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A framework for asymptotic integration of differential systems

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Abstract. A new method for asymptotic integration of linear systems of ordinary differential equations is proposed and studied. It is based on the introduction of a certain integral equation that pinpoints sufficient conditions for asymptotic integration. These conditions serve as a framework from which new and old theorems are derived. In particular the fundamental theorems of Levinson and Hartman–Wintner are shown to follow from one and the same scheme. The new theorems in asymptotic integration are shown to be best possible in a certain sense. Examples are given that are not amenable to other techniques.

1. Introduction

The asymptotic theory of n dimensional linear differential systems

$$Y' = A(t)Y,$$

where A, Y are $n \times n$ matrix functions asks for a representation of fundamental solutions $Y(t)$ of (1.1) in the vicinity of $t = \infty$. Its importance can hardly be overestimated for more reasons than one. Firstly for its own sake. Secondly for the reason that the asymptotic behaviour of solutions of nonlinear problems require quite often asymptotic integration of a linearized problem. A comprehensive account of this "nonanalytic theory" is given in the textbook [3]. Since the appearance of [3] many new results were published. See, e.g., [1,2,14,15,17].

We assume that the differential equation $Y' = A(t)Y$ is given in the form

$$Y' = \big(D(t) + R(t)\big)Y, \tag{1.1}$$

$$D(t) = \operatorname{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}, \qquad R(t) = \big(r_{jk}(t)\big)_{j,k=1}^{n}, \tag{1.2}$$

which is widely discussed in the literature. It is also observed that the diagonal elements and the off-diagonal elements of $R(t)$ play different roles. Hence it makes sense to place all diagonal elements of Eq. (1.1) in $D(t)$ while the perturbation term $R(t)$ consists only of off-diagonal terms. This convention will be assumed throughout the rest of our work.

Since a fundamental solution of the unperturbed equation $Y' = DY$ is

$$\Phi(t) = \exp\left(\int_{t_0}^{t} D(s)\,\mathrm{d}s\right), \tag{1.3}$$

one may hope that an asymptotic representation of a fundamental solution of (1.3) be given by

$$Y(t) = \bigl(I + Q(t)\bigr)\exp\left(\int_{t_0}^{t} D(s)\,\mathrm{d}s\right) \tag{1.4}$$

with $Q(t) \to 0$ as $t \to \infty$. Another option is to look for a solution $Y(t)$ of (1.1) that is represented as $Y = \Phi(I + P)$, i.e.,

$$Y(t) = \exp\left(\int_{t_0}^{t} D(s)\,\mathrm{d}s\right)\bigl(I + P(t)\bigr) \tag{1.5}$$

with another unknown perturbation P such that $P(t) \to 0$ as $t \to \infty$. The distinction between (1.4) and (1.5) makes it worthwhile to repeat the definition of [6] of an almost diagonal system.

Definition. Let $D(t) \in C[a,\infty)$ be a diagonal matrix and let $R(t) \in C[a,\infty)$ be such that its diagonal elements are all zero. We say that the system (1.1) is "right almost diagonal" if it possesses an asymptotic representation (1.5) with $P(t) \in C[a,\infty]$ and $\lim P(t) = 0$ as $t \to \infty$. Similarly, if representation (1.4) holds, the system (1.1) will be called "left almost diagonal".

The integration of (1.1) is strongly related to the Levinson dichotomy conditions: *for each pair of integers $\alpha \neq \beta$ and for all τ and t such that $a \leqslant \tau < t < \infty$, either*

$$\int_a^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s \to -\infty \quad \textit{as } t \to \infty \quad \textit{and} \quad \int_\tau^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s \leqslant K_1 \tag{Dic1}$$

or

$$\int_\tau^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s \geqslant K_2, \tag{Dic2}$$

where K_1 and K_2 are some constants.

Note that the dichotomy conditions may be written in other forms. For example, [3] uses another equivalent formulation.

Levinson utilized an extra similarity transformation which rediagonalizes $D + R$. This was further enhanced in [11,12] in which it was shown how to transform (1.1) via repeated diagonalizations

$$Y_0 \equiv Y, \qquad Y_{j-1} = (I + Q_j)Y_j, \quad j = 1, \ldots, N, \tag{1.6}$$

so that $Q_j(\infty) = 0$ into a system

$$Y_N' = \bigl(D_N(t) + R_N(t)\bigr)Y_N \tag{1.7}$$

and so that the final formula of asymptotic integration is given by

$$Y(t) = \prod_{j=1}^{N} (I + Q_j(t)) \exp\left(\int^t D_N(s)\,\mathrm{d}s\right). \tag{1.8}$$

This ultimately derived differential equation (1.7) for Y_N is shown to satisfy the conditions of Levinson's theorem, thus becoming asymptotically integrable. However, $D_N(t)$ does not coincide necessarily with $D(t)$ which consists of the eigenvalues of the unperturbed system.

Our approach differs from the established trend that is manifested in [3] in three essential features.

(i) The first is that we strive to show that (1.1) is almost diagonal without having to resort to additional transformations of the form (1.6).

(ii) Secondly, the result of our asymptotic integration is formula (1.4) rather than (1.8). While (1.4) and (1.8) seem similar, there is an essential difference between them. In (1.4) there appear the eigenvalues of the original diagonal matrix D and so they preserve the original physical meaning, which is in contrast to (1.8). (In the setting of quantum mechanics, the "physical meaning" of the eigenvalues of a system, given by the elements of D, are of great importance since they are proportional to the energy levels of a quantum mechanical system.) Another drawback of (1.8) is that the calculation of the eigenvalues of D_N may be a laborious task. The same difficulty is inherent in the Levinson's theorem, where the eigenvalues of $D + R$ need to be calculated.

(iii) The third feature is that we do not utilize Levinson's theorem as our main weapon. Indeed, we derive Levinson's theorem as a byproduct of our method and consequently reconfirm that the off diagonal elements in (1.1) need not be absolutely integrable in order to affect asymptotic integration. Some recent reconfirmation of this is given by [1,14,15,5,17].

We benefitted in this work from our study [5] where $|\int_\tau^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s| \leqslant B$ for all $\tau,\ t,\ \alpha,\ \beta$. However we could not use in here the precise scheme of asymptotic integration in [5] as the kernel of a certain integral equation utilized in the process would have come out to be unbounded. Consequently we had to introduce substantial modifications to the technique in [5] and produce an asymptotic formula of the form (1.4) rather than (1.5).

The order of events in this paper is as follows. In Section 2 we derive an integral equation for the perturbation matrix Q. In Section 3 we formulate Theorem 1 that is the point of departure from which new and old theorems spring. We also show in Section 3 without any extra linear similarity transformations how Levinson's theorem as well as the Hartman–Wintner theorem follow from one and same formulas albeit by using different estimates of the same integrals. In Section 4 we apply integrations by parts that bring out the important fact that quotients of the type $r_{\alpha\beta}/(\lambda_\alpha - \lambda_\beta)$ play an important role in asymptotic integration. Section 5 is devoted to examples and comparisons.

2. Some formal calculations

Let Φ be the solution (1.3) of the unperturbed equation $Y' = DY$, i.e., $\Phi' = D\Phi$. With this Φ we put $Y = \Phi(I + P)$ into (1.1) to get

$$D\Phi(I + P) + \Phi(I + P)' = (D + R)\Phi(I + P),$$

i.e.,

$$P' = \Phi^{-1} R\Phi(I + P). \tag{2.1}$$

Let

$$K(t) = \Phi^{-1} R\Phi = \exp\biggl(-\int_{t_0}^{t} D(s)\,\mathrm{d}s\biggr) R(t) \exp\biggl(\int_{t_0}^{t} D(s)\,\mathrm{d}s\biggr). \tag{2.2}$$

Here $R(t) = (r_{jk}(t))_{j,k=1}^{n}$, $K(t) = (r_{jk}(t) \exp \int_{t_0}^{t} -(\lambda_j - \lambda_k)\,\mathrm{d}s)_{j,k=1}^{n}$. From now on $\int_{t_1}^{t} (\lambda_\alpha(s) - \lambda_\beta(s))\,\mathrm{d}s$ will be written in an abbreviated form $\int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s$.

