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Singular Eigenvalue Problems for the Equation $y^{(n)} + \lambda p(x)y = 0$

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Abstract. The paper studies singular eigenvalue problems for the equation $y^{(n)} + \lambda p(x)y = 0$ with boundary conditions imposed on the derivatives $y^{(i)}$ at the points $x = a$ and $x = \infty$. We look for singular problems which are analogous to regular problems on a finite interval. It is characterized when each eigenfunction has a finite number of zeros and when the spectrum is discrete or continuous, respectively.

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1. Introduction

Eigenvalue problems for the differential equation

$$y^{(n)} + \lambda p(x)y = 0 \tag{1.1}$$

where $p(x)$ is a continuous, one-signed function on $[a, b]$ and various boundary value conditions, like

$$\begin{aligned} y^{(i)}(a) &= 0, \quad i = 0, \dots, k-1, \\ y^{(i)}(b) &= 0, \quad i = 0, \dots, n-k-1, \end{aligned} \tag{1.2}$$

or

$$\begin{aligned} y^{(i)}(a) &= 0, \quad i = 0, \dots, k-1, \\ y^{(i)}(b) &= 0, \quad i = k, \dots, n-1, \end{aligned} \tag{1.3}$$

are well known ([10], [7], [4]). A typical result is that there exists an infinite sequence or eigenvalues λ_i,

$$\begin{gathered} 0 < (-1)^{n-k-1}\lambda_1 \operatorname{sgn}[p] < (-1)^{n-k-1}\lambda_2 \operatorname{sgn}[p] < \cdots, \\ |\lambda_i| \to \infty \quad \text{as} \quad i \to \infty, \end{gathered}$$

and the i-th eigenfunction has precisely $i - 1$ simple zeros in (a, b).

Naito [11] investigated equation (1.1) with even n, $\lambda > 0, p(x) > 0$ on $[a, \infty)$ and the singular boundary conditions $y(a) = 0,\ y^{(i)}(\infty) = 0,\ i = 1, \dots, n-1,$ and proved the existence of eigenvalues and eigenfunctions with similar properties. In this work we assume that $p(x)$ is one signed on $[a,\infty)$, n is an arbitrary integer and ask for which other singular eigenvalue problems on the infinite domain $[a,\infty)$ the spectrum $\{\lambda_i\}$ and the zeros of the eigenfunctions behave as described above.

Several questions arise:

(1) Is there a natural singular boundary value problem which is similar to the regular problems (1.1), (1.2) or (1.1), (1.3)?

(2) Which assumptions ensure eigenfunctions with finite number of zeros in (a,∞)?

(3) Which assumptions ensure a discrete spectrum?

(4) Are these conditions also necessary?

The answers are not obvious and the generalization from regular boundary conditions (1.2) or (1.3) to singular boundary conditions by sending the endpoint b to $+\infty$ is not straightforward. Even for second order equations, this may lead to limit circle/limit point problems. Various boundary conditions of mathematical physics (like "the solution is bounded at a singular point" or "the solution is at most of polynomial growth") lead to either discrete or continuous spectra.

A central point in this work is to understand the difference between regular and singular boundary value problems. The second point is to show that one and the same set of singular boundary conditions may lead to essentially different phenomena for various coefficients $p(x)$. Our results are formulated and summarized in Theorems 1, 2 and 3 later in this section. The theorems are highlighted by several examples in the last section. A few technical details which are common to singular and regular boundary value problems are borrowed from [2] (or [4]).

We begin with a careful analysis of questions (2), (3) and (4) which will lead us to the formulation of suitable boundary value problems. Our starting point is the requirement that the sequence of eigenvalues of the wanted problem is unbounded and that each eigenfunction has a finite number of zeros in (a,∞). Since $y^{(n)} = -\lambda p(x)y$, it follows that not only y but also each of its derivatives has a finite number of zeros. So there exists some $x_0 = x_0(\lambda) > a$, such that $y^{(i)}(x) \neq 0$ on $[x_0,\infty)$ for all $i = 0,\ldots,n$. Consequently, by the lemma of Kiguradze, there exists a certain integer q such that y (or $-y$) satisfies

$$\begin{aligned} y^{(i)} &> 0, \quad i = 0,\ldots,q, \\ (-1)^{i-q}y^{(i)} &> 0, \quad i = q,\ldots,n, \quad x_0 \leqslant x < \infty. \end{aligned} \tag{1.4}$$

Since we expect that our boundary conditions will include vanishing of some derivatives at both endpoints a and ∞, the cases $q = 0$ and $q = n$, i.e.,

$$(-1)^{(i)}y^{(i)} > 0, \quad i = 0,\ldots,n, \tag{1.5}$$

and

$$y^{(i)} > 0, \quad i = 0,\ldots,n, \tag{1.6}$$

respectively, must be excluded. Indeed, if (1.5) holds at some point $x = x_1$ this prevents the vanishing of any derivative for $x < x_1$, while a solution which satisfies

(1.6) at $x = x_1$ can have no derivative which vanishes in $(x_1, \infty]$. Thus, we assume from now and on that $1 \leqslant q \leqslant n-1$. Moreover, by $y^{(n)} = -\lambda py$ and (1.4) it follows that the integer q must satisfy

$$\operatorname{sgn}[-\lambda p] = (-1)^{n-q}. \tag{1.7}$$

Inequalities (1.4) are equivalent to the $(q,\ n-q)$-disfocality of equation (1.1) on $[x_0, \infty)$, namely, that for every α, β, $x_0 \leqslant \alpha < \beta$, the boundary conditions

$$\begin{aligned} y^{(i)}(\alpha) &= 0, \quad i = 0, \ldots, q-1, \\ y^{(i)}(\beta) &= 0, \quad i = q, \ldots, n-1, \end{aligned} \tag{1.8}$$

have no solution $y \not\equiv 0$. Next recall the following result about disfocality [2, Theorem 7.11]:

If equation (1.1) *is* $(q,\ n-q)$*-disfocal on an interval and* $1 \leqslant k \leqslant q$, $k \equiv q$ (mod 2), *then the equation*

$$y^{(n)} + \lambda\left(\binom{n-1}{q}\Big/\binom{n-1}{k}\right)p(x)y = 0 \tag{1.9}$$

is $(k,\ n-k)$*-disfocal on the same interval. If* $n-1 \geqslant k \geqslant q$, $k \equiv q$ (mod 2), *then the equation*

$$y^{(n)} + \lambda\left(\binom{n-1}{q-1}\Big/\binom{n-1}{k-1}\right)p(x)y = 0 \tag{1.10}$$

is $(k,\ n-k)$*-disfocal there.*

According to our assumptions, equations (1.1) is $(q, n-q)$-disfocal for some integer q and for arbitrary large values of λ on some $[x_0(\lambda), \infty)$. It follows from the last remark that (1.1) is also $(k, n-k)$-disfocal on some other infinite ray $[x_0(\lambda'), \infty)$, where the suitable λ' is determined by (1.9) or by (1.10). Consequently, equation (1.1) must be eventually $(k, n-k)$-disfocal for *every* value of λ and *every* k such that $\operatorname{sgn}[-\lambda p] = (-1)^{n-k}$. In other words, equation (1.1) must be eventually disconjugate for every value of λ.

This last property is completely characterized [3]:

Proposition 1. *Equation* (1.1) *is eventually disconjugate for every value of* λ *if and only if*

$$\lim_{x\to\infty}\left(x^{n-\alpha-1}\int_x^{\infty} s^{\alpha}|p(s)|ds\right) = 0 \tag*{$(1.11)_\alpha$}$$

for some $\alpha \leqslant n-1$.

If $(1.11)_\alpha$ *holds for some* α, *then* $(1.11)_\beta$ *holds for every* β, $\beta < n-1$. *However, the implication* $(1.11)_\alpha \to (1.11)_\beta$ *is in general false for* $\alpha < n-1$, $\beta = n-1$.

It is clear that $\alpha = n-1$ plays a special role in $(1.11)_\alpha$. Indeed, for $\alpha = n-1$, $(1.11)_\alpha$ reduces into the integrability condition

$$\int^{\infty} s^{n-1}|p(s)|ds < \infty, \tag{1.12}$$

which is a well known necessary and sufficient condition for every solution of equation (1.1) to be asymptotic to some polynomial of order smaller than n [1]. It will turn out that the difference between $(1.11)_\alpha$ with $\alpha < n-1$ and (1.12) plays a critical role in our eigenvalue problems.

Inequalities (1.4) and their close relation with the boundary conditions (1.8) suggest to consider singular boundary conditions of the form

$$\begin{aligned} y^{(i)}(a) &= 0, \quad i = 0, \dots, k-1, \\ \lim_{x\to\infty} y^{(i)}(x) &= 0, \quad i = k, \dots, n-1, \end{aligned} \tag{1.13}$$

for some fixed integer k, $1 \leqslant k \leqslant n-1$, $\operatorname{sgn}[-\lambda p] = (-1)^{n-k}$. Henceforth, we study the eigenvalue problem (1.1), (1.13).

Up to here the analysis relied only on the assumption that each eigenfunction of the problem that we look for has only a finite number of zeros. The next stage deals with the question when do the eigenvalues form a discrete set.

