, the last block of initial values is $\alpha_i = 0$, $k_q \leqslant i \leqslant n_q - 1 = n-1$. We claim that the corresponding innermost integrals are

$$\int_a^{s_i} (s_{i+1} - a)^{P_i - P_{i+1} - 1} \cdots \int_a^{s_{n-2}} (s_{n-1} - a)^{P_{n-2} - P_{n-1} - 1} \int_a^{s_{n-1}} \mathrm{d}s_n \cdots \mathrm{d}s_{i+1} = \text{const.}(s_i - a)^{P_i}, \quad i = k_q - 1, \ldots, n_q - 1 = n-1. \tag{A 13}$$

For $i = n_q - 1 = n - 1$, this is true, as

$$\int_a^{s_{n-1}} \mathrm{d}s_n = s_{n-1} - a = (s_{n-1} - a)^{P_{n-1}},$$

since $P_{n-1} = 1$. Suppose that (A 13) is true for an integer i. For $i-1$, the corresponding integral is

$$\int_a^{s_{i-1}} (s_i - a)^{P_{i-1} - P_i - 1} \times \text{const.}(s_i - a)^{P_i}\, \mathrm{d}s_i = \text{const.}(s_{i-1} - a)^{P_{i-1}},$$

and (A 13) is verified for $i = k_q - 1, \ldots, n_q - 1 = n - 1$.

In the following block, $\alpha_i \neq 0$, $i = n_{q-1}, \ldots, k_q - 1$, so the calculation is immediate,

$$\int_a^{s_i} \int_a^{s_{i+1}} \cdots \int_a^{s_{k_q - 2}} (s_{k_q - 1} - a)^{P_{k_q - 1}}\, \mathrm{d}s_{k_q - 1} \cdots \mathrm{d}s_{i+1} = \text{const.}(s_i - a)^{k_q - i - 1 + P_{k_q - 1}}$$

for $i = n_{q-1} - 1, \ldots, k_q - 2$. Advancing from block to block, we finally get that the integral (A 12) is

$$\text{const.}(x - a)^{(k_1 - 1) + P_{k_1 - 1} + (k_2 - n_1 - 1) + P_{k_2 - 1} + (k_3 - n_2 - 1) + \cdots + P_{k_q - 1}},$$

where the exponent is positive. This completes the proof that $v_0 \equiv u_0$. By (A 1), it now follows that $v_i \equiv u_i$ for all $i = 0, \ldots, n-1$. □

In the proof of proposition 3.1, we need the existence of the derivatives

$$\frac{\partial v_j}{\partial \alpha_\ell} = \frac{\partial N_j[u]}{\partial \alpha_\ell}$$

precisely for those components for which $\alpha_\ell \neq 0$ in the initial value (3.1). The following theorem proves that in a neighbourhood of an eigenfunction of (1.1), (1.6), the solution of (1.10) with initial values (3.1) is differentiable with respect to the non-vanishing initial values.

THEOREM A.3. *Let $v_i(x, \alpha_0, \ldots, \alpha_{n-1})$, $i = 0, \ldots, n-1$, denote the solution of the initial-value problem (1.10), (3.1) and let $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{n-1}$ be such that the solution of the system (1.10), with initial values*

$$(v_0(a), \ldots, v_{n-1}(a)) = (N_0[u](a), \ldots, N_{n-1}[u](a)) = (\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{n-1}),$$

corresponds to an eigenfunction of (1.1), (1.8). If assumption 1.2 about the powers p_i *is satisfied and* $\tilde{\alpha}_\ell \neq 0$*, then the derivatives*

$$\frac{\partial v_0}{\partial \alpha_\ell}, \ldots, \frac{\partial v_{n-1}}{\partial \alpha_\ell}$$

exist at $\alpha_\ell = \tilde{\alpha}_\ell$ *and are continuous on the interval* $a \leqslant x \leqslant b$*. The same holds for the derivatives* $\partial v_i / \partial \lambda$*.*

Proof. Consider an initial-value problem

$$\boldsymbol{v}' = \boldsymbol{F}(x, \boldsymbol{v}), \qquad \boldsymbol{v}(a) = \boldsymbol{\alpha},$$

where $\boldsymbol{v} = (v_0, \ldots, v_{n-1})$, $\boldsymbol{F} = (F_0, \ldots, F_{n-1})$ and $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1})$. If the $\partial F_i / \partial v_j$ are continuous, then, by Peano's theorem, $\boldsymbol{z} = \partial \boldsymbol{v} / \partial \alpha_\ell$ is the solution of the variational system

$$\boldsymbol{z}' = \left(\frac{\partial F_i}{\partial v_j} \right) \boldsymbol{z}, \qquad z_k(a) = \delta_{k,\ell}$$