By the notation (2.2), Eq. (2.1) becomes

$$P' = K + KP, \tag{2.3}$$

or, componentwise,

$$p'_{jk}(t) = r_{jk}(t)\,\mathrm{e}^{-\int_{t_0}^{t} (\lambda_j - \lambda_k)\,\mathrm{d}s} + \sum_{h=1}^{n} r_{jh}(t)\,\mathrm{e}^{-\int_{t_0}^{t} (\lambda_j - \lambda_h)\,\mathrm{d}s} p_{hk}(t), \quad j, k = 1, \ldots, n. \tag{2.4}$$

Instead of this differential equation, we shall consider an integrated version. The limits of integration of individual terms may be different, so at present we denote them formally as ℓ_{jk}. The exact value of each ℓ_{jk} will be determined in the sequel. Accordingly,

$$p_{jk}(t) = \int_{\ell_{jk}}^{t} r_{jk}(t_1)\,\mathrm{e}^{-\int_{t_0}^{t_1} (\lambda_j - \lambda_k)\,\mathrm{d}s}\,\mathrm{d}t_1 + \sum_{h=1}^{n} \int_{\ell_{jh}}^{t} r_{jh}(t_1)\,\mathrm{e}^{-\int_{t_0}^{t_1} (\lambda_j - \lambda_h)\,\mathrm{d}s} p_{hk}(t_1)\,\mathrm{d}t_1. \tag{2.5}$$

The integrated version obviously implies the original differential equation. Formally (2.5) will be written as

$$P(t) = \int_{(L)}^{t} K(t_1)\,\mathrm{d}t_1 + \int_{(L)}^{t} K(t_1)P(t_1)\,\mathrm{d}t_1, \tag{2.6}$$

where (L) denotes the matrix of lower limits (ℓ_{jk}). Next we integrate again by parts the last term of (2.6):

$$\begin{aligned} P(t) &= \int_{(L)}^{t} K(t_1)\,\mathrm{d}t_1 + \int_{(L)}^{t} K(t_1)P(t_1)\,\mathrm{d}t_1 \\ &= \int_{(L)}^{t} K(t_1)\,\mathrm{d}t_1 + \biggl[\int_{(L)}^{t_1} K(t_2)\,\mathrm{d}t_2\biggr] P(t_1)\bigg|_{t_1=(L)}^{t} - \int_{(L)}^{t} \biggl[\int_{(L)}^{t_2} K(t_1)\,\mathrm{d}t_1\biggr] P'(t_2)\,\mathrm{d}t_2 \\ &= \int_{(L)}^{t} K(t_1)\,\mathrm{d}t_1 + \biggl[\int_{(L)}^{t} K(t_2)\,\mathrm{d}t_2\biggr] P(t) - \int_{(L)}^{t} \biggl[\int_{(L)}^{t_1} K(t_2)\,\mathrm{d}t_2\biggr] \bigl[K(t_1) + K(t_1)P(t_1)\bigr]\,\mathrm{d}t_1. \end{aligned} \tag{2.7}$$

Denote

$$M_1(t) = \int_{(L)}^{t} K(t_1)\,\mathrm{d}t_1,$$
$$M_2(t) = \int_{(L)}^{t} M_1(t_1)K(t_1)\,\mathrm{d}t_1 = \int_{(L)}^{t}\left[\int_{(L)}^{t_1} K(t_2)\,\mathrm{d}t_2\right]K(t_1)\,\mathrm{d}t_1.$$

With this notation (2.6) becomes $P(t) = M_1(t) + \int_{(L)}^{t} K(t_1)P(t_1)\,\mathrm{d}t_1$ while (2.7) may be rewritten as

$$\bigl(I - M_1(t)\bigr)P(t) = M_1(t) - M_2(t) + \int_{(L)}^{t} M_1(t_1)K(t_1)P(t_1)\,\mathrm{d}t_1. \tag{2.8}$$

So far we followed the scheme in [5] which was specialized to differential systems where

$$\left|\int_{\tau}^{t} \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s\right| \leqslant B, \quad \text{for all } \tau, t,\ \alpha, \beta = 1, \ldots, n. \tag{2.9}$$

However when (2.9) is violated, the kernel of the integral equation (2.5) cannot be made bounded. Consequently we introduce a matrix $Q(t) = (q_{ij}(t))$ related to $P(t)$ via

$$\Phi(I + P) = (I + Q)\Phi \tag{2.10}$$

and proceed to find an asymptotic representation of the form $Y = (I + Q)\Phi$ (i.e., (1.4)) rather than $Y = \Phi(I + P)$ (that is (1.5)). The relation (2.10) is equivalent to

$$P(t) = \Phi^{-1}(t)Q(t)\Phi(t). \tag{2.11}$$

It will turn out in the sequel that the resulting integral equation for Q has a bounded kernel under fairly broad conditions when the lower limits $\ell_{\alpha\beta}$ are properly chosen. It is noteworthy that by (2.11) the elements $p_{ij}(t) = q_{ij}(t)\exp\int_{t_0}^{t}(\lambda_i - \lambda_j)\,\mathrm{d}s$ need not be bounded although $q_{ij}(t) \to 0$ as $t \to \infty$.

One may wonder if there should be any advantage to the representation (1.4) over (1.5) and vice versa. In addition to the advantages of the representation (1.5) in wave propagation problems like quantum mechanics and acoustics [8,9], the representation (1.5) has definite merits when tackling the "connection problem". See [7].

After the substitution of (2.11) into Eq. (2.8) it becomes

$$\bigl(I - M_1(t)\bigr)\Phi^{-1}(t)Q(t)\Phi(t) = M_1(t) - M_2(t) + \int_{(L)}^{t} M_1(t_1)K(t_1)\Phi^{-1}(t_1)Q(t_1)\Phi(t_1)\,\mathrm{d}t_1.$$

Multiplication by $\Phi(t)$ from the left-hand side and by $\Phi^{-1}(t)$ from the right-hand side leads to

$$\begin{aligned}\bigl(I - \Phi(t)M_1(t)\Phi^{-1}(t)\bigr)Q(t) &= \Phi(t)M_1(t)\Phi^{-1}(t) - \Phi(t)M_2(t)\Phi^{-1}(t)\\ &\quad + \int_{(L)}^{t} \Phi(t)\bigl[M_1(t_1)K(t_1)\Phi^{-1}(t_1)Q(t_1)\Phi(t_1)\bigr]\Phi^{-1}(t)\,\mathrm{d}t_1.\end{aligned} \tag{2.12}$$

This is our basic integral equation for the unknown $Q(t)$. Note that a direct substitution of $Y = (I+Q)\Phi$ into the differential equation (1.1) leads to a differential equation

$$Q' = (D+R)Q - QD + R,$$

which is far less convenient than the differential equation (2.3). Consequently, the derivation of the integral equation (2.12) for Q is simpler. A different point of departure is utilized in [19, p. 165].