Proposition 2. *Let equation* (1.1) *be eventually disconjugate for every value of* λ. *A necessary condition for the problem* (1.1), (1.13) *to have a discrete set of eigenvalues is* (1.12). *If* (1.12) *does not hold* (*but* $(1.11)_\alpha$ *holds for some* $\alpha < n-1$), *then* (1.1), (1.13) *has a nontrivial solution for every* $\lambda \neq 0$.

In fact we prove a result about the behaviour of the solutions of (1.1) near ∞, which is a key ingredient of this work and implies Proposition 2 straightforwardly:

Proposition 2′. *If* (1.12) *holds then the* $n-k$ *singular boundary conditions*

$$\lim_{x\to\infty} y^{(i)}(x) = 0, \quad i = k, \dots, n-1, \tag{1.14}$$

are satisfied by precisely k *linearly independent solutions of* (1.1). *If* (1.12) *does not hold but* $(1.11)_\alpha$ *holds for some* $\alpha < n-1$ *and* $\lambda \neq 0$, *then* (1.14) *is satisfied by* $k+1$ *independent solutions of* (1.1).

This implies Proposition 2 immediately. Indeed, if (1.12) does not hold and $\lambda \neq 0$, then some combination of the mentioned $k+1$ solutions satisfies the k boundary conditions of (1.13) at $x = a$, hence boundary conditions (1.13) have a nontrivial solution. Consequently (1.12) is necessary for a discrete spectrum of (1.1), (1.13).

Proposition 2′ will be proved at the end of the section. Now we can formulate the main results of this work which describe the behavior of eigenvalue problem (1.1), (1.13) under assumptions (1.12) and $(1.11)_\alpha$, respectively. We have already concluded:

Theorem 1. (a) *A necesssary and sufficient condition for equation* (1.1) *to have a solution with a finite number of zeros for arbitrary large values of* λ (*other than* (1.5), (1.6)) *is that* $(1.11)_\alpha$ *holds for some* α, $\alpha \leqslant n-1$.

(b) *A necessary condition for the eigenvalue problem* (1.1), (1.13) *to have a discrete sequence of eigenvalues is* (1.12).

Our main results are the following:

Theorem 2 (Discrete Spectrum Case). *When* (1.12) *holds, then the eigenvalue problem* (1.1), (1.13) *has a sequence of eigenvalues* $\{\lambda_i\}$ *which satisfies*

$$0 < (-1)^{n-k-1}\lambda_1 \operatorname{sgn}[p] < (-1)^{n-k-1}\lambda_2 \operatorname{sgn}[p] < \cdots,$$
$$|\lambda_i| \to \infty \quad as \quad i \to \infty.$$

To each eigenvalue λ_i *there corresponds an essentially unique eigenfunction, and it has precisely* $i-1$ *simple zeros in* (α, ∞).

Theorem 3 (Continuous Spectrum Case). *Suppose that* $(1.11)_\alpha$ *holds for some* $\alpha < n-1$ *but* (1.12) *fails to hold. Then for every* $\lambda \neq 0$, *such that* $\operatorname{sgn}[-\lambda p] = (-1)^{n-k}$, *there exists an essentially unique eigenfunction* $y(x, \lambda)$ *of the eigenvalue problem* (1.1), (1.13).

Moreover, there exists a sequence a numbers λ_i, $\operatorname{sgn}[-\lambda_i p] = (-1)^{n-k}$, *such that for* $|\lambda_{i-1}| < |\lambda| \leqslant |\lambda_i|$, $y(x, \lambda)$ *has exactly* $i-1$ *simple zeros in* (a, ∞).

In the discrete case (1.1), (1.13) describe a proper eigenvalue problem similar to the regular problem (1.1), (1.3). In the continuous case (1.1), (1.13) is not a typical eigenvalue problem in spite of its external shape. Its distinctive feature is the stepwise growth of the number of zeros in spite of the continuous spectra. Theorems 2 and 3 are proved in Sections 2 and 3, respectively. The difference and similarity between the two cases will be discussed in Section 4. Examples 1, 2 and 3 of Section 5 demonstrate the Discrete Spectrum Case while Example 4 shows the Continuous Spectrum Case.

We close the section by the proof of Proposition 2′.

Proof of Proposition 2′. This proposition brings forward the essential difference between the $n-k$ regular boundary conditions $y^{(i)}(b) = 0$, $i = k, \ldots, n-1$, of (1.3) and the analogous $n-k$ singular boundary conditions (1.14). While the $n-k$ regular conditions are obviously satisfied by exactly k solutions, we show that the number of solutions which satisfy the $n-k$ singular boundary conditions (1.14) depends on the behaviour of $p(x)$. For this we quote a detailed description of the behaviour of all solutions of (1.1) near $+\infty$:

Proposition 3. *There exists a basis of solutions* $\{u_0, \ldots, u_{n-1}\}$ *of equation* (1.1) *such that if the equation is eventually* $(q, n-q)$*-disfocal for some* $1 \leqslant q \leqslant n-1$, $\operatorname{sgn}[-\lambda p] = (-1)^{n-q}$, *then the two solutions* u_{q-1}, u_q *and all their linear combinations satisfy inequalities* (1.4). *Moreover, their Wronskian* $W(u_{q-1}, u_q)$ *is positive on* (a, ∞). *The elements of this basis depend analytically on* λ.

If $(-1)^n \lambda p > 0$ *then the solution* u_0 (*which corresponds to* $q = 0$) *satisfies* (1.5). *If* $\lambda p < 0$ *then, for* $q = n$, *the solution* u_{n-1} *satisfies* (1.6).

The proof of Proposition 3 is given for an arbitrary equation of type (1.1) in [2, Chapter 8]. For nonoscillatory equations see [9]. Each u_i is analytic in λ since it is obtained as a limit of solutions which are analytic in λ and converge uniformly on compact intervals.

Now we return to the proof of Proposition 2′. If a solution y satisfies inequalities (1.4) (for example, u_{q-1} and u_q do this), then $y, y', \dots, y^{(q-1)}$ are positive and increasing, $y^{(q)}, -y^{(q+1)}, \dots, (-1)^{n-q-1}y^{(n-1)}$ are all positive and decreasing. Therefore

$$\begin{aligned} &y, y', \dots, y^{(q-2)} \nearrow \infty, \\ &|y^{(q+1)}|, |y^{(q+2)}|, \dots, |y^{(n-1)}| \searrow 0 \quad \text{as } x \to \infty. \end{aligned} \tag{1.15}$$

However, for the increasing $y^{(q-1)}$ and the decreasing $y^{(q)}$, both possibilities

$$y^{(q-1)} \nearrow C > 0 \quad \text{or} \quad y^{(q-1)} \nearrow +\infty, \tag{1.16}$$

$$y^{(q)} \searrow 0 \quad \text{or} \quad y^{(q)} \searrow c > 0, \tag{1.17}$$

can occur. In every case the bounds

$$0 < Ax^{q-1} \leqslant y(x) \leqslant Bx^q, \quad x_0 \leqslant x < \infty,$$

hold for some suitable $0 < A < B$.

Any solution which satisfies (1.15) with $q = k-2, k-4, \dots$, automatically satisfies the singular boundary conditions (1.14), while (1.15) with $q = k+2, k+4, \dots$, makes (1.14) impossible. Hence $u_0, \dots, u_{k-2}$ and their nontrivial combinations always satisfy the boundary conditions (1.14) while $u_{k+1}, \dots, u_{n-1}$ never satisfy (1.14).

This leaves us with u_{k-1}, u_k, which obey inequalities (1.4) with $q = k$. Do they satisfy boundary conditions (1.14)? While we know that both u_{k-1}, u_k satisfy

$$y, y', \dots, y^{(k-2)} \to \infty, \qquad y^{(k+1)}, y^{(k+2)}, \dots, y^{(n-1)} \to 0,$$

the behaviour of the $(k-1)$th and kth derivatives of u_{k-1}, u_k at $+\infty$ is not determined yet due to (1.16), (1.17) with $q = k$.