(see [9]). Unfortunately, our system (1.10) does not satisfy Peano's assumption, since the coefficients of the corresponding linearized system (3.5),

$$\begin{aligned} z'_{i-1} &= (a_i^{-1/p_i} p_i^{-1} |v_i|^{1/p_i - 1}) z_i, \quad i = 1, \ldots, n-1, \\ z'_{n-1} &= \lambda b \left(a_0^{-r/p_0} \frac{r}{p_0} |v_0|^{r/p_0 - 1} \right) z_0, \end{aligned}$$

are discontinuous at points where v_i vanishes if $p_i > 1$. Hence Peano's theorem is not directly applicable at some points. Such singularities certainly occur at the endpoints a, b because of the boundary values (1.8) and possibly at other points of (a, b) where some v_i vanish. At all other points, the usual proof of Peano's theorem holds.

We start our discussion near $x = a$. The role of the point $x = a$ is different to that of the other points, since the initial values are specified at a. It is clear by the initial values that at a,

$$\frac{\partial}{\partial \alpha_\ell} (v_k(a)) = \delta_{k,\ell}.$$

However, this is of no help, since we need the existence and continuity of $\partial \boldsymbol{v} / \partial \alpha_\ell$. It was shown earlier that the zeros of $v_i = N_i[u]$ do not accumulate. Therefore, $v_i \neq 0$ for all i on some $(a, a + \varepsilon)$ and $\partial \boldsymbol{v} / \partial \alpha_\ell$ exists there. We shall prove its continuity as $x \to a$ by direct calculation.

Let

$$\begin{aligned} \boldsymbol{v}(x, \alpha_\ell) &= \boldsymbol{v}(x, \alpha_0, \ldots, \alpha_\ell, \ldots, \alpha_{n-1}), \\ \boldsymbol{v}(x, \alpha_\ell + h) &= \boldsymbol{v}(x, \alpha_0, \ldots, \alpha_\ell + h, \ldots, \alpha_{n-1}) \end{aligned}$$

be two solutions of (1.10), (3.1) that correspond to two different initial values (with further reference to initial values α_j, $j \neq \ell$, omitted). In (3.3), we agreed to name the non-zero initial values, for simplicity, $\alpha_k, \ldots, \alpha_{n-2}$. Here, however, we need their exact identity, so we split the initial value into blocks of consecutive zero and non-zero components, as in (A 6). Since we are interested in $\alpha_\ell = \tilde{\alpha}_\ell \neq 0$, we necessarily have $n_r \leqslant \ell \leqslant k_{r+1} - 1$ for some r.

Our calculations are analogous to those in the proof of theorem A.2, with the powers r_i of (A 1) replaced by the corresponding powers in (1.10), i.e. $r_i = 1/p_i$, $i = 1, \dots, n-1$, and $r_0 = r/p_0$, so that $r_0 \cdots r_{n-1} = 1$. Let us start with $i - 1 \neq \ell$. First note that $v_j(a, \alpha_\ell) = v_j(a, \alpha_\ell + h)$ for all $j \neq \ell$. For $k_j \leqslant i \leqslant n_j - 1$, $i - 1 \neq \ell$, due to $v_{i-1}(a, \alpha_\ell + h) - v_{i-1}(a, \alpha_\ell) = 0$, we obtain, as in (A 9),

$$\begin{aligned}\frac{v_{i-1}(s_{i-1}, \alpha_\ell + h) - v_{i-1}(s_{i-1}, \alpha_\ell)}{h}&\\ \approx \text{const.} \int_a^{s_{i-1}} (s_i - a)^{P_{i-1} - P_i - 1} &\frac{v_i(s_i, \alpha_\ell + h) - v_i(s_i, \alpha_\ell)}{h} \, \mathrm{d}s_i\end{aligned}$$

near $x = a$. If $n_j \leqslant i \leqslant k_{j+1} - 1$, $i - 1 \neq \ell$, then again $v_{i-1}(a, \alpha_\ell + h) - v_{i-1}(a, \alpha_\ell) = 0$, and in this case we have, analogously with (A 10), that

$$\frac{v_{i-1}(s_{i-1}, \alpha_\ell + h) - v_{i-1}(s_{i-1}, \alpha_\ell)}{h} \approx \text{const.} \int_a^{s_{i-1}} \frac{v_i(s_i, \alpha_\ell + h) - v_i(s_i, \alpha_\ell)}{h} \, \mathrm{d}s_i.$$