Consider the term $\Phi(t)M_1(t)\Phi^{-1}(t)$ which appears on both sides of (2.12).

$$\begin{aligned}\widehat{M_1}(t) &\equiv \Phi(t)M_1(t)\Phi^{-1}(t) = \Phi(t)\int_{(L)}^{t} K(t_1)\,\mathrm{d}t_1\,\Phi^{-1}(t)\\ &= \Phi(t)\int_{(L)}^{t}\Phi^{-1}(t_1)R(t_1)\Phi(t_1)\,\mathrm{d}t_1\Phi^{-1}(t) = \int_{(L)}^{t}\Phi(t)\Phi^{-1}(t_1)R(t_1)\Phi(t_1)\Phi^{-1}(t)\,\mathrm{d}t_1.\end{aligned}$$

A typical entry of $\widehat{M_1}$, say entry (α,β), is

$$\int_{\ell_{\alpha\beta}}^{t} r_{\alpha\beta}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1. \tag{2.13}$$

Next we turn to the integral term in (2.12),

$$\int_{(L)}^{t}\Phi(t)M_1(t_1)K(t_1)\Phi^{-1}(t_1)Q(t_1)\Phi(t_1)\Phi^{-1}(t)\,\mathrm{d}t_1. \tag{2.14}$$

By substitution of the proper expressions for $M_1(t_1)$ and $K(t_1)$ and reorganizing the functions Φ, Φ^{-1}, it becomes

$$\begin{aligned}&\int_{(L)}^{t}\Phi(t)\left[\int_{(L)}^{t_1}\Phi^{-1}(t_2)R(t_2)\Phi(t_2)\,\mathrm{d}t_2\right]\left[\Phi^{-1}(t_1)R(t_1)\Phi(t_1)\right]\left[\Phi^{-1}(t_1)Q(t_1)\Phi(t_1)\right]\Phi^{-1}(t)\,\mathrm{d}t_1\\ &\quad=\int_{(L)}^{t}\Phi(t)\Phi^{-1}(t_1)\left[\int_{(L)}^{t_1}\Phi(t_1)\Phi^{-1}(t_2)R(t_2)\Phi(t_2)\Phi^{-1}(t_1)\,\mathrm{d}t_2\right]R(t_1)Q(t_1)\Phi(t_1)\Phi^{-1}(t)\,\mathrm{d}t_1.\end{aligned} \tag{2.15}$$

The internal integral is precisely $\widehat{M_1}(t_1)$ whose elements had been calculated in (2.13). Consequently the (α,γ) term of (2.15) is

$$\int_{\ell_{\alpha\gamma}}^{t}\left(\sum_{\beta=1}^{n}\left[\int_{\ell_{\alpha\beta}}^{t_1} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right]\left[R(t_1)Q(t_1)\right]_{\beta\gamma}\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right)\mathrm{d}t_1. \tag{2.16}$$

A typical term (β,γ) of $R(t_1)Q(t_1)$ is $\sum_{\nu=1}^{n} r_{\beta\nu}(t_1)q_{\nu\gamma}(t_1)$. When we substitute it in (2.16), we obtain

$$\int_{\ell_{\alpha\gamma}}^{t}\sum_{\nu=1}^{n}\sum_{\beta=1}^{n}\left(\left[\int_{\ell_{\alpha\beta}}^{t_1} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right)q_{\nu\gamma}(t_1)\,\mathrm{d}t_1. \tag{2.17}$$

Consider an individual term in the double sum (2.17) that contains the sole element $q_{\nu\gamma}(t_1)$ of the matrix $Q(t_1)$. It is given by

$$\int_{\ell_{\alpha\gamma}}^{t} \sum_{\beta=1}^{n} \left(\left[\int_{\ell_{\alpha\beta}}^{t_1} r_{\alpha\beta}(t_2)\, e^{\int_{t_2}^{t_1} (\lambda_\alpha - \lambda_\beta)\,ds}\, dt_2 \right] r_{\beta\nu}(t_1)\, e^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\gamma)\,ds} \right) q_{\nu\gamma}(t_1)\, dt_1. \tag{2.18}$$

The term $\Phi M_2 \Phi^{-1}$ on the right-hand side of (2.12) is calculated similarly.

$$\Phi(t) M_2(t) \Phi^{-1}(t) = \Phi(t) \int_{(L)}^{t} \left[\int_{(L)}^{t_1} \Phi^{-1}(t_2) R(t_2) \Phi(t_2)\, dt_2 \right] \left[\Phi^{-1}(t_1) R(t_1) \Phi(t_1) \right] dt_1 \Phi^{-1}(t)$$

$$= \int_{(L)}^{t} \Phi(t) \Phi^{-1}(t_1) \left[\int_{(L)}^{t_1} \Phi(t_1) \Phi^{-1}(t_2) R(t_2) \Phi(t_2) \Phi^{-1}(t_1)\, dt_2 \right] R(t_1) \Phi(t_1) \Phi^{-1}(t)\, dt_1. \tag{2.19}$$

Notice that (2.19) is obtained from (2.15) by substituting the identity matrix instead of $Q(t_1)$ in (2.15) or $\delta_{\nu\gamma}$ instead of $q_{\nu\gamma}(t_1)$. Hence the (α, γ) term of (2.19) is

$$\int_{\ell_{\alpha\gamma}}^{t} \left(\sum_{\beta=1}^{n} \left[\int_{\ell_{\alpha\beta}}^{t_1} r_{\alpha\beta}(t_2)\, e^{\int_{t_2}^{t_1} (\lambda_\alpha - \lambda_\beta)\,ds}\, dt_2 \right] r_{\beta\gamma}(t_1)\, e^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\gamma)\,ds} \right) dt_1. \tag{2.20}$$

3. A framework for asymptotic approximation

The existence of the asymptotic representation (1.4) depends on the availability of a solution Q of (2.12) such that $Q(t) \to 0$ as $t \to \infty$. This will be provided in the following general principle. Our later tasks will be to show that (i) it may be reduced to practical criteria and (ii) it includes many known results about asymptotic integration.

Theorem 1. *If there exists constants $\ell_{\alpha\beta} \leqslant \infty$ such that for all α, β, γ, ν, $\alpha \neq \beta$, $\beta \neq \nu$,*

$$\int_{\ell_{\alpha\beta}}^{t} r_{\alpha\beta}(t_1)\, e^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\,ds}\, dt_1 \to 0, \tag{3.1}$$

$$\int_{\ell_{\alpha\gamma}}^{t} \left| \left[\int_{\ell_{\alpha\beta}}^{t_1} r_{\alpha\beta}(t_2)\, e^{\int_{t_2}^{t_1} (\lambda_\alpha - \lambda_\beta)\,ds}\, dt_2 \right] r_{\beta\nu}(t_1)\, e^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\gamma)\,ds} \right| dt_1 \to 0 \tag{3.2}$$

as $t \to \infty$, then Eq. (1.1) *has an asymptotic representation* (1.4) *where Q is a solution of Eq.* (2.12) *such that $Q(t) \to 0$ as $t \to \infty$.*

Proof. Recall that R is an off-diagonal matrix, so $r_{\alpha\alpha} = 0$.

Conditions (3.1) and (3.2) ensure that the elements of $\Phi M_1 \Phi^{-1}$, $\Phi M_2 \Phi^{-1}$ of the integral equation (2.12) tend to 0 as $t \to \infty$. Consequently we multiply (2.12) from the left-hand side by the inverse matrix $S(t) = (I - \Phi(t) M_1(t) \Phi^{-1}(t))^{-1}$ and get

$$Q(t) = V(t) + \mathcal{L}[Q](t), \tag{3.3}$$

where

$$\mathcal{L}[Q](t) = S(t)\int_{(L)}^{t} \Phi(t)\big[M_1(t_1)K(t_1)\Phi^{-1}(t_1)Q(t_1)\Phi(t_1)\big]\Phi^{-1}(t)\,\mathrm{d}t_1$$

and

$$V(t) = S(t)\big(\Phi(t)M_1(t)\Phi^{-1}(t) - \Phi(t)M_2(t)\Phi^{-1}(t)\big).$$

Since $S(t) \to I,\ \Phi M_1\Phi^{-1}, \Phi M_2\Phi^{-1} \to 0$, we have $V(t) \to 0$ as $t \to \infty$.

