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Rearrangement of a Conditionally Convergent Series

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One of the surprising results in an elementary calculus course is that a rearrangement of a conditionaly convergent series may change its sum, even its very convergence. Observing typical textbook examples of this phenomenon, it turns out that during the rearrangement some of the terms are moved arbitrarily large distances from their original locations. Is this necessary? The answer is positive. Indeed, we can assert:

Let $\sum a_n$ be a convergent series. Suppose that, in a rearrangement of the series, there is a fixed positive integer p such that each term of the series that is shifted forward is shifted at most p places. Then the rearranged series converges to the same sum as the original one.

Note that there is a clear difference between forward shifts and backward shifts.

The proof is straightforward. Let $S_n = a_1 + \cdots + a_n$ and $T_n = a_{\pi(1)} + \cdots + a_{\pi(n)}$ be the partial sums of the original and rearranged series, respectively, where π denotes the corresponding permutation of the positive integers. Then T_n consists of the terms of S_n, with the possible exception of $2p$ terms: some of the last p terms of S_n could be moved forward by π and excluded from T_n, in which event they would be replaced by at most p terms $a_{n_1}, \ldots, a_{n_p}$, with $n < n_1 < \cdots < n_p$, that are moved backward, possibly from arbitrarily large distances. Thus,

$$|T_n - S_n| \le |a_{n-p+1}| + \cdots + |a_n| + |a_{n_1}| + \cdots + |a_{n_p}|.$$

Due to the convergence of $\sum a_n$, for each $\varepsilon > 0$ it is the case that $|a_k| < \varepsilon/2p$ once $k > N(\varepsilon)$. Consequently, $|T_n - S_n| < \varepsilon$ whenever $n > N(\varepsilon) + p$. ∎

This proposition intends to give the students the feeling that conditionaly convergent series, as well as math professors, may tolerate perturbations, as long as they remain bounded. However, both series and professors may undergo surprising metamorphoses when the perturbations go on and on.

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