Let us apply the details of Proposition 3 for the two solutions u_{k-1}, u_k. Since $W(u_{k-1}, u_k) > 0$, u_{k-1}/u_k is monotone and tends to some limit L, finite or infinite. By interchanging their role or replacing u_{k-1} by $u_{k-1} - Lu_k$, we may assume without loss of generality that $u_{k-1}/u_k \to 0$, i.e., u_{k-1} is the "small" solution which satisfies (1.4) and u_k is a "large" one. It follows that also $u_{k-1}^{(i)}/u_k^{(i)} \to 0$ for $i = 0, \dots, n-1$, in particular $u_{k-1}^{(k)}/u_k^{(k)} \to 0$. This shows what happens in (1.17) with $q = k$: At least the "small" u_{k-1} must always satisfy $u_{k-1}^{(k)} \to 0$. It remains to decide the behaviour of the "large" u_k, whether

$$u_k^{(k)} \searrow 0 \quad \text{or} \quad u_k^{(k)} \searrow c > 0.$$

These two possibilities are easily distinguished. As mentioned before, a solution (and every solution) of (1.1) is asymptotic to a polynomial, or more precisely

$$\begin{aligned} u_j^{(i)} &\approx \frac{x^{j-i-1}}{(j-i-1)!}, \quad i = 0, \dots, j-1, \\ u_j^{(i)} &\approx o(x^{j-i-1}), \qquad i = j, \dots, n-1, \end{aligned}$$

if and only if (1.12) holds or if $\lambda = 0$. (Note that the case $\lambda = 0$ is the same as $p(x) \equiv 0$, hence it always belongs to assumption (1.12)). The sufficiency follows from well known results about perturbations of arbitrary linear equations. For its history see [1, p. 54]. The necessity of (1.12) is discussed in [8, p. 136]. For our problem this means that $u_k^{(k)} \to c > 0$ if and only if (1.12) holds, otherwise $u_k^{(k)} \to 0$. Thus, u_{k-1}, u_k behave as following: If (1.12) holds then u_{k-1} satisfies

$$\begin{aligned} &u_{k-1}^{(i)} \to \infty, && i = 0, \dots, k-2, \\ &u_{k-1}^{(k-1)} \to C > 0, && \\ &u_{k-1}^{(i)} \to 0, && i = k, \dots, n-1, \end{aligned} \tag{1.18}$$

and u_k satisfies

$$\begin{aligned} &u_k^{(i)} \to \infty, && i = 0, \dots, k-1, \\ &u_k^{(k)} \to c > 0, && \\ &u_k^{(i)} \to 0, && i = k+1, \dots, n-1, \end{aligned} \tag{1.19}$$

while if (1.12) does not hold (but $(1.11)_\alpha$ holds) and $\lambda \neq 0$, both u_{k-1}, u_k satisfy

$$\begin{aligned} &u^{(i)} \to \infty, && i = 0, \dots, k-1, \\ &u^{(i)} \to 0, && i = k, \dots, n-1. \end{aligned} \tag{1.20}$$

We summarize our analysis of the singular boundary conditions (1.14):

If (1.12) *holds then precisely the k solutions* $u_0, \dots, u_{k-1}$ *satisfy the singular boundary conditions* (1.14). *If* (1.12) *fails to hold but* $(1.11)_\alpha$ *holds and* $\lambda \neq 0$, *then the* $k+1$ *solutions* $u_0, \dots, u_{k-1}$, u_k *satisfy the singular boundary conditions* (1.14).

This completes the proof of Proposition 2′. □

2. The Discrete Spectrum Case

The proof of Theorem 2 is divided into a long sequence of steps, each one proving another detail. The equivalence of eventual disconjugacy for all λ and $(1.11)_\alpha$ had been already proved and summarized in Proposition 1. The necessity of the integrability condition (1.12) for discrete eigenvalues was proved in Proposition 2. Its sufficiency will be treated here.

(i) A basis of solutions. When the integrability condition (1.12) holds we need a description of a basis of solutions which is more detailed than that of Proposition 3:

Suppose that (1.12) *holds and an interval* $[\Lambda_1, \Lambda_2]$ *is given. There exists a point* $x_0 = x_0(\Lambda_1, \Lambda_2)$ *and a basis of solutions* $\{u_0(x,\lambda), \dots, u_{n-1}(x,\lambda)\}$ *such that each* $u_\ell(x,\lambda)$, $\ell = 0, \dots, n-1$, *is continuous for* $(x,\lambda) \in [x_0, \infty) \times [\Lambda_1, \Lambda_2]$, *it satisfies*

$$\frac{1}{2} \leqslant \frac{u_\ell(x,\lambda)}{x^\ell/\ell!} \leqslant 2 \quad \text{on } [x_0, \infty) \times [\Lambda_1, \Lambda_2] \tag{2.1}$$

and

$$\lim_{x \to \infty} \frac{u_\ell(x,\lambda)}{x^\ell/\ell!} = 1, \tag{2.2}$$

and the convergence in (2.2) *is uniform in* $\lambda \in [\Lambda_1, \Lambda_2]$. *Moreover,* $u_\ell(x,\lambda)/x^\ell$ *is uniformly continuous on* $[a,\infty) \times [\Lambda_1, \Lambda_2]$, $a > 0$.

This is shown by a standard fixed point argument and we only emphasize the uniform estimates (2.1), (2.2) which are critical for the next steps of the proof. In the following discussion let ℓ be a fixed integer, $0 \leqslant \ell \leqslant n-1$. $u_\ell(x,\lambda)$ will be characterized as the unique solution of the integral equation

$$u(x) = \frac{x^\ell}{\ell!} + (-1)^{n-\ell}\lambda \int_{x_0}^{x} \frac{(x-\tau)^{\ell-1}}{(\ell-1)!} \left(\int_\tau^\infty \frac{(s-\tau)^{n-\ell-1}}{(n-\ell-1)!} p(s)u(s)\, ds \right) d\tau. \quad (2.3)$$

Let the right hand side of (2.3) be denoted by $T[u]$ and choose x_0 sufficiently large so that

$$\max(|\Lambda_1|, |\Lambda_2|) \int_{x_0}^\infty s^{n-1}|p(s)|\, ds < 1/10.$$

Let B be the space of continuous functions on $[x_0, \infty)$ with the norm

$$\|u\| = \sup_{[x_0,\infty)} \left| \frac{u(x)}{x^\ell/\ell!} \right|$$

and $K \subset B$ the convex cone

$$K = \left\{ u \,\middle|\, \frac{1}{2} \leqslant \frac{u(x)}{x^\ell/\ell!} \leqslant 2 \text{ on } [x_0,\infty) \right\}.$$

For $u \in K$ we have $u(x) \leqslant (2/\ell!)x^\ell$, so

$$\left| \lambda \int_\tau^\infty \frac{(s-\tau)^{n-\ell-1}}{(n-\ell-1)!} p(s)u(s)\, ds \right| \leqslant \max(|\Lambda_1|, |\Lambda_2|) \frac{2}{\ell!} \int_\tau^\infty s^{n-1}|p(s)|\, ds < 1/5$$

for $\tau \geqslant x_0$. Hence it is easily seen that $T[u] \in K$. Moreover, T is a contraction of K since

$$\begin{aligned} \|T[u] - T[v]\| = \sup_{x \in [x_0,\infty)} &\left| \frac{\max(|\Lambda_1|, |\Lambda_2|)}{x^\ell/\ell!} \right. \\ &\left. \int_{x_0}^x \frac{(x-\tau)^{\ell-1}}{(\ell-1)!} \left(\int_\tau^\infty \frac{(s-\tau)^{n-\ell-1}}{(n-\ell-1)!} p(s)s^\ell \frac{u(s)-v(s)}{s^\ell} ds \right) d\tau \right| \\ &\leqslant \|u-v\| \sup_{x \in [x_0,\infty)} \left| \frac{1}{x^\ell/\ell!} \int_{x_0}^x \frac{(x-\tau)^{\ell-1}}{(\ell-1)!} \frac{1}{5} d\tau \right| \\ &\leqslant \frac{1}{5} \|u-v\|, \end{aligned}$$

so that T has a unique fixed point in K which is a solution of the integral equation (2.3). It is straightforward to see that the solution of the integral equation (2.3) is a solution of the differential equation (1.1) such that

$$u^{(\ell)}(x) = 1 + (-1)^{n-\ell}\lambda \int_x^\infty \frac{(s-x)^{n-\ell-1}}{(n-\ell-1)!} p(s)u(s)\, ds \to 1$$

and $u^{(i)} \to 0$, $i = \ell + 1, \ldots, n - 1$ as $x \to \infty$. This is the required solution $u_\ell(x, \lambda)$. Once $u_\ell(x, \lambda)$ is defined on $[x_0, \infty)$, we extend it to $[a, x_0]$. The same argument is repeated for each $\ell = 0, \ldots, n - 1$.

According to the well known proof of the fixed point theorem for contractive operators, the iteration $z_0(x) = x^\ell$, $z_i = T[z_{i-1}]$, $i = 1, \ldots$ converges to the unique fixed point u_ℓ so that $\|z_i - u_\ell\| \leqslant (1/5)\|z_{i-1} - u_\ell\|$, i.e.,

$$\sup_{x \in [x_0, \infty)} \left| \frac{z_i(x) - u_\ell(x, \lambda)}{x^\ell/\ell!} \right| \leqslant \frac{1}{5} \sup_{x \in [x_0, \infty)} \left| \frac{z_{i-1}(x) - u_\ell(x, \lambda)}{x^\ell/\ell!} \right|.$$

This means that the sequence $z_i(x)/x^\ell$, $i = 1, 2, \ldots$, converges uniformly on $[x_0, \infty)$ to $u_\ell(x, \lambda)/x^\ell$. Since each iteration depends analytically on λ, so does the limit function.

(2.1) and (2.2) follow from the above estimates. The uniform convergence of (2.2) is shown similarly. First take $x_1 \geqslant x_0$ such that $\max(|\Lambda_1|, |\Lambda_2|) \int_{x_1}^{\infty} s^{n-1}|p(s)|\, ds < \varepsilon$ and fix x_1. Then by the above estimates,

$$\left| \frac{u_\ell(x, \lambda)}{x^\ell/\ell!} - 1 \right| < \varepsilon$$

for all $x \in [x_1, \infty)$, $\lambda \in [\Lambda_1, \Lambda_2]$ and the uniform convergence of (2.2) follows.