Suppose that $Q(t)$ is indeed a bounded solution of (2.12) on some interval $[a,\infty)$. For any bounded matrix valued function $A(t)$ let $\|A(t)\| = \sum_{ij}|a_{ij}(t)|$ and let $\|\!|A|\!\| = \sup_{t\in[a,\infty)}\|A(t)\|$. In the integral equation (3.3) the terms $q_{\nu\gamma}$ of Q appear inside integrals of the type (2.18). We bound (2.18) from above by

$$\sum_{\beta=1}^{n}\left|\int_{\ell_{\alpha\gamma}}^{t}\left|\left[\int_{\ell_{\alpha\beta}}^{t_1} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right|\mathrm{d}t_1\right| \times \|\!|Q|\!\|.$$

Denote

$$N(t) = \max_{\alpha,\beta,\gamma,\nu}\left|\int_{\ell_{\alpha\gamma}}^{t}\left|\left[\int_{\ell_{\alpha\beta}}^{t_1} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right|\mathrm{d}t_1\right|.$$

Thus we get

$$\|Q(t)\| \leqslant \|V(t)\| + n^2\|S(t)\|N(t)\|\!|Q|\!\|.$$

Now, by (3.2) $N(t)$ may be made as small as we want for sufficiently large values of t. Take a large enough so that $n^2\|\!|S|\!\|N(t) \leqslant \rho < 1$ on $[a,\infty)$. Then, under the present assumptions

$$\|\!|Q|\!\| \leqslant \frac{\|\!|V|\!\|}{1-\rho}.$$

With these values of a and ρ we complete the proof of the existence of Q by a standard iteration. Define the sequence

$$Q_0 = V, \qquad Q_j = \mathcal{L}[Q_{j-1}], \quad j = 1,2,\ldots.$$

Then

$$\|Q_{j+1}(t) - Q_j(t)\| \leqslant \|\mathcal{L}[Q_j - Q_{j-1}]\| \leqslant \rho\|\!|Q_j - Q_{j-1}|\!\|,$$

and also

$$\|\!|Q_{j+1} - Q_j|\!\| \leqslant \rho\|\!|Q_j - Q_{j-1}|\!\|.$$

Hence, the sequence $Q_j(t)$ converges uniformly on $C[a,\infty)$ to a limit function $Q(t)$. It is evident that $Q \in C[a,\infty)$ and it is the unique solution of the integral equation (3.3). Consequently, also $Q \in C^1$. Since $V(t) \to 0$, $S(t) \to I$ and $N(t) \to 0$, it follows that $\lim_{t\to\infty} Q(t) = 0$. □

What is a reasonable choice of the limits of integration $\ell_{\alpha\beta}$? We shall always choose the limits of integration of each term in (2.5), (2.13) and (2.16) so that the kernel $\exp \int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s$ that it contains will be bounded by the corresponding dichotomy condition.

If (α,β) *satisfies the dichotomy condition* (Dic1)*, the limit of integration of the corresponding term will be* $\ell_{\alpha\beta} = t_0$*. If* (α,β) *satisfies the dichotomy condition* (Dic2)*, we choose* $\ell_{\alpha\beta} = \infty$*.*

The relations (3.1) (and (3.2)) demonstrate the lack of symmetry that is inherently built into Theorem 1, namely, that various perturbation terms are required to satisfy different smallness conditions. If (α,β) satisfies the dichotomy condition (Dic1), ($(\alpha,\beta) \in$ (Dic1), for short) then $\ell_{\alpha\beta} = t_0$ and

$$I(t) = \int_{t_0}^{t} r_{\alpha\beta}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1 \tag{3.4}$$

has the kernel $\exp \int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s$ which is bounded from above by e^{K_1} for $t_1 \leqslant t$ (and tends to 0 as $t \to \infty$). If $(\alpha,\beta) \in$ (Dic2) (i.e., (α,β) satisfies the dichotomy condition (Dic2)) then the integral in (3.1) is

$$I(t) = \int_{t}^{\infty} r_{\alpha\beta}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1 \tag{3.5}$$

and for $t_1 \geqslant t$ the kernel is bounded by $|\exp \int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s| \leqslant \mathrm{e}^{-K_2}$.

We wish to show that some well known results follow, naturally, from our general framework.

3.1. *Comparison with Levinson's theorem*

Levinson's theorem as presented by [3] claims that if the dichotomy condition holds and $R \in L^1$ then (1.1) has the asymptotic solution (1.4).

Note that in the traditional formulation of Levinson's theorem it is not assumed that R is off-diagonal, as we do. However, even if the diagonal terms of R are moved into D and λ_i is replaced by $\lambda_i + r_{ii}$ it makes no essential difference in the asymptotic solution (1.4) since $r_{ii} \in L^1$.

We show that this basic result can be deduced from our framework. For this one must verify that (3.1) and (3.2) hold. Let us start with (3.1).

If (α,β) satisfies (Dic2) then the integral in (3.1) is

$$\left| \int_{t}^{\infty} r_{\alpha\beta}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1 \right| \leqslant \mathrm{e}^{-K_2} \int_{t}^{\infty} |r_{\alpha\beta}(t_1)|\,\mathrm{d}t_1 \to 0, \tag{3.6}$$

since $\mathrm{Re} \int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s \leqslant -K_2$ for $t \leqslant t_1$.

If (α, β) satisfies (Dic1) we formulate the calculation of (3.1) as a lemma for further applications:

Lemma. *If* $(\alpha, \beta) \in$ (Dic1) *and* $r_{\alpha\beta} \in L^1$ *then*

$$\int_{t_0}^{t} \left| r_{\alpha\beta}(t_1)\, \mathrm{e}^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s} \right| \mathrm{d}t_1 \to 0 \quad \textit{as } t \to \infty. \tag{3.7}$$

Proof. Here $t_0 \leqslant t_1 \leqslant t < \infty$ and $\int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s \leqslant K_1$. We split the integral into two parts

$$I_1 + I_2 = \int_{t_0}^{T} \left| r_{\alpha\beta}(t_1)\, \mathrm{e}^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s} \right| \mathrm{d}t_1 + \int_{T}^{t} \left| r_{\alpha\beta}(t_1)\, \mathrm{e}^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s} \right| \mathrm{d}t_1$$

and for any given, small $\varepsilon > 0$ we choose a fixed T such that

$$I_2 \leqslant \int_{T}^{t} \left| r_{\alpha\beta}(t_1) \right| \mathrm{e}^{K_1}\, \mathrm{d}t_1 \leqslant \mathrm{e}^{K_1} \int_{T}^{\infty} \left| r_{\alpha\beta}(t_1) \right| \mathrm{d}t_1 \leqslant \frac{\varepsilon}{2}$$

for all $t \geqslant T$. With this fixed T

$$I_1 = \int_{t_0}^{T} \left| r_{\alpha\beta}(t_1)\, \mathrm{e}^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s} \right| \mathrm{d}t_1 = \left| \mathrm{e}^{\int_{T}^{t} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s} \right| \int_{t_0}^{T} \left| r_{\alpha\beta}(t_1)\, \mathrm{e}^{\int_{t_1}^{T} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s} \right| \mathrm{d}t_1.$$

According to (Dic1) the first factor converges to 0 as $t \to \infty$ while the second one is bounded by $\mathrm{e}^{K_1} \int_{t_0}^{\infty} |r_{\alpha\beta}(t_1)|\, \mathrm{d}t_1$. $\square$

The integral in (3.2) may have four different forms:

$$\int_{t}^{\infty} \left| \left[\int_{t_1}^{\infty} r_{\alpha\beta}(t_2)\, \mathrm{e}^{\int_{t_2}^{t_1} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s}\, \mathrm{d}t_2 \right] r_{\beta\nu}(t_1)\, \mathrm{e}^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\gamma)\, \mathrm{d}s} \right| \mathrm{d}t_1$$
$$\text{if } (\alpha, \beta) \in \text{(Dic2)},\ (\alpha, \gamma) \in \text{(Dic2)} \tag{3.8}$$

$$\int_{t}^{\infty} \left| \left[\int_{t_0}^{t_1} r_{\alpha\beta}(t_2)\, \mathrm{e}^{\int_{t_2}^{t_1} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s}\, \mathrm{d}t_2 \right] r_{\beta\nu}(t_1)\, \mathrm{e}^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\gamma)\, \mathrm{d}s} \right| \mathrm{d}t_1$$
$$\text{if } (\alpha, \beta) \in \text{(Dic1)},\ (\alpha, \gamma) \in \text{(Dic2)} \tag{3.9}$$

$$\int_{t_0}^{t} \left| \left[\int_{t_1}^{\infty} r_{\alpha\beta}(t_2)\, \mathrm{e}^{\int_{t_2}^{t_1} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s}\, \mathrm{d}t_2 \right] r_{\beta\nu}(t_1)\, \mathrm{e}^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\gamma)\, \mathrm{d}s} \right| \mathrm{d}t_1$$
$$\text{if } (\alpha, \beta) \in \text{(Dic2)},\ (\alpha, \gamma) \in \text{(Dic1)} \tag{3.10}$$

and

$$\int_{t_0}^{t} \left| \left[\int_{t_0}^{t_1} r_{\alpha\beta}(t_2)\, \mathrm{e}^{\int_{t_2}^{t_1} (\lambda_\alpha - \lambda_\beta)\, \mathrm{d}s}\, \mathrm{d}t_2 \right] r_{\beta\nu}(t_1)\, \mathrm{e}^{\int_{t_1}^{t} (\lambda_\alpha - \lambda_\gamma)\, \mathrm{d}s} \right| \mathrm{d}t_1$$
$$\text{if } (\alpha, \beta) \in \text{(Dic1)},\ (\alpha, \gamma) \in \text{(Dic1)}. \tag{3.11}$$

For (3.8) and (3.9) $t \leqslant t_1 < \infty$, so $t_1 \to \infty$ as $t \to \infty$. Consequently the internal integral is o(1) as $t \to \infty$ either as in (3.6) (for (3.8)) or by (3.7) (for (3.9)). Now the estimate for the outer integral follows easily.