Now we turn to $i - 1 = \ell$. Recall that the present theorem is only about the case $\alpha_\ell = \tilde{\alpha}_\ell \neq 0$, so we consider only $n_r \leqslant \ell \leqslant k_{r+1} - 1$. If $n_r \leqslant \ell < \ell + 1 \leqslant k_{r+1} - 1$, then, as $v_\ell(a, \alpha_\ell + h) - v_\ell(a, \alpha_\ell) = h$, the above equation is modified as

$$\begin{aligned}\frac{v_\ell(s_\ell, \alpha_\ell + h) - v_\ell(s_\ell, \alpha_\ell)}{h} - 1&\\ \approx \text{const.} \int_a^{s_\ell} &\frac{v_{\ell+1}(s_{\ell+1}, \alpha_\ell + h) - v_{\ell+1}(s_{\ell+1}, \alpha_\ell)}{h} \, \mathrm{d}s_{\ell+1},\end{aligned}$$

while if $\ell + 1 = k_{r+1}$ (next block!), then

$$\begin{aligned}\frac{v_\ell(s_\ell, \alpha_\ell + h) - v_\ell(s_\ell, \alpha_\ell)}{h} - 1&\\ \approx \text{const.} \int_a^{s_\ell} &(s_{\ell+1} - a)^{P_{k_{r+1}-1} - P_{k_{r+1}} - 1}\\ &\times \frac{v_{\ell+1}(s_{\ell+1}, \alpha_\ell + h) - v_{\ell+1}(s_{\ell+1}, \alpha_\ell)}{h} \, \mathrm{d}s_{\ell+1}.\end{aligned}$$

If $\ell \neq 0$, then, combining the last integrals together, we have, as in theorem A.2,

$$\begin{aligned}&\frac{v_0(x, \alpha_\ell + h) - v_0(x, \alpha_\ell)}{h}\\ &\quad\approx \text{const.} \int_a^x \cdots \int_a^{s_{\ell-1}} \biggl(1 + \text{const.}\\ &\qquad\times \int_a^{s_l} \cdots \frac{v_0(s_n, \alpha_\ell + h) - v_0(s_n, \alpha_\ell)}{h} \, \mathrm{d}s_n \cdots \mathrm{d}s_{\ell-1}\biggr) \, \mathrm{d}s_\ell \cdots \mathrm{d}s_1.\end{aligned}$$

The inner structure of the iterated integral is as in theorem A.2, according to the blocks defined above and is omitted here. It is important only that the index ℓ

belongs to a block of type $n_r \leqslant \ell \leqslant k_{r+1}-1$. Therefore, the first integral is

$$\int_a^x \cdots \int_a^{s_{k_1-1}} (s_{k_1}-a)^{P_{k_1-1}-P_{k_1}-1} \cdots \int_a^{s_{n_r-1}} \cdots \int_a^{s_{\ell-1}} \mathrm{d}s_\ell \cdots \mathrm{d}s_1 \\ = \text{const.}(x-a)^{k_1-1+P_{k_1-1}+(k_2-n_1-1)+\cdots+P_{k_{r-1}-1}+(\ell-n_r-1)}$$

and

$$\left|\frac{v_0(x,\alpha_\ell+h)-v_0(x,\alpha_\ell)}{h}\right| \\ \leqslant \text{const.}(x-a)^{k_1-1+P_{k_1-1}+(k_2-n_1-1)+\cdots+P_{k_{r-1}-1}+(\ell-n_r-1)} \\ + \text{const.} \int_a^x \cdots \int_a^{s_{n-1}} \left|\frac{v_0(s_n,\alpha_\ell+h)-v_0(s_n,\alpha_\ell)}{h}\right| \mathrm{d}s_n \cdots \mathrm{d}s_1.$$

The innermost integral $\int^{s_{n-1}} |\cdots|$ is naturally increasing, so we replace its upper limit by $x \geqslant s_{n-1}$. There remains

$$\left|\frac{v_0(x,\alpha_\ell+h)-v_0(x,\alpha_\ell)}{h}\right| \\ \leqslant \text{const.}(x-a)^{k_1-1+P_{k_1-1}+(k_2-n_1-1)+\cdots+P_{k_{r-1}-1}+(\ell-n_r-1)} \\ + \text{const.}(x-a)^{k_1-1+P_{k_1-1}+\cdots+P_{k_q-1}-1} \int_a^x \left|\frac{v_0(s,\alpha_\ell+h)-v_0(s,\alpha_\ell)}{h}\right| \mathrm{d}s.$$

By Gronwall's inequality,

$$\left|\frac{v_0(x,\alpha_\ell+h)-v_0(x,\alpha_\ell)}{h}\right| \\ \leqslant \text{const.}(x-a)^{k_1-1+P_{k_1}+(k_2-n_1-1)+\cdots+P_{k_{r-1}}+(\ell-n_r-1)} o(1),$$

with the details of the last term unimportant. Note that all 'const.' terms above are independent of h.

We have already mentioned that $\partial v_0/\partial\alpha_\ell$ exists on $(a, a+\varepsilon)$. By the last estimates, we conclude that $\partial v_0(x)/\partial\alpha_\ell \to 0$ as $x \downarrow a$. A similar argument holds for the other $\partial v_i(x)/\partial\alpha_\ell$ and for $i=\ell$, $\partial v_\ell(x)/\partial\alpha_\ell \to 1$. This completes the continuity of $\partial \boldsymbol{v}/\partial\alpha_\ell$ near $x=a$.