By the above discussion we get that $u_\ell(x, \lambda)/x^\ell$ is uniformly continuous on some $[x_1, \infty) \times [\Lambda_1, \Lambda_2]$. Next consider the quantities $u_\ell^{(i)}(x_1, \lambda)$, $i = 0, \ldots, n - 1$, as initial values at the fixed point x_1. Since they depend continuously on λ, it follows by standard properties of initial value problems that $u_\ell(x, \lambda)$ is uniformly continuous also on $[a, x_1] \times [\Lambda_1, \Lambda_2]$. Thus $u_\ell(x, \lambda)/x^\ell$ is uniformly continuous on the whole $[a, \infty) \times [\Lambda_1, \Lambda_2]$, $a > 0$.

(ii) The solution $y(x, \lambda)$. Consider a set of $n - 1$ singular boundary conditions

$$\begin{aligned} y^{(i)}(a) &= 0, \quad i = 0, \ldots, k - 2, \\ \lim_{x \to \infty} y^{(i)}(x) &= 0, \quad i = k, \ldots, n - 1, \end{aligned} \tag{2.4}$$

which is generated from (1.13) by the omission of one boundary condition at $x = a$, namely $y^{(k-1)}(a) = 0$. For every λ there exists a solution $y(x, \lambda)$ of (1.1), (2.4). Indeed, it was already seen that the $n - k$ boundary conditions at $x = \infty$ are satisfied by each of the k solutions $u_0, \ldots, u_{k-1}$ and there exists some linear combination of them which satisfies also the $k - 1$ boundary conditions of (2.4) at $x = a$.

The purpose of $y(x, \lambda)$ is to characterize the eigenvalues as those values of λ for which the omitted n-th boundary condition $y^{(k-1)}(a) = 0$ is satisfied. The main steps of the proof are as following: first we show that $y(x, \lambda)$ is essentially unique and it depends analytically on λ. Next it is shown that $y(x, \lambda)$ may have only simple zeros in (a, ∞) and that their numbers increase indefinitely as λ grows. Finally it is proved that the suitable sequence of eigenvalues $\{\lambda_i\}$ indeed exists and the number of zeros of the corresponding $y(x, \lambda_i)$ is determined.

(iii) The uniqueness of $y(x, \lambda)$ and its dependence on λ. To investigate $y(x, \lambda)$, we adopt a technique which is widely used in [2]. For each solution y we wish to count the number of sign changes among the terms of the sequence $y(x)$, $y'(x), \ldots, y^{(n)}(x)$. As some of these terms may vanish at certain isolated values of x, we rather do this on a small right hand side neighbourhood of x, and define $V(x)$ as the number of sign changes in the sequence

$$y(x+\varepsilon), y'(x+\varepsilon), \ldots, y^{(n)}(x+\varepsilon),$$

as $\varepsilon \searrow 0$. The function $V(x)$ is well defined on the domain of definition of the solution y, it is integer valued, and since the first and last terms in the sequence are related by $y^{(n)} = -\lambda p y$, $V(x)$ is even or odd according if $-\lambda p$ is positive or negative. That is,

$$(-1)^{V(x)} = \operatorname{sgn}[-\lambda p]. \tag{2.5}$$

The main property of the sign changes is that *for each solution y of* (1.1), *the corresponding function $V(x)$ is a nonincreasing step function of x.*

This property results from a simple variation on the Budan-Fourier theorem [5, p. 83]. We only outline the idea of the proof as we follow closely [5]. Let y be a solution of (1.1). The corresponding function $V(x)$ is clearly constant on any interval where $y^{(i)}(x) \neq 0$, $i = 0, \ldots, n$. What happens as x crosses a point x_0 such that x_0 is a zero of $y^{(q)}$ of multiplicity m and $y^{(q-1)}(x_0) \neq 0$, $q \geqslant 1$, $0 < m < n-q$? The first and the last terms of the sequence

$$y^{(q-1)}(x), y^{(q)}(x), \ldots, y^{(q+m)}(x),$$

do not change their signs as x grows from $x_0 - \varepsilon$ to $x_0 + \varepsilon$, so the parity of $V(x)$ remains fixed. The signs of the terms $y^{(q)}(x)$, $y^{(q+1)}(x), \ldots, y^{(q+m)}(x)$ alternate at $x = x_0 - \varepsilon$ and all these terms have the same sign at $x = x_0 + \varepsilon > 0$, so $V(x)$ decreases by m as x grows from $x = x_0 - \varepsilon$ to $x = x_0 + \varepsilon$. Between the terms $y^{(q-1)}(x)$, $y^{(q)}(x)$ there is either a loss or a gain of change of sign as x grows, depending on the sign of $y^{(q-1)}(x_0) y^{(q+m)}(x_0) \neq 0$. Hence $V(x)$ decreases by m or by $m \pm 1$ (or is possibly unchanged if $m = 1$). In any case the decrease is by an even number.

If x_0 is a zero of y then also $y^{(n)}(x_0) = 0$ and we have to discuss the case when

$$\begin{aligned} &y^{(q)}(x_0) = \cdots = y^{(n)}(x_0) = y(x_0) = \cdots = y^{(r-1)}(x_0) = 0, \\ &y^{(q-1)}(x_0) \neq 0, y^{(r)}(x_0) \neq 0, \quad r \leqslant q, \end{aligned}$$

in a cyclic order. This is discussed similarly. Hence $V(x)$ is a nonincreasing step function of x.

The role of $V(x)$ is clarified if we calculate it for an eigenfunction y of (1.1), (1.13) (if such y exists). By the boundary conditions at $x = a$, it is clear that y, $y', \ldots, y^{(k)}$ all have the same sign on a small right neighborhood of a, so $V(a) \leqslant n - k$. On the other hand, by the boundary conditions at $x = \infty$ and since the $y^{(i)}$-s have no zeros near ∞, it is clear that $y^{(i)} y^{(i+1)} < 0$, $i = k, \ldots, n-1$, near $x = \infty$. So $V(x) \geqslant n - k$ for large values of x. Since $V(x)$ in nonincreasing,

$V(x) \equiv n-k$ on (a,∞). This enforces that

$$\operatorname{sgn}[-\lambda p] = (-1)^{n-k} \tag{2.6}$$

which is compatible with our choice in (1.13) and the sign of the eigenvalues λ, if they exist, is determined by the boundary conditions (1.13).

Now we calculate $V(x)$ for the solution $y(x,\lambda)$ which is defined by (2.4). By the $k-1$ boundary conditions of (2.4) at $x=a$, it is clear that $V(a) \leqslant n-k+1$ and by the $n-k$ boundary conditions at $x=\infty$, it follows that $V(x) \geqslant n-k$ for large values of x. Since $V(x)$ is nonincreasing, $n-k \leqslant V(x) \leqslant V(a) \leqslant n-k+1$. In addition, the parity of $V(x)$ is determined by (2.5) and (2.6), so it follows that

$$V(x) \equiv n-k \quad \text{on } [a,\infty) \quad \text{for } y=y(x,\lambda). \tag{2.7}$$

Since $y=y(x,\lambda)=\sum_{i=0}^{k-1} c_i u_i(x,\lambda)$, it is asymptotic to some polynomial of order h, $h \leqslant k-1$. Therefore $y^{(i)}(x,\lambda) \to 0$, $i=h+1,\dots,n-1$ as $x \to \infty$ and since these derivatives are eventually non vanishing, we have $y^{(i)}(x,\lambda)\, y^{(i+1)}(x,\lambda) < 0$ for $i=h+1,\dots,n-1$. Consequently $V(x) \geqslant n-(h+1)$. On the other hand $V(x) \equiv n-k$, so $h \geqslant k-1$. Thus $h=k-1$ and

$$y(x,\lambda) \approx \text{const } x^{k-1}. \tag{2.8}$$

Consequently $y(x,\lambda)$ may be normalized by

$$y^{(k-1)}(\infty) = 1. \tag{2.9}$$

The recent considerations imply that the solution $y(x,\lambda)$ is essentially unique. For if there would be two such linearly independent solutions, some linear combination of them would violate (2.8) but would still satisfy boundary conditions (2.4), a contradiction.

Once it is known that $y(x,\lambda)=\sum_{i=0}^{k-1} c_i u_i$ is essentially unique, it may be represented as a determinant

$$y(x,\lambda) = \begin{vmatrix} u_0(x,\lambda) & \cdots & u_{k-1}(x,\lambda) \\ u_0(a,\lambda) & \cdots & u_{k-1}(a,\lambda) \\ \vdots & & \\ u_0^{(k-2)}(a,\lambda) & \cdots & u_{k-1}^{(k-2)}(a,\lambda) \end{vmatrix}. \tag{2.10}$$

This verifies the analytic dependence of $y(x,\lambda)$ on λ. In fact, $y(x,\lambda)$ is real analytic, since our analysis holds only for λ restricted by (2.6). We also see by (2.8) that the cofactor of $u_{k-1}(x,\lambda)$ in the determinant (2.10) must be nonzero, and after the normalization (2.9) it is identically 1.

(iv) The zeros of $y(x,\lambda)$ are simple. Suppose that $y(x,\lambda)$ (or one of its derivatives) has at a point x_0 of (a,∞) a zero of multiplicity $m \geqslant 2$. According to the previous section $V(x)$ decreases at x_0 by an even integer which is at least $m-1$, i.e., at least by 2. But this is impossible since $V(x) \equiv n-k$ for $y=y(x,\lambda)$, so $y(x,\lambda)$ and its derivatives cannot have any multiple zero in (a,∞).