For (3.10) $t_2 \geqslant t_1$, $(\alpha, \beta) \in$ (Dic2), so the internal integral is bounded by

$$\left| \int_{t_1}^{\infty} r_{\alpha\beta}(t_2)\, \mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s}\, \mathrm{d}t_2 \right| \leqslant \int_{t_0}^{\infty} \left| r_{\alpha\beta}(t_2) \right| \mathrm{e}^{-K_2}\, \mathrm{d}t_2$$

and $\int_{t_0}^{t} |r_{\beta\nu}(t_1)\, \mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha - \lambda_\gamma)\,\mathrm{d}s}|\, \mathrm{d}t_1 \to 0$ as in the lemma, since $(\alpha, \gamma) \in$ (Dic1). For (3.11) $t_2 \leqslant t_1$, $(\alpha, \beta) \in$ (Dic1), so the internal integral is bounded by

$$\left| \int_{t_0}^{t_1} r_{\alpha\beta}(t_2)\, \mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s}\, \mathrm{d}t_2 \right| \leqslant \int_{t_0}^{\infty} \left| r_{\alpha\beta}(t_2) \right| \mathrm{e}^{K_1}\, \mathrm{d}t_2$$

and the estimate of the double integral is completed as above.

The recent calculations hint that addition of integrable terms to Eq. (1.1) has no influence on the asymptotic behaviour of its solutions.

Theorem 1′. *If the assumptions of Theorem* 1 *hold and the elements of the matrix* $S(t) = (s_{ij}(t))$ *are in* L^1*, then the equation*

$$Y' = \big(D(t) + R(t) + S(t)\big)Y$$

has as well an asymptotic representation (1.4).

Proof. All one has to show is that if $r_{\alpha\beta}$ is replaced by $r_{\alpha\beta} + s_{\alpha\beta}$ and $s_{\alpha\beta} \in L^1$, then (3.1) and (3.2) continue to hold. This is verified along the same lines as the previous calculations. □

3.2. The results of Hartman–Wintner and Behncke–Remling

Hartman and Wintner [13] proved that if

$$\begin{aligned} &\text{(a)} \quad \left|\mathrm{Re}(\lambda_\alpha - \lambda_\beta)\right| \geqslant c > 0, \quad \alpha \neq \beta, \\ &\text{(b)} \quad R(t) \in L^p, \quad 1 < p \leqslant 2, \end{aligned} \tag{3.12}$$

then (1.1) has the asymptotic solution (1.4).

Behncke and Remling [1] proved that if

$$\begin{aligned} &\text{(a)} \quad \left|\mathrm{Re}(\lambda_\alpha - \lambda_\beta)\right| \geqslant ct^{-a_{\alpha\beta}}, \quad c > 0,\ a_{\alpha\beta} = a_{\beta\alpha},\ a_{\alpha\beta} < 1,\ \alpha \neq \beta, \\ &\text{(b)} \quad r_{\alpha\beta} t^{b_{\alpha\beta}} \in L^p, \quad p > 1, \\ &\text{(c)} \quad p' b_{\alpha\beta} \geqslant a_\alpha, \quad a_\alpha = \max_\beta a_{\alpha\beta} < 1,\ \frac{1}{p} + \frac{1}{p'} = 1, \end{aligned} \tag{3.13}$$

then

$$y_\alpha(t) = \big(e_\alpha + \mathrm{o}(1)\big) \exp \int_{t_0}^{t} \big(\lambda_\alpha(s) + r_{\alpha\alpha}(s) + \mathrm{o}\big(s^{-a_\alpha}\big)\big)\, \mathrm{d}s.$$

(In [1] R is not assumed to be off-diagonal.) Since $a_\alpha = \max_j(a_{\alpha j}) < 1$ and $o(t^{-a_\alpha})$ is not necessarily in L^1 ($a_{\alpha\beta},\ b_{\alpha\beta}$ may be even negative!), this approximation is not necessarily an asymptotic integration in the sense of (1.4). Nevertheless, it is closely related to the Hartman–Wintner theorem and a small variation of the conditions will yield a similar result with a strict estimate, without the $o(t^{-a_\alpha})$ term.

We start to establish the validity of (3.1) under assumptions similar to (3.13). If $\operatorname{Re}(\lambda_\alpha - \lambda_\beta) \geqslant ct^{-a_{\alpha\beta}} > 0$ then $(\alpha,\beta) \in$ (Dic2) and we take $\ell_{\alpha\beta} = \infty$. Then for $t \leqslant t_1 < \infty$, $\int_{t_1}^t (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s \leqslant \int_{t_1}^t cs^{-a_{\alpha\beta}}\,\mathrm{d}s < 0$ and (3.1) is bounded from above by

$$\int_t^\infty |r_{\alpha\beta}|(t_1)\,\mathrm{e}^{\int_{t_1}^t cs^{-a_{\alpha\beta}}\,\mathrm{d}s}\,\mathrm{d}t_1 = \int_t^\infty |r_{\alpha\beta}|(t_1)\,\mathrm{e}^{c(t^{-a_{\alpha\beta}+1}-t_1^{-a_{\alpha\beta}+1})/(-a_{\alpha\beta}+1)}\,\mathrm{d}t_1$$
$$\leqslant \left(\int_t^\infty \left(|r_{\alpha\beta}|(t_1)t_1^{b_{\alpha\beta}}\right)^p\mathrm{d}t_1\right)^{1/p}\left(\int_t^\infty t_1^{-b_{\alpha\beta}p'}\,\mathrm{e}^{p'c(t^{-a_{\alpha\beta}+1}-t_1^{-a_{\alpha\beta}+1})/(-a_{\alpha\beta}+1)}\,\mathrm{d}t_1\right)^{1/p'}. \quad (3.14)$$

The first integral tends to 0 since $r_{\alpha\beta}t^{b_{\alpha\beta}} \in L^p$. For the second integral we need the asymptotic approximation

$$\int_t^\infty s^\mu\,\mathrm{e}^{(t^{\nu+1}-s^{\nu+1})/(\nu+1)}\,\mathrm{d}s \approx t^{\mu-\nu}. \quad (3.15)$$

(Show by the l'Hospital rule that $\lim \int_t^\infty s^\mu\,\mathrm{e}^{-s^{\nu+1}/(\nu+1)}\,\mathrm{d}s/t^{\mu-\nu}\,\mathrm{e}^{-t^{\nu+1}/(\nu+1)} = 1$.) According to this estimate with $\mu = -p'b_{\alpha\beta}$, $\nu = -a_{\alpha\beta}$, the second factor of (3.14) behaves as $t^{(-p'b_{\alpha\beta}-(-a_{\alpha\beta}))/p'}$ when $t \to \infty$ and it tends to 0 provided that $a_{\alpha\beta} - p'b_{\alpha\beta} < 0$.