At an internal point $c \in (a,b)$, the situation is simpler, since we know that v_i may have only simple zeros. If $p_i \leqslant 1$, then the term $|v_i|^{1/p_i-1}$ in (3.5) is regular and causes no difficulty. On the other hand, if $p_i > 1$, then, near $x=c$,

$$|v_i|^{1/p_i-1} \approx |x-c|^r, \quad \text{with } r > -1,$$

and the system (3.5) is of the form

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}x} = |x-c|^r A(x)\boldsymbol{z}, \tag{A 14}$$

with a continuous $A(x)$. A change of variable $s=(x-c)^{r+1*}$ transforms it into

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}s} = \frac{1}{r+1}\tilde{A}(s)\boldsymbol{z}(s),$$

with a differentiable solution $\boldsymbol{z}(s)$ at $s = 0$. Consequently, $\boldsymbol{z}(x) = \boldsymbol{z}(c + s^{1/(r+1)})$ is continuous (but with unbounded derivative!) at $x = c$ in spite of the mild singularity.

Finally, we come to the endpoint $x = b$. By the boundary values (1.8), we know that $n - k$ of the v_i vanish at $x = b$, so system (3.5) is singular there. Let

$$\begin{aligned} N_j[u](b) &= 0, \quad j = \mu, \mu + 1, \dots \mu + \ell, \\ N_{\mu+\ell+1}[u](b) &\neq 0 \end{aligned}$$

be a block of consecutive vanishing v_j. Then, as in (A 7) and (A 8), we obtain that, for all i, $\mu \leqslant i \leqslant \mu + \ell$,

$$v_i(x) \approx \text{const.}(x - b)^{P_i},$$

with

$$P_i = 1 + \frac{1}{p_{i+1}} + \frac{1}{p_{i+1}p_{i+2}} + \cdots + \frac{1}{p_{i+1}p_{i+2}\cdots p_{i+\ell}}.$$

Here, we have replaced r_i by $1/p_i$, $i = 1, \dots, n - 1$, and r_0 by r/p_0. However, as remarked after the statement of assumption 1.2, we will, for convenience, write $1/p_0$ in place of r/p_0. Thus, in (3.5),

$$v_i^{1/p_i - 1} \approx (x - b)^{P_i(1/p_i - 1)}$$

near b. If $p_i \leqslant 1$ for all i, $\mu \leqslant i \leqslant \mu + \ell$, this expression is continuous and the corresponding equation in (3.5) presents no difficulty. Negative powers of $(x - b)$ and a singularity occur in (3.5) when $p_i > 1$. As in the previous discussion, the solution of (3.5) will nevertheless be continuous at $x = b$ if $P_i(1/p_i - 1) > -1$ whenever $v_i(b) = 0$. Now

$$\begin{aligned} P_i\left(\frac{1}{p_i} - 1\right) &= \left(1 + \frac{1}{p_{i+1}} + \frac{1}{p_{i+1}p_{i+2}} + \cdots + \frac{1}{p_{i+1}p_{i+2}\cdots p_{i+\ell}}\right)\left(\frac{1}{p_i} - 1\right) \\ &= -\left(1 + \frac{1}{p_{i+1}} + \frac{1}{p_{i+1}p_{i+2}} + \cdots + \frac{1}{p_{i+1}p_{i+2}\cdots p_{i+\ell}}\right) \\ &\quad + \left(\frac{1}{p_i} + \frac{1}{p_ip_{i+1}} + \cdots + \frac{1}{p_ip_{i+1}\cdots p_{i+\ell-1}} + \frac{1}{p_ip_{i+1}\cdots p_{i+\ell}}\right). \end{aligned}$$

One possible way to ensure that this is bigger than -1 is to assume that

$$\frac{1}{p_i} \geqslant \frac{1}{p_{i+1}} \geqslant \frac{1}{p_{i+2}} \geqslant \cdots \geqslant \frac{1}{p_{i+\ell-1}} \geqslant \frac{1}{p_{i+\ell}}$$

for all i, $\mu \leqslant i \leqslant \mu + \ell$. Then the expression above is not less than

$$-1 + \frac{1}{p_ip_{i+1}\cdots p_{i+\ell}} > -1.$$

The lack of symmetry between the endpoints a and b is only an artefact of our choice of initial value. We may interchange the roles of a and b and make analogous assumptions for blocks of v_i vanishing at $x = a$. This is precisely our assumption 1.2.

The proof for $\partial v_i/\partial\lambda$ is similar. □

Acknowledgments

This research was supported by the Fund for the Promotion of Research at the Technion.

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(*Issued 13 December 2002*)