(v) The zeros of $y(x,\lambda)$ are uniformly bounded for $\lambda \in [\Lambda_1,\Lambda_2]$. Recall that $y(x,\lambda)=\sum_{i=0}^{k-1} c_i(\lambda) u_i(x,\lambda)$. By the normalization (2.9) of $y(x,\lambda)$ and the property

(2.2) of the basis, we have $c_{k-1}(\lambda) \equiv 1$. Moreover, according to the representation (2.10), $c_i(\lambda)$ are continuous and so $|c_i(\lambda)| \leqslant M$, $i = 0, \dots, k-2$, $\lambda \in [\Lambda_1, \Lambda_2]$. It was proved in (i) that each $u_\ell(x,\lambda)/x^\ell$, $\ell = 1, 2, \dots$, is uniformly continuous on $[a,\infty) \times [\Lambda_1, \Lambda_2]$, $a > 0$. Consequently, also

$$y(x,\lambda)/x^{k-1} = \sum_{i=0}^{k-1} c_i(\lambda) \frac{u_i(x,\lambda)}{x^i} x^{-(k-1-i)}$$

is uniformly continuous on $[a,\infty) \times [\Lambda_1, \Lambda_2]$.

Now we wish to show that the zeros of $y(x,\lambda)$ are uniformly bounded for every $\lambda \in [\Lambda_1, \Lambda_2]$. Let $x_2 = \max(x_0, 4Mk!)$. Then, using (2.1), we have on $[x_2, \infty)$

$$\begin{aligned}
|y(x,\lambda) - u_{k-1}(x,\lambda)| &= \left| \sum_{i=0}^{k-2} c_i(\lambda) u_i(x,\lambda) \right| \\
&\leqslant x^{k-1} \sum_{i=0}^{k-2} |c_i(\lambda)| \left| \frac{u_i(x,\lambda)}{x^i} \right| x^{-(k-1-i)} \\
&\leqslant x^{k-1} \sum_{i=0}^{k-2} M(2/i!) x_2^{-1} < k(2M/x_2) x^{k-1} \leqslant \frac{x^{k-1}}{2(k-1)!}.
\end{aligned}$$

Consequently, by the left hand side of (2.1),

$$y(x,\lambda) > u_{k-1}(x,\lambda) - \frac{x^{k-1}}{2(k-1)!} > 0$$

on $[x_2, \infty)$ for all $\lambda \in [\Lambda_1, \Lambda_2]$ and all the zeros of $y(x,\lambda)$, if any, are in $[a, x_2)$.

(vi) The zeros of $y(x,\lambda)$ for small λ and for large λ. In (v) we verified that $y(x,\lambda)/x^{k-1}$ is uniformly continuous on $[a,\infty) \times [0,\Lambda]$. But $y(x,0) = (x-a)^{k-1}/(k-1)! \neq 0$ on (a,∞) and has precisely $k-1$ zeros at $x = a$, so $y(x,\lambda)$ may have no zero in (a,∞) for sufficiently small values of λ.

For sufficiently large values of λ, we apply the following result [2, Lemma 10.2]:

Given a family of solutions $y(x,\lambda)$ such that none of them satisfies (1.5) (*if* $\lambda p(x) < 0$) *or* (1.6) (*if* $(-1)^n \lambda p(x) < 0$), *then on any given interval $y(x,\lambda)$ has a zero provided that $|\lambda|$ is sufficiently large.*

Our family $y(x,\lambda)$ is of this type since $V(x) \equiv n-k \neq 0, n$ for $y(x,\lambda)$. Consequently $y(x,\lambda)$ has arbitrary many zeros in (a,∞) for sufficiently large values of λ.

(vii) The existence of eigenvalues. We follow a method used in [11]. For each $i = 1, 2, \dots$ let

$$L_i = \{\lambda | (-1)^{n-k} \lambda p < 0, y(x,\lambda) \text{ has at least } i \text{ simple zeros in } (a,\infty)\}.$$

L_i is nonempty, since according to (vi), $y(x,\lambda)$ has arbitrarily many zeros for sufficiently large values of λ. L_i is an open set since the number of the simple zeros of $y(x,\lambda)$ in (a,∞) is preserved under a small change of λ. For sake of simplicity let us assume that the relevant values of λ in L_i are positive and put $\lambda_i = \inf L_i$. (In the opposite case, when $\lambda < 0$, take $\lambda_i = \sup L_i$). Our aim is to show that λ_i is an eigenvalue of (1.1), (1.13), $y(x, \lambda_i)$ is the corresponding eigenfunction and it has precisely $i-1$ zeros in (a, ∞).

By the discussion above, $0 < \lambda_i < \infty$ and, due to $L_i \subset L_{i+1}$, we have $\lambda_i \leqslant \lambda_{i+1}$. From now on let i be fixed. Choose a sequence $\mu_i(j) \in L_i$, $j = 1, 2, \ldots,$ which converges to $\lambda_i = \inf L_i$ as $j \to \infty$. By the definition of L_i each of the solutions $y(x, \mu_i(j))$ has at least i zeros in (a, ∞), all of them simple, say $a < z_1(j) < \cdots < z_i(j)$. Since $\mu_i(j) \in [0, \lambda_i + 1]$ for all j, it follows by (v) that all the zeros $z_1(j), \ldots, z_i(j), j = 1, 2, \ldots$ are bounded in a fixed interval $[a, x_2]$, where x_2 is independent of j (but is determined, of course, by λ_i).

Let us extract convergent subsequences of the sequences $\{z_1(j)\}, \ldots, \{z_i(j)\}$ which will be denoted, without loss of generality, by the same notation. Then $z_1(j) \to z_1, \ldots, z_i(j) \to z_i$ as $j \to \infty$. Of course $a \leqslant z_1 \leqslant \cdots \leqslant z_i \leqslant x_2$. Since $y(x, \mu_i(j)) \to y(x, \lambda_i)$ as $j \to \infty$ as well as their derivatives, it follows that $z_1, \ldots, z_i$ are i zeros of $y(x, \lambda_i)$, multiplicities counted. But by (iv) the zeros of $y(x, \lambda_i)$ in (a, ∞) are simple, so $a \leqslant z_1 < \cdots < z_i$. If $z_1 \neq a$, then $y(x, \lambda_i)$ has at least i simple zeros in $(a, x_2 + 1)$. Due to the continuous dependence on λ, $y(x, \lambda)$ has also i simple zeros for all λ sufficiently close of λ_i, contradicting the definition of λ_i as $\inf L_i$. Consequently we must have $z_1(j) \searrow z_1 = a$ and $y(x, \lambda_i)$ may have at most $i-1$ zeros in (a, ∞).

Every $y(x, \lambda)$ has, according to (2.4), at least $k-1$ zeros at $x = a$ and since $z_1(j) \searrow a$ as $j \to \infty$, it follows that $y(x, \lambda_i)$ has (at least) k zeros at $x = a$. No zero other than $z_1(j)$ can tend to $x = a$ as $j \to \infty$ otherwise $y(x, \lambda_i)$ would have more than k zeros at $x = a$, which contradicts (2.7). So the zero of $y(x, \lambda_i)$ at $x = a$ is exactly of multiplicity k. Thus $y(x, \lambda_i)$ is an eigenfunction of (1.1), (1.13), and it has precisely $i-1$ zeros in (a, ∞). This characterization shows that $\lambda_i < \lambda_{i+1}$.

The sequence λ_i is unbounded. If, on the contrary, $\lambda_i \in [0, \Lambda]$ for all i, then $\lambda_i \to \tilde{\lambda}$ and we could extract from the sequence $y(x, \lambda_i)$ a subsequence which converges to a nontrivial solution $y(x, \tilde{\lambda})$ of (1.1) with infinitely many zeros in some $[a, x_2]$, which is impossible.

(viii) These are all the eigenvalues. Are these λ_i-s all the eigenvalues of (1.1), (1.13) or perhaps there are additional ones which are not tractable by the process of (vii)? More explicitly, it is pointed out in (vii) that when $\lambda \in L_i$ decreases toward $\inf L_i$, the first simple zero of $y(x, \lambda)$ in (a, ∞) tends to a. Is this the only way that zeros of $y(x, \lambda)$ meet $x = a$ or are there more values of λ, other than the numbers $\inf L_i$, for which some zeros may meet $x = a$ and thus generate additional eigenvalues and eigenfunctions?

This question is not elaborated in [11]. It may be shown as in [2, Lemma 10.6] that there exists only one eigenfunction which has precisely $i-1$ zeros in (a, ∞). Hence the above scenario never happens and all eigenvalues are indeed obtained by (vii).

This completes the proof of Theorem 2 (Discrete Spectrum Case). □

3. The Continuous Spectrum Case

In this section we prove Theorem 3. It is assumed that $(1.11)_\alpha$ holds for some $\alpha < n-1$ but (1.12) does not hold and that $\lambda \neq 0$. In this case all solutions of (1.1) are nonoscillatory but none of them is asymptotic to any polynomial.