If $\operatorname{Re}(\lambda_\alpha - \lambda_\beta) \leqslant -ct^{-a_{\alpha\beta}} < 0$ then $(\alpha,\beta) \in$ (Dic1) and we take $\ell_{\alpha\beta} = t_0$. In this case we are interested in

$$\int_{t_0}^t |r_{\alpha\beta}|(t_1)\,\mathrm{e}^{\int_{t_1}^t -cs^{-a_{\alpha\beta}}\,\mathrm{d}s}\,\mathrm{d}t_1$$
$$\leqslant \left(\int_{t_0}^t \left(|r_{\alpha\beta}|(t_1)t_1^{b_{\alpha\beta}}\right)^p\mathrm{d}t_1\right)^{1/p}\left(\int_{t_0}^t t_1^{-b_{\alpha\beta}p'}\,\mathrm{e}^{-p'c(t^{-a_{\alpha\beta}+1}-t_1^{-a_{\alpha\beta}+1})/(-a_{\alpha\beta}+1)}\,\mathrm{d}t_1\right)^{1/p'}. \quad (3.16)$$

Now the first integral is bounded and the second one behaves as $t^{(-p'b_{\alpha\beta}-(-a_{\alpha\beta}))/p'} \to 0$ when $t \to \infty$, due to the estimate

$$\int_1^t s^\mu\,\mathrm{e}^{-(t^{\nu+1}-s^{\nu+1})/(\nu+1)}\,\mathrm{d}s \approx t^{\mu-\nu}, \quad (3.17)$$

which is similarly verified.

For $p = 1$ (3.14) is replaced by

$$\int_t^\infty |r_{\alpha\beta}|(t_1)\,\mathrm{e}^{\int_{t_1}^t cs^{-a_{\alpha\beta}}\,\mathrm{d}s}\,\mathrm{d}t_1$$
$$\leqslant \int_t^\infty |r_{\alpha\beta}|(t_1)t_1^{b_{\alpha\beta}}\,\mathrm{d}t_1 \times \max_{t_1\geqslant t}\left(t_1^{-b_{\alpha\beta}}\exp\left(c\frac{t^{-a_{\alpha\beta}+1}-t_1^{-a_{\alpha\beta}+1}}{-a_{\alpha\beta}+1}\right)\right)$$
$$= \int_t^\infty |r_{\alpha\beta}|(t_1)t_1^{b_{\alpha\beta}}\,\mathrm{d}t_1 \times t^{-b_{\alpha\beta}},$$

so (3.13(c)) is replaced by $b_{\alpha\beta} > 0$.

The relation (3.2) may again have the four forms (3.8)–(3.11) and it is treated as in our proof of the Levinson theorem. First we show that the internal integrals either tend to 0 or are, at least, bounded. At the second step we see that the double integrals tend to 0 if $p' b_{\beta\nu} > a_{\alpha\gamma}$, i.e., if

$$p' \min\{b_{\beta\nu}\} > \max\{a_{\alpha\gamma}\},$$

or $b_{\beta\nu} > 0$ in the case $p = 1$.

This yields a strict asymptotic result in the style of [1]. (We do not attempt to reproduce the precise result of [1] as it contains a somewhat nontraditional asymptotic statement). It can be formalized in the following theorem the proof of which is now superfluous.

Theorem 2. *If*

(a) $|\mathrm{Re}(\lambda_\alpha - \lambda_\beta)| \geqslant ct^{-a},\ c > 0,\ a < 1,\ \alpha \neq \beta,$
(b) $r_{\alpha\beta} t^b \in L^p,\ p > 1,$
(c) $p' b > a,\ 1/p + 1/p' = 1,$

or if

(a) $|\mathrm{Re}(\lambda_\alpha - \lambda_\beta)| \geqslant ct^{-a},\ c > 0,\ a < 1,\ \alpha \neq \beta,$
(b) $r_{\alpha\beta} t^b \in L^1,\ b > 0,$

then Eq. (1.1) *has an asymptotic solution* (1.4).

The Hartman–Wintner theorem is of course included. Note that $a,\ b$ may be even negative as long as (c) holds.

4. An explicit criteria

Theorem 1 formulates a general principle for asymptotic integration. In this section we obtain explicit, simple criteria which will enable us to verify that the assumptions (3.1) and (3.2) really hold. This is done through systematic integrations by parts of the integrals in (3.1) and (3.2). under the additional assumption that R is differentiable.

Theorem 3. *The conditions of Theorem* 1 *hold under the following assumptions*: *for every* $\alpha \neq \beta, \nu$

$$\frac{r_{\alpha\beta}}{\lambda_\alpha - \lambda_\beta} \to 0 \quad \text{as } t \to \infty, \tag{4.1}$$

$$\left(\frac{r_{\alpha\beta}}{\lambda_\alpha - \lambda_\beta}\right)' \in L^1, \tag{4.2}$$

$$r_{\beta\nu}(t) \int_t^\infty \left| \left(\frac{r_{\alpha\beta}}{\lambda_\alpha - \lambda_\beta}\right)' \right| \mathrm{d}t_1 \in L^1 \tag{4.3}$$

and for (α, β) *such that* $\mathrm{Re} \int^t (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s \to -\infty$ (*i.e.*, $(\alpha, \beta) \in$ (Dic1)) *also*

$$r_{\beta\nu}(t)\, \mathrm{e}^{\int_{t_0}^t (\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s} \in L^1, \tag{4.4}$$

$$r_{\beta\nu}(t)\int_{t_0}^{t}\left(\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}\right)'\mathrm{e}^{\int_{t_2}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\in L^1. \tag{4.5}$$

Proof. We begin to show that (4.1) and (4.2) imply (3.1). By integration by parts

$$\int_{\ell_{\alpha\beta}}^{t} r_{\alpha\beta}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1 = \frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(\ell_{\alpha\beta})\,\mathrm{e}^{\int_{\ell_{\alpha\beta}}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}-\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t)+\int_{\ell_{\alpha\beta}}^{t}\left(\frac{r_{\alpha\beta}(t_1)}{\lambda_\alpha-\lambda_\beta}\right)'\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1. \tag{4.6}$$

(Expressions of the form $h(t)/g(t)$ will be abbreviated for simplicity by $\frac{h}{g}(t)$.)

If $(\alpha,\beta)\in$ (Dic1) then $\ell_{\alpha\beta}=t_0$ and the right-hand side of (4.6) is

$$\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t_0)\,\mathrm{e}^{\int_{t_0}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}-\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t)+\int_{t_0}^{t}\left(\frac{r_{\alpha\beta}(t_1)}{\lambda_\alpha-\lambda_\beta}\right)'\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1. \tag{4.7}$$

As $t\to\infty$ the first term goes to 0 by (Dic1), the second by (4.1) and the third by (4.2) and the lemma.

If $(\alpha,\beta)\in$ (Dic2) then $\ell_{\alpha\beta}=\infty$, $r_{\alpha\beta}/(\lambda_\alpha-\lambda_\beta)(\infty)=0$ by (4.1) and $|\mathrm{e}^{\int_{\infty}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}|\leqslant \mathrm{e}^{-K_2}$ by (Dic2), so the first term on the right-hand side of (4.6) is 0 and there remains

$$-\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t)-\int_{t}^{\infty}\left(\frac{r_{\alpha\beta}(t_1)}{\lambda_\alpha-\lambda_\beta}\right)'\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1. \tag{4.8}$$

The first term tends to zero by (4.1). In the integral $|\exp\int_{t_1}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s|$ is bounded by e^{-K_2} and $(r_{\alpha\beta}/(\lambda_\alpha-\lambda_\beta))'\in L^1$ by (4.2), hence (4.8) tends to 0 as $t\to\infty$.

Next we show that (3.2) is implied by (4.3)–(4.5). The discussion of (3.2) is divided again into four cases as in (3.8)–(3.11).

Case I. $(\alpha,\beta)\in$ (Dic2), $(\alpha,\gamma)\in$ (Dic2). In this case the integral in (3.2) is

$$\int_{t}^{\infty}\left|\left[\int_{t_1}^{\infty} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right]r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right|\mathrm{d}t_1 = \int_{t}^{\infty}\left|\left[-\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t_1)-\int_{t_1}^{\infty}\left(\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}\right)'\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right]r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right|\mathrm{d}t_1.$$

Here $|\exp\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s|\leqslant \mathrm{e}^{-K_2}$ for $t_1\leqslant t_2$, $|\exp\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s|\leqslant \mathrm{e}^{-K_2}$ for $t\leqslant t_1$, and thanks to (4.1),

$$\left|\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t_1)\right|\leqslant\int_{t_1}^{\infty}\left|\left(\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}\right)'\right|\mathrm{d}t_2.$$

So the last double integral is bounded by

$$\int_{t}^{\infty}\left[\int_{t_1}^{\infty}\left|\left(\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}\right)'\right|\mathrm{d}t_2\right]|r_{\beta\nu}(t_1)|\,\mathrm{d}t_1\,(1+\mathrm{e}^{-K_2})\,\mathrm{e}^{-K_2},$$

and assumption (4.3) guarantees that (3.2) tends to 0.

Case II. $(\alpha,\beta)\in(\text{Dic2}),\ (\alpha,\gamma)\in(\text{Dic1})$. In this case (3.2) is

$$\int_{t_0}^{t}\left|\left[\int_{t_1}^{\infty} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right|\mathrm{d}t_1.$$

Since $(\alpha,\gamma)\in(\text{Dic1})$, it is sufficient according to the lemma to require that the integral (considered as a function of t_1) satisfies

$$\left[\int_{t_1}^{\infty} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\in L^1.$$

By (4.8) this becomes

$$\left[\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t)+\int_{t}^{\infty}\left(\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}\right)'\mathrm{e}^{\int_{t_2}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t)\in L^1.$$

As in Case I, this is guaranteed by assumption (4.3).

Case III. $(\alpha,\beta)\in(\text{Dic1}),\ (\alpha,\gamma)\in(\text{Dic2})$. In this case (3.2) is, according to (4.6),

$$\begin{aligned}
&\int_{t}^{\infty}\left|\left[\int_{t_0}^{t_1} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right|\mathrm{d}t_1\\
&\quad=\int_{t}^{\infty}\left|\left[\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t_0)\,\mathrm{e}^{\int_{t_0}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}-\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t_1)\right.\right.\\
&\qquad\left.\left.+\int_{t_0}^{t_1}\left(\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}\right)'\mathrm{e}^{\int_{t_2}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right|\mathrm{d}t_1.
\end{aligned}$$

Here, due to (4.1) and (4.2),

$$\left|\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t_1)\right|\leqslant\int_{t_1}^{\infty}\left|\left(\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}\right)'\right|\mathrm{d}t_2$$

and $|\exp\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s|\leqslant \mathrm{e}^{-K_2}$ for $t\leqslant t_1$. Consequently the three assumptions (4.3)–(4.5) imply (3.2).

Case IV. $(\alpha,\beta)\in(\text{Dic1}),\ (\alpha,\gamma)\in(\text{Dic1})$. Now the integral in (3.2) is

$$\int_{t_0}^{t}\left|\left[\int_{t_0}^{t_1} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\,\mathrm{e}^{\int_{t_1}^{t}(\lambda_\alpha-\lambda_\gamma)\,\mathrm{d}s}\right|\mathrm{d}t_1.$$

Since $(\alpha,\gamma)\in(\text{Dic1})$, it is sufficient according to the lemma to require that the internal integral (considered as a function of t_1) satisfies

$$\left[\int_{t_0}^{t_1} r_{\alpha\beta}(t_2)\,\mathrm{e}^{\int_{t_2}^{t_1}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_2\right] r_{\beta\nu}(t_1)\in L^1.$$

According to (4.7) this is equivalent to

$$\left[\mathit{const}\cdot \mathrm{e}^{\int_{t_0}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s} - \frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}(t) + \int_{t_0}^{t}\left(\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}\right)' \mathrm{e}^{\int_{t_2}^{t}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s}\,\mathrm{d}t_1\right] r_{\beta\nu}(t)\in L^1$$

which is implied by the conditions (4.3)–(4.5). □

Remarks.

(i) If $r_{\alpha\beta}/(\lambda_\alpha-\lambda_\beta)$ is monotone then (4.1) implies (4.2) and (4.3) may be written as $r_{\alpha\beta}r_{\beta\nu}(t)/(\lambda_\alpha-\lambda_\beta)\in L^1$.

(ii) If $\lambda_\alpha\equiv\lambda_\beta$ the theorem holds provided that $r_{\alpha\beta}\equiv 0$. In this case (4.4) and (4.5) are irrelevant.

(iii) Conditions (4.3), (4.4) and (4.5) are not too severe requirements since in each of them $r_{\beta\nu}$ is multiplied by a factor which tends to zero.

(iv) Conditions (4.1) and (4.2) stem from the requirement (3.1) of Theorem 1. Conditions (4.3)–(4.5) guarantee that requirement (3.2) holds.

5. Some examples and comparisons

The following examples are designed to bring out the various manners that Theorem 3 could be applied. Consider Eq. (1.1) with

$$D=\mathrm{diag}\{c_i t^{p_i}\},\qquad R=\left(a_{ij}t^{q_{ij}}\right),\quad a_{ii}\equiv 0,\ i,j=1,\dots,n. \tag{5.1}$$

Here and throughout this section $c_i,\ a_{ij}$ are complex valued constants and $p_i,\ q_{ij}$ are real valued, positive or negative. Among the diagonal terms there may be both real valued and complex valued terms, i.e., the differences $\mathrm{Re}(\lambda_\alpha-\lambda_\beta)$ may be either zero or nonzero. This is a generic case, since even eigenvalues of real matrices are, in general, complex valued.

Let us check when do the assumptions (4.1)–(4.5) hold. (4.1) means

$$\frac{r_{\alpha\beta}}{\lambda_\alpha-\lambda_\beta}=\frac{a_{\alpha\beta}t^{q_{\alpha\beta}}}{c_\alpha t^{p_\alpha}-c_\beta t^{p_\beta}}=\mathrm{O}\left(t^{q_{\alpha\beta}-\max(p_\alpha,p_\beta)}\right)\to 0.$$

This holds when

$$q_{\alpha\beta}<\max(p_\alpha,p_\beta). \tag{5.2}$$

The relation (4.2) leads to the same condition (5.2). Condition (4.3) becomes

$$q_{\alpha\beta}+q_{\beta\nu}-\max(p_\alpha,p_\beta)<-1. \tag{5.3}$$

(4.4) and (4.5) are assumed only when $\int^t \mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s\to-\infty$. But

$$\int^t \mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s=\int^t \mathrm{Re}\left(c_\alpha s^{p_\alpha}-c_\beta s^{p_\beta}\right)\mathrm{d}s\to-\infty$$

as $t \to \infty$ may happen only if $\max(p_\alpha, p_\beta) \geqslant -1$ (and c_α, c_β have suitable arguments). If $\max(p_\alpha, p_\beta) > -1$ then

$$r_{\beta\nu}(t)\,\mathrm{e}^{\int^t(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s} = a_{\beta\nu}t^{q_{\beta\nu}}\,\mathrm{e}^{\int^t(c_\alpha s^{p_\alpha}-c_\beta s^{p_\beta})\,\mathrm{d}s} \in L^1$$

holds obviously by the fast exponential decay. The formula (4.5) behaves similarly. This leaves us with only two assumptions, (5.2) and (5.3). If $\max(p_\alpha, p_\beta) = -1$ and $\mathrm{Re}(\lambda_\alpha - \lambda_\beta) \approx cs^{-1}$ with $c < 0$ (i.e., $(\alpha, \beta) \in$ (Dic1)), then condition (4.4),

$$r_{\beta\nu}(t)\,\mathrm{e}^{\int^t(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s} = \mathit{const}\cdot t^{q_{\beta\nu}}t^c \in L^1,$$

holds if we assume $q_{\beta\nu} \leqslant -1$ for all ν. For such (α, β) assumptions (5.2) and (5.3) are replaced by

$$q_{\alpha\beta} < -1 \quad \text{and} \quad q_{\beta\nu} \leqslant -1,$$

respectively.