It was proved in Proposition 2′ that for every $\lambda \neq 0$, $\operatorname{sgn}[-\lambda p] = (-1)^{n-k}$, the $k+1$ solutions $u_0, \ldots, u_k$ satisfy the $n-k$ singular boundary conditions of (1.13) at $x = \infty$ and some linear combination of them satisfies the k boundary conditions of (1.13) at $x = a$. Thus, for every $\lambda \neq 0$ the boundary value problem (1.1), (1.13) has a solution, say $y(x,\lambda) = \sum_{i=0}^{k} c_i u_i$. For $\lambda = 0$ this does not hold, since only k solutions satisfy (1.14) and, as mentioned in the proof of Proposition 2′, $\lambda = 0$ belongs to the discrete case.

There are similarities and differences between the discrete and continuous cases. For the present $y(x,\lambda)$, we see as in (2.7) that

$$V(x) \equiv n-k, \quad a \leqslant x < \infty, \tag{3.1}$$

and the zero of $y(x,\lambda)$ at $x = a$ must be precisely of order k. Consequently we conclude that $y(x,\lambda)$ is essentially unique and it may be represented as

$$y(x,\lambda) = \begin{vmatrix} u_0(x,\lambda) & \cdots & u_k(x,\lambda) \\ u_0(a,\lambda) & \cdots & u_k(a,\lambda) \\ \vdots & & \\ u_0^{(k-1)}(a,\lambda) & \cdots & u_k^{(k-1)}(a,\lambda) \end{vmatrix}. \tag{3.2}$$

As in Section 2(iv), the zeros of this $y(x,\lambda)$ are simple, too.

The normalization of $y(x,\lambda)$ is completely different from that in the discrete case. Since

$$y^{(k-1)}(x,\lambda) \to \infty, \qquad y^{(k)}(x,\lambda) \to 0 \quad \text{as } x \to \infty,$$

no normalization at $x = \infty$ makes sense. Instead, a proper normalization may be

$$y^{(k)}(a,\lambda) = 1.$$

In contradiction with the discrete case, we do not claim here uniform continuity of $y(x,\lambda)$ on $[a,\infty)$. All we can say is that when $\lambda \to \tilde{\lambda}$, $y(x,\lambda)$ converges to $y(x,\tilde{\lambda})$, uniformly on compact intervals. See [3, Example 1].

We neither claim that the zeros are uniformly bounded for $\lambda \in [\Lambda_1, \Lambda_2]$. In fact, the opposite is true and this is a key ingredient in the understanding of the Continuous Spectrum Case. This is discussed next.

(i) How do the zeros of $y(x,\lambda)$ vary? As in Section 2(vi) we conclude that $y(x,\lambda)$ has arbitrary many zeros in (a,∞) for sufficiently large values of λ. Where do these zeros come from as λ varies?

Since $y(x,\lambda)$ is continuous in λ and its zeros in (a,∞) are always simple, these zeros are continuous functions of λ. Due to their simplicity, the zeros cannot appear by splitting from double zeros (or disappear by coalescing into double zeros). Thus their number may change as λ varies only as they enter or leave the interval (a,∞) through its endpoints. But since the zero of $y(x,\lambda)$ at $x = a$ is precisely of order k for every λ, no additional zero may approach a. Thus a zero is added to $y(x,\lambda)$ in (a,∞) only when it enters the interval through $x = \infty$.

Let $z_1^{(0)}(\lambda) < z_2^{(0)}(\lambda) < \cdots < z_h^{(0)}(\lambda)$ be all the zeros of $y(x,\lambda)$ which approach $+\infty$ as λ tends to some $\tilde{\lambda}$. Due to the k boundary conditions $y^{(i)}(a) = 0$, $i = 0,\ldots,k-1$, the derivatives $y^{(i)}(x,\lambda)$, $i = 1,\ldots,k$, also have zeros

$z_1^{(i)} < z_2^{(i)} < \cdots < z_h^{(i)}$ such that

$$z_1^{(i)} < z_1^{(i-1)} < z_2^{(i)} < z_2^{(i-1)} < \cdots < z_h^{(i)} < z_h^{(i-1)}, \quad i = 1, \ldots, k.$$

Since their total number is unknown yet, we agree that $z_m^{(i)}$ denotes the zero of $y^{(i)}(x, \lambda)$ which is closest to $z_m^{(i-1)}$ on its left hand side. Due to the $n-k$ boundary conditions of (1.13) at $x = \infty$, the higher derivatives $y^{(i)}(x, \lambda)$, $i = k+1, \ldots, n$, also have zeros which are ordered now so that

$$z_1^{(i-1)} < z_1^{(i)} < z_2^{(i-1)} < z_2^{(i)} < \cdots < z_h^{(i-1)} < z_h^{(i)}, \quad i = k+1, \ldots, n.$$

Our first claim is that all these zeros tend to $+\infty$ as $\lambda \to \tilde{\lambda}$. Suppose on the contrary that this is false and for some t, $1 \leqslant t \leqslant k$, $z_1^{(t-1)} \to \infty$ while $z_1^{(t)}$ which is on its left hand side remains bounded by some constant K as some subsequence of λ-s converges to $\tilde{\lambda}$. Obviously $y^{(t-1)}(x, \lambda) y^{(t)}(x, \lambda) < 0$ on $(z_1^{(t)}, z_1^{(t-1)})$ and as $\lambda \to \tilde{\lambda}$, we get

$$y^{(t-1)}(x, \tilde{\lambda}) y^{(t)}(x, \tilde{\lambda}) < 0 \quad \text{on } [K, \infty).$$

Thus $y(x, \tilde{\lambda})$ satisfies inequalities (1.4) for some $q \leqslant t-1 < k$, in contradiction with (3.1). A similar argument applies to the zeros of the derivatives of order bigger than k.

Let all the zeros of $y^{(i)}(x, \lambda)$, $i = 0, \ldots, n-1$, which approach $x = +\infty$ as $\lambda \to \tilde{\lambda}$ (and none but them) be in some interval $[M, \infty)$. By using the k boundary conditions at $x = a$ and the $n-k$ boundary conditions at $x = \infty$, we have seen that each $y^{(t)}(x, \lambda)$, $t = 1, \ldots, n$, has at least h zeros in $[M, \infty)$. If between two consecutive zeros of $y^{(t-1)}(x, \lambda)$ there is more than one zero of $y^{(t)}(x, \lambda)$ or if $y^{(t)}(x, \lambda)$ has any zero in $(z_h^{(t-1)}, \infty)$, then by the same argument we conclude that also $y^{(t+1)}(x, \lambda), \ldots, y^{(n)}(x, \lambda)$ have strictly more than h zeros in $[M, \infty)$. Since $y^{(n)} = -\lambda p y$, this leads to a contradiction with the number of zeros of $y^{(t-1)}(x, \lambda)$. Hence, between each two consecutive zeros of $y^{(t-1)}(x, \lambda)$ which tend to $+\infty$, $y^{(t)}(x, \lambda)$ has exactly one zero and no other zeros in $[M, \infty)$.

The next aim is to show that in fact $h = 1$, i.e., only one zero of $y(x, \lambda)$ may approach $+\infty$ as $\lambda \to \tilde{\lambda}$. Suppose on the contrary that $h \geqslant 2$ and $z_1^{(i)}(\lambda), \ldots, z_h^{(i)}(\lambda) \to \infty$ as $\lambda \to \tilde{\lambda}$, $i = 0, \ldots, n$. $y(x, \lambda)$ eventually satisfies inequalities (1.4) with $q = k$ (since $V(x) = n-k$):

$$y^{(i)}(x, \lambda) > 0 \quad \text{on } (z_h^{(i)}(\lambda), \infty), \quad i = 0, \ldots, k,$$
$$(-1)^{i-k} y^{(i)}(x, \lambda) > 0, \quad \text{on } (z_h^{(i)}(\lambda), \infty), \quad i = k, \ldots, n.$$

On the other hand, since equation (1.1) is $(k, n-k)$-disfocal for every λ, there exists a solution u such that

$$u^{(i)} > 0, \quad i = 0, \ldots, k,$$
$$(-1)^{i-k} u^{(i)} > 0, \quad i = k, \ldots, n,$$

on some $[M_2, \infty)$, where M_2 is some fixed constant, and M_2 is the same for every λ in $[\tilde{\lambda} - 1, \tilde{\lambda} + 1]$.

Let λ be so close to $\tilde{\lambda}$ that $z_1^{(k)}(\lambda) > M_2$, i.e., all the diverging zeros are in $[M_2, \infty)$. First we discuss the case when h is an even integer, say $h = 2r \geqslant 2$. Then $y(x, \lambda) < 0$ in $(z_1^{(0)}, z_2^{(0)}) \cup \cdots \cup (z_{2r-1}^{(0)}, z_{2r}^{(0)})$ while, of course, $u > 0$ there. Consider $y(x, \lambda) + \alpha u$ as α grows, starting from 0. For $\alpha = 0$ the zeros of $y + \alpha u$ are the $h = 2r$ simple zeros of $y(x, \lambda)$ and as α grows, the zeros that emerge from the $z_j^{(0)}$-s come nearer until one of the pairs coalesces inside some interval $(z_{2q-1}^{(0)}, z_{2q}^{(0)})$. For the same α, $y + \alpha u > 0$ on $[M_2, z_1^{(0)})$ and on $[z_{2r}^{(0)}, \infty)$. A similar thing happens for the other derivatives of $y + \alpha u$. Let A be the smallest value of α for which two zeros of one of the derivatives meet. It is impossible that this will happen for all the derivatives at once since as x crosses a double zero, $V(x)$ decreases by 2 and this cannot happen n times for the solution $y + Au$. Therefore some derivative $(y + Au)^{(t-1)}$ still has h distinct zeros while the consecutive one, $(y + Au)^{(t)}$ has only $h - 2$ changes of sign in $[M, \infty)$, contradiction!

When h is odd, say $h = 2r + 1 \geqslant 3$, the situation is slightly different. Now $y(x, \lambda) > 0$ in $(z_1^{(0)}, z_2^{(0)}) \cup \cdots \cup (z_{2r-1}^{(0)}, z_{2r}^{(0)}) \cup (z_{2r+1}^{(0)}, \infty)$ and we follow the zeros of $y(x, \lambda) - \alpha u$ as α grows, starting from 0 till a multiple zero appears. The only difference from the previous case is that here a zero coming from ∞ may appear in $(z_{2r+1}^{(0)}, \infty)$ or a zero which emerges from $z_{2r+1}^{(0)}$ may disappear at ∞ for a certain α. Except this detail, a contradiction is encountered for the first value of α for which some $(y + \alpha u)^{(t)}$ has a double zero.

Thus, it is verified that the number of zeros of $y(x, \lambda)$ in (a, ∞) varies with λ only when one simple zero approaches $x = \infty$. See Example 4.

(ii) $y(x, \lambda)$ **for small values of** λ. The behaviour of $y(x, \lambda)$ for small values of λ is tricky. Let $w(x) = \lim_{\lambda \to 0} y(x, \lambda)$ where λ tends to zero, possibly through some suitable subsequence. Note that $y(x, \lambda)$ is not defined at all for $\lambda = 0$. $w(x)$ is obviously a solution of $y^{(n)} = 0$ and it satisfies the k boundary conditions of (1.13) at $x = a$, i.e., it is a polynomial of order k at least. It is evident that such polynomial cannot satisfy also the $n - k$ boundary conditions of (1.13) at $x = \infty$. This is not surprising since, as $y(x, \lambda)$ is not uniformly continuous on $[a, \infty)$, there is no reason to expect that boundary conditions at $x = \infty$ are preserved by the limit $\lim_{\lambda \to 0} y(x, \lambda)$.

At every fixed point x_0, $y^{(i)}(x_0, \lambda) \to w^{(i)}(x_0)$, $i = 0, \ldots, n$, as $\lambda \to 0$. If $w(x)$ is a polynomial of order h, we have for large values of x_0,

$$\lim_{\lambda \to 0} y^{(i)}(x_0, \lambda) = w^{(i)}(x_0) > 0, \quad i = 0, \ldots, h.$$

If $h > k$, $y(x_0, \lambda), \ldots, y^{(h)}(x_0, \lambda) > 0$ contradict (3.1). Hence $w(x)$ is exactly of order k and due to the boundary conditions at $x = a$ and the normalization, we have

$$\lim_{\lambda \to 0} y(x, \lambda) = w(x) = (x - a)^k / k!.$$

As in (i), we conclude that $y(x, \lambda)$ may have at most one zero $z_1 = z_1(\lambda)$ that tends to $+\infty$ as $\lambda \to 0$. If $y(x, \lambda_i)$ has exactly one zero in (a, ∞) for small values of λ, it follows by the definition that $\lambda_1 = 0$. Otherwise, if $y(x, \lambda)$ has no zero in (a, ∞) for small values of λ, then $\lambda_1 \neq 0$. In fact, we guess that this is always the case.

(iii) The existence of eigenvalues. The other details of Theorem 3 are verified similarly to those of Theorem 2. For example, for large values of λ, $y(x, \lambda)$ has arbitrary many zeros. It turns out that λ_i are those values of λ for which a single simple zero of $y(x, \lambda)$ enters (a, ∞) through $x = \infty$. Also the sequence λ_i cannot be bounded. Otherwise, if $\lambda_i \to \tilde{\lambda}$, the limit solution $\lim_{i\to\infty} y(x, \lambda_i) = y(x, \tilde{\lambda})$ would have infinitely many zeros in (a, ∞), which contradicts the eventual disconjugacy of equation (1.1).

This completes the proof of Theorem 3. □

4. Comparison of the Discrete and Continuous Cases

In both the Discrete Spectrum Case and the Continuous Spectrum Case we utilized families of solutions $y(x, \lambda)$. At first sight they seem different: In the discrete case, $y(x, \lambda)$ is defined by the $n - 1$ boundary conditions (2.4), while in the continuous case, $y(x, \lambda)$ satisfies the n boundary conditions (1.13). However, the correct way to examine these solutions is not through the number of boundary conditions but rather by their codimensions, i.e., the dimensions of the solution spaces which are determined by them.

In the discrete case the $k - 1$ regular boundary conditions of (2.4) at $x = a$ are obviously satisfied by $n - k + 1$ solutions; the $n - k$ singular boundary conditions of (2.4) at $x = \infty$ are satisfied, according to Proposition $2'$, precisely by the k solutions $u_0, \ldots, u_{k-1}$. Hence the $n - 1$ boundary conditions (2.4) have in this case codimensions $n - k + 1$, k, respectively.

For the continuous case the k conditions of (1.13) at $x = a$ are satisfied by $n - k$ solutions, and the $n - k$ singular conditions of (1.13) at $x = \infty$ are satisfied precisely by the $k + 1$ solutions $u_0, \ldots, u_{k-1}, u_k$. Now the n boundary conditions (1.13) which define $y(x, \lambda)$ in the continuous case, have codimensions $n - k$, $k + 1$, respectively. Thus, the two boundary conditions which define the solutions $y(x, \lambda)$ in the two cases are related by the interchanges $a \longleftrightarrow \infty$ and $k - 1 \longleftrightarrow k$.

The duality between the roles of the endpoints $x = a$ and $x = \infty$ goes further on. For the discrete case zeros of $y(x, \lambda)$ enter the interval (a, ∞) through $x = a$ while in the continuous case this happens through $x = \infty$. In the discrete case $y(x, \lambda)$ is normalized by $y^{(k-1)}(\infty) = 1$ while in the continuous case the natural normalization is $y^{(k)}(a) = 1$.

There is another way to observe the duality between the boundary conditions which define the corresponding solutions $y(x, \lambda)$ in the two cases. Recall that in the continuous case, the singular boundary conditions of (1.13) are designated to select the $k + 1$ solutions $u_0, \ldots, u_{k-1}, u_k$. But the same solution space can be selected also by a different set of boundary conditions. It was seen by (1.20) that in the continuous case u_{q-1}, u_q, $q \equiv (\operatorname{mod} k)$, satisfy

$$u^{(i)} \to \infty, \quad i = 0, \ldots, q - 1,$$
$$u^{(i)} \to 0, \quad i = q \ldots, n - 1.$$

Therefore, the $n - k - 1$ singular boundary conditions

$$\lim_{x\to\infty} y^{(i)}(x) = 0, \quad i = k + 1, \ldots, n - 1,$$

allow $q \leqslant k$, i.e., $u_0, \dots, u_{k-1}$, u_k and reject $q \geqslant k+2$, i.e., u_{k+1}, $u_{k+2}, \dots, u_{n-1}$. Consequently, in the continuous case the $n-1 = k + (n-k-1)$ conditions

$$\begin{aligned} y^{(i)}(a) &= 0, \quad i = 0, \dots, k-1, \\ \lim_{x\to\infty} y^{(i)}(x) &= 0, \quad i = k+1, \dots, n-1, \end{aligned} \tag{4.1}$$

define the same $y(x, \lambda)$ as do the n conditions (1.13). Thus, (4.1) is the continuous case analogue to the $n-1 = (k-1) + (n-k)$ boundary conditions (2.4) of the discrete case and it is again achieved by the interchange $k-1 \longleftrightarrow k$.

From the proof of Theorems 2 and 3 it turns out that in both cases $y(x, \lambda_i)/x^k \to 0$ as $x \to \infty$. Indeed, in Theorem 2 we must have $y(x, \lambda) = \sum_{i=0}^{k-1} c_i u_i(x, \lambda) \approx x^{k-1}/(k-1)!$. In the proof of Proposition 2′ we saw that in the continuous case $y^{(k)}(x, \lambda) = \sum_{i=0}^{k} c_i u_i^{(k)}(x, \lambda) \to 0$. Therefore,

Corollary. *In both the discrete and the continuous spectrum cases the boundary conditions* (1.13) *can be replaced by*

$$y^{(i)}(a) = 0, \quad i = 0, \dots, k-1,$$

$$\lim_{x\to\infty} y(x)/x^k = 0.$$

5. Examples

We start with some equations which satisfy the integrability condition (1.12) and Theorem 2.