Consider, for example, the system

$$Y' = \left[\begin{pmatrix} \mathrm{i}t^{p_1} & 0 & 0 \\ 0 & \mathrm{i}t^{p_2} & 0 \\ 0 & 0 & t^{p_3} \end{pmatrix} + \begin{pmatrix} 0 & c_{12}t^{q_{12}} & c_{13}t^{q_{13}} \\ c_{21}t^{q_{21}} & 0 & c_{23}t^{q_{23}} \\ c_{31}t^{q_{31}} & c_{32}t^{q_{32}} & 0 \end{pmatrix}\right] Y, \quad t_0 \leqslant t < \infty, \tag{5.4}$$

where c_{jk} are complex constants, $p_i,\ q_{ij}$ are real valued.

Levinson's methods cannot be applied to system (5.4). This is so on two counts. The first count is that the off diagonal terms are not necessarily in L^1. The second count being that two elements of the diagonal matrix may coalesce as $t \to \infty$. Therefore a continuously differentiable invertible transformation that diagonalizes the system cannot be guaranteed. Because of same reasons the results in [6] cannot be applied. The methods in [3,11,12] cannot conclude either that the system is almost diagonal. Neither can [14,1] be applied to (5.4) as $\mathrm{Re}(\lambda_1 - \lambda_2) \equiv 0$.

Another example is

$$Y' = \left[\begin{pmatrix} \mathrm{i}t^{p_1} & 0 & 0 \\ 0 & \mathrm{i}t^{p_1} & 0 \\ 0 & 0 & t^{p_3} \end{pmatrix} + \begin{pmatrix} 0 & 0 & c_{13}t^{q_{13}} \\ 0 & 0 & c_{23}t^{q_{23}} \\ c_{31}t^{q_{31}} & c_{32}t^{q_{32}} & 0 \end{pmatrix}\right] Y, \quad t_0 \leqslant t < \infty, \tag{5.5}$$

with two eigenvalues of the diagonal matrix being identical. In this case the first inequality of (5.2), $q_{12} < \max(p_1, p_2)$, may be omitted. Neither is this system amenable to the methods mentioned above.

The conditions (5.2) and (5.3) are best possible in the following sense. Consider for example the 2×2 system

$$Y' = \begin{pmatrix} t^p & t^{q_{12}} \\ t^{q_{21}} & 2t^p \end{pmatrix} Y, \quad p > -1,\ 1 \leqslant t < \infty, \tag{5.6}$$

(which, of course, could be investigated by other methods). The transformation $t = ((p+1)s)^{1/(p+1)}$ takes it into

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = \begin{pmatrix} 1 & c_{12}s^{(q_{12}-p)/(p+1)} \\ c_{21}s^{(q_{21}-p)/(p+1)} & 2 \end{pmatrix} Y, \tag{5.7}$$

which, according to (5.2) and (5.3), is asymptotically integrable if $q_{12} < p, q_{21} < p$ and

$$\frac{q_{12}-p}{p+1}+\frac{q_{21}-p}{p+1}<-1. \tag{5.8}$$

(5.8) cannot be relaxed. For, let $\alpha = (q_{12}-p+q_{21}-p)/(p+1)$. If (5.7) is asymptotically integrable, it has according to (1.4) a fundamental solution

$$Y(s) = (I+\mathrm{o}(1))\begin{pmatrix}\exp(s) & 0\\ 0 & \exp(2s)\end{pmatrix}C_1.$$

On the other hand, by Levinson's theorem, it has a fundamental solution

$$Y(s) = (I+\mathrm{o}(1))\begin{pmatrix}\exp(\int\lambda_1(s)\,\mathrm{d}s) & 0\\ 0 & \exp(\int\lambda_2(s)\,\mathrm{d}s)\end{pmatrix}C_2,$$

where $\lambda_1(s) = 1+\mathrm{O}(s^{\alpha})$, $\lambda_2(s) = 2+\mathrm{O}(s^{\alpha})$ are the eigenvalues of the coefficient matrix of (5.7). It is easily seen that the two representations are compatible only if $\alpha < -1$. This means that (5.8) is necessary. Consequently conditions (5.3) cannot be relaxed if $p_\alpha = p_\beta$.

Theorem 3 has obvious advantages as its conditions can be tested in a straight forward manner. It also has drawbacks. For example, for the system

$$Y' = \begin{pmatrix} t^p & t^q(1+\cos(\mathrm{e}^t))/2\\ t^q(1+\cos(\mathrm{e}^t))/2 & 2t^p\end{pmatrix}Y, \tag{5.9}$$

$p < q$, the large rate of growth of the derivative of the off diagonal elements precludes us from applying Theorem 3. On the other hand (5.9) can be shown to satisfy the conditions of Theorem 1. The integral in (3.1) reduces either to (3.4) or (3.5) with

$$|r_{12}(t_1)|, |r_{21}(t_1)| \leqslant t^q, \qquad \int_{t_1}^t(\lambda_2-\lambda_1)\,\mathrm{d}s = \frac{t^{p+1}-t_1^{p+1}}{p+1}$$

and both (3.4) and (3.5) tend to 0 by (3.15) and (3.17), respectively. In (3.2) the internal integral is bounded and the external integral is estimated again by (3.15) and (3.17).

Theorem 1 is in particular convenient for use in some other cases. It appears that when $\int_{t_1}^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s \to -\infty$ or when $\int_{t_1}^t \mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s \to +\infty$, then the estimation of the integrals (3.1) and (3.2) can be done by the l'Hospital rule and no differentiability of $r_{\alpha\beta}/(\lambda_\alpha-\lambda_\beta)$ is needed. In the first case $(\alpha,\beta) \in$ (Dic1), (3.1) becomes (3.4) and this integral is bounded by

$$|I(t)| \leqslant \int_{t_0}^t |r_{\alpha\beta}(t_1)| \exp\biggl(\int_{t_1}^t \mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s\biggr)\,\mathrm{d}t_1 = \frac{\int_{t_0}^t |r_{\alpha\beta}(t_1)|\exp(-\int_a^{t_1}\mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s)\,\mathrm{d}t_1}{\exp(-\int_a^t \mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s)}.$$

The denominator tends to $+\infty$. If $|r_{\alpha\beta}(t)|\exp(-\int_a^t \mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s) \notin L^1$ then by the l'Hospital rule ("∞/∞") this tends to $\lim -|r_{\alpha\beta}(t)|/\mathrm{Re}(\lambda_\alpha-\lambda_\beta)$, provided that the last limit exists. Otherwise, if $|r_{\alpha\beta}(t)|\exp(-\int_a^t\mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s) \in L^1$, we get $I(t) = \mathrm{O}(\exp\int_a^t \mathrm{Re}(\lambda_\alpha-\lambda_\beta)\,\mathrm{d}s)$.

If $\mathrm{Re}\int^t(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s \to +\infty$ (this is the case when $(\beta,\alpha) \in$ (Dic1)), then the integral (3.1) is bounded by

$$\begin{aligned}|I(t)| &\leqslant \int_t^\infty |r_{\alpha\beta}(t_1)| \exp\biggl(\int_{t_1}^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s\biggr)\,\mathrm{d}t_1 \\ &= \frac{\int_t^\infty |r_{\alpha\beta}(t_1)| \exp(-\int_a^{t_1} \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s)\,\mathrm{d}t_1}{\exp(-\int_a^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s)}.\end{aligned}$$

Now the denominator tends to 0 by (Dic1). If

$$r_{\alpha\beta}(t_1)\exp\biggl(-\int_{t_1}^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s\biggr) \in L^1$$

then by the l'Hospital rule ("$0/0$") this tends to $\lim |r_{\alpha\beta}(t_1)|/\mathrm{Re}(\lambda_\alpha - \lambda_\beta)$. Otherwise, if

$$r_{\alpha\beta}(t_1)\exp\biggl(-\int_{t_1}^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s\biggr) \notin L^1,$$

our quantity is undefined.

On the other hand, if for a pair (α,β) $|\int_{t_1}^t \mathrm{Re}(\lambda_\alpha - \lambda_\beta)\,\mathrm{d}s|$ is bounded for all $t_1,\ t$, then the above estimates are not helpful. Here the generic assumption $r_{\alpha\beta} \in L^1$ is natural and it implies that $I(t) = \mathrm{O}(\int_t^\infty |r_{\alpha\beta}|\,\mathrm{d}s)$. As an alternative, extra differentiability is needed for the integration by parts utilized in Theorem 3 to take place.

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