Example 1.

$$\begin{aligned} &y^{(4)} - \lambda x^{-8} y = 0, \\ &y(1) = y'(1) = y''(\infty) = y'''(\infty) = 0. \end{aligned} \tag{5.1}$$

The equation satisfies (1.12) with $n = 4$ and a basis of solutions with convenient asymptotic behaviour as $x \to \infty$ is

$$\begin{aligned} u_0 &= x^3[\sinh(\alpha/x) - \sin(\alpha/x)] \to \alpha^3/3, \\ u_1 &= x^3[\cosh(\alpha/x) - \cos(\alpha/x)] \to \alpha^2 x, \\ u_2 &= x^3 \sinh(\alpha/x) \to \alpha x^2, \\ u_3 &= x^3 \cosh(\alpha/x) \to x^3, \quad \alpha = \lambda^{1/4}. \end{aligned}$$

Hence we look for a combination $y = c_0 u_0 + c_1 u_1$ with a double zero at $x = 1$. This happens when $\lambda = \alpha^4 = \left(\frac{\pi}{2} + \pi n\right)^4$, $n = 1, 2, \dots$.

Similarly, for the problem

$$\begin{aligned} &y^{(4)} + \lambda x^{-8} y = 0, \\ &y(1) = y'(\infty) = y''(\infty) = y'''(\infty) = 0, \end{aligned} \tag{5.2}$$

$\lambda > 0$, the solution $y(x,\lambda) = x^3[\cosh(\alpha/x)\sin(\alpha/x) - \cos(\alpha/x)\sinh(\alpha/x)]$, $\alpha = \lambda^{1/4}$, satisfies the three boundary conditions at $x = \infty$. $y(1) = 0$ holds when $\tan(\alpha) = \tanh(\alpha)$, $\alpha > 0$, and the eigenvalues are $\lambda_n = (\alpha_n)^4 \approx (\frac{\pi}{2} + \pi n)^4$.

Example 2. Consider the eigenvalue problem

$$y'' + \lambda(\cosh x)^{-2} y = 0, \quad 0 \leqslant x < \infty,$$
$$y(0) = y'(\infty) = 0. \tag{5.3}$$

Here $(\cosh x)^{-2} \approx 4e^{-2x}$ satisfies (1.12). The equation is transformed by $t = \tanh x = (e^{2x} - 1)/(e^{2x} + 1)$, $y(x) = v(t)$, into the Legendre equation

$$(1 - t^2)v'' - 2tv' + \lambda v = 0, \quad 0 \leqslant t < 1.$$

For $\lambda = 2n(2n-1)$, the Legendre polynomial $P_{2n-1}(t)$ satisfies $P_{2n-1}(0) = 0$, $P_{2n-1}(1) \neq 0$ and has exactly $n - 1$ simple zeros in (0, 1). Consequently

$$\lambda_n = 2n(2n-1) \qquad y_n(x) = P_{2n-1}\left(\frac{e^{2x} - 1}{e^{2x} + 1}\right), \quad n = 1, 2, \ldots$$

Example 3. The boundary value problem

$$y'' + \lambda(x^2 + 1)^{-2} y = 0, \quad 0 \leqslant x < \infty,$$
$$y(0) = y'(\infty) = 0, \tag{5.4}$$

satisfies (1.12). The general solution of the equation is

$$\sqrt{x^2 + 1}[c_0 \cos(\sqrt{\lambda + 1}\arctan x) + c_1 \sin(\sqrt{\lambda + 1}\arctan x)]$$

[6, 2.365], and its bounded solution is $\sqrt{x^2+1}\sin(\sqrt{\lambda+1}(\arctan x - \frac{\pi}{2}))$, i.e., $y = \sqrt{x^2+1}\sin(\sqrt{\lambda+1}\arcsin(x^2+1)^{-1/2})$. It satisfies $y(0) = 0$ if $\sqrt{\lambda+1} = 2n$, $n = 1, 2, \ldots$ With the notation $\sin\theta = (x^2+1)^{-1/2}$, this is $y = \sin(2n\theta)/\sin\theta = U_{2n-1}(\cos\theta)$, where U_k denotes the Chebychev polynomials of the second type. Hence,

$$\lambda_n = 4n^2 - 1, \qquad u_n(x) = U_{2n-1}\left(\frac{x}{\sqrt{x^2+1}}\right), \quad n = 1, 2, \ldots$$

Now we turn to an example which satisfies $(1.11)_\alpha$ and demonstrates Theorem 3.

Example 4. The differential equation

$$y'' + \frac{\lambda}{x^2 \log x} y = 0, \quad 1 < x < \infty, \tag{5.5}$$

satisfies $(1.11)_\alpha$ for every $\alpha < 1$ but not (1.12). (5.5) has a solution

$$y(x,\lambda) = \log x\left[1 + \sum_{k=1}^{\infty} \frac{(1-\lambda)(2-\lambda)\cdots(k-\lambda)}{k!(k+1)!}(\log x)^k\right] \tag{5.6}$$

and for every integer $\lambda = n$, this solution is

$$y(x,n) = \log x\, L_{n-1}^{(1)}(\log x), \tag{5.7}$$

where $L_m^{(\alpha)}(t)$ denotes the Laugerre polynomials. For every $\lambda \neq 0$ the solution $y(x,\lambda)$ satisfies the boundary conditions $y(1) = y'(\infty) = 0$ and for every λ, $n-1 < \lambda \leqslant n$, $y(x,\lambda)$ has precisely $n-1$ simple zeros in $(1,\infty)$. Hence $\lambda_n = n$, $n = 1, 2, \ldots$

Proof. The substitution $t = \log x$, $y(x) = tv(t)$ transforms Equation (5.5) into

$$tv'' + (2-t)v' + (\lambda - 1)v = 0, \quad 0 < t < \infty.$$

This is Laguerre's differential equation of type $\alpha = 1$ (with λ replaced by $\lambda - 1$), and its only solution which is regular at $t = 0$ is

$$v(t,\lambda) = 1 + \sum_{k=1}^{\infty} \frac{(1-\lambda)(2-\lambda)\cdots(k-\lambda)}{k!(k+1)!} t^k.$$

All other solutions have a logarithmic singularity at $t = 0$. This yields (5.6). When λ is an integer, say $\lambda = n$, v reduces to a Laguerre polynomial and $y(x,n) = \log x L_{n-1}^{(1)}(\log x)$.

It is shown in [3] that for noninteger λ, $y(x,\lambda) \approx x/(\log x)^{\lambda}$ as $x \to \infty$. This, together with (5.7), shows that $y(x,\lambda)$ satisfies the boundary conditions $y(1) = y'(\infty) = 0$ for every $\lambda > 0$.

By known properties of Laguerre polynomials, $y(x,n)$ has precisely $n-1$ simple zeros on $(1,\infty)$. By Sturm's comparison theorem it follows that as λ increases, the number of zeros of $y(x,\lambda)$ in $(1,\infty)$ does not decrease. Hence our claim will be completed if we show that $y(x,\lambda)$, $n < \lambda < n+\varepsilon$, has in $(1,\infty)$ precisely one more zero than $y(x,n)$ has, namely n zeros. To this purpose we rewrite (5.6) as

$$\begin{aligned} y(x,\lambda) = {} & \sum_{k=0}^{n-1} \frac{(1-\lambda)(2-\lambda)\cdots(k-\lambda)}{k!(k+1)!} (\log x)^{k+1} \\ & + (-1)^n (\lambda-1)(\lambda-2)\cdots(\lambda-n)(\log x)^{n+1} \\ & \times \sum_{k=n}^{\infty} \frac{(n+1-\lambda)\cdots(k-\lambda)}{k!(k+1)!} (\log x)^{k-n} \\ \equiv {} & P_\lambda(x) + Q_\lambda(x). \end{aligned} \tag{5.8}$$

Let x_0 be an arbitrary fixed point larger than the zeros of $L_{n-1}^{(1)}(\log x)$. Then on the compact interval $[1, x_0]$,

$$P_\lambda(x) \to \log x L_{n-1}^{(1)}(\log x), \qquad Q_\lambda(x) = O(|\lambda - n|) \quad \text{as } \lambda \to n.$$

Therefore $y(x,\lambda)$ has exactly $n-1$ simple zeros on $(1, x_0]$ as $\lambda \to n$.

Now consider $y(x,\lambda)$ on $[x_0,\infty)$. By the choice of x_0 and for $n < \lambda < n+1$, $(-1)^{n-1} P_\lambda(x) > 0$ on $[x_0,\infty)$, and in fact

$$0 < (-1)^{n-1} P_\lambda(x) \leqslant A(\log x)^n \quad \text{on } [x_0,\infty).$$

On the other hand, for $n < \lambda < n+1$ all terms of $Q_\lambda(x)$ are of the same sign, so

$$(-1)^n Q_\lambda(x) \geqslant B(\log x)^{n+1} > 0 \quad \text{for } x > 1.$$

Also $Q_\lambda(x) = O(\lambda - n)$ on any compact set. Thus, at $x = x_0$, $Q_\lambda(x_0)/P_\lambda(x_0)$ is as small as we wish provided that we take a λ sufficiently close to n. For this fixed λ, $Q_\lambda(x)/P_\lambda(x) \to -\infty$ as $x \to \infty$. Consequently, $y(x,\lambda) = P_\lambda(x)[1 + Q_\lambda(x)/P_\lambda(x)]$ must have a zero in (x_0, ∞) as $\lambda \to n^+$. So our $y(x,\lambda)$ has at least n zeros in $(1,\infty)$. □

The boundary value problem $y(a) = y'(\infty) = 0$, $a > 1$, can be treated as well. Here one needs to consider also the second solution of (5.5), $z(x,\lambda) = y(x,\lambda)\int y^{-2}(x,\lambda), dx$. See [3] for details.

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