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ON THE APPROXIMATION OF THE JACOBI POLYNOMIALS

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ABSTRACT. New approximations of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are provided on the interval $(1,\infty)$. The approximations are given explicitly in terms of some expressions derived from a coefficient of a related hypergeometric equation and in terms of certain perturbation terms. The perturbation terms are essentially resolvent series that are absolutely convergent. These series converge uniformly for all positive n,α and β in some semi-infinite interval and for x in the interval $[1,\infty)$. They are shown to converge faster then a geometric series, where the ratio of successive terms is $\pi^2/32$. We thus also demonstrate that it is possible to approximate the Jacobi polynomials in the vicinity of $x = 1$ as well as on the entire interval $[1,\infty)$ without resorting to Bessel functions. The asymptotic approximation of $P_n^{(\alpha+an,\beta+bn)}(x)$ on $[1,\infty)$, that is related to the Racah coefficients and to a one-dimensional quantum walk, is also pointed out.

1. Introduction. It was pointed out in, e.g., [**9**] that the problem of approximating special functions uniformly for various ranges of the independent variable and other parameters, on which the special function depends, is one of the most difficult problems of the theory. In this article we take a fresh look at the problem of approximating the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ uniformly with respect to the parameters n,α,β and x where α and β vary in the interval $[1,\infty)$, $n = 0,1,2,\dots$, and x varies on the interval $[1,\infty)$. We will employ rapidly convergent approximations, whenever asymptotic formulas in the classical sense, fail to deliver.

Jacobi polynomials are widely studied in the literature. Some of the historic methods applied are Darboux's method and Hilb's type formulas. Reference [**15**] produces uniform approximations for the Jacobi polynomials on $[1,\infty)$ utilizing transformations of the dependent

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and independent variables of solutions of second order linear ODE's. There is a substantial interest in the asymptotic approximation of the Jacobi polynomials as $\alpha \to \infty$ or as $\beta \to \infty$ and especially as $n \to \infty$. See [**14, 15**] and the references therein. For applications of the Jacobi polynomials in physics, see [**2, 3**]. The difficulty in deriving these approximations depends on whether x is inside the interval $(-1, 1)$ or not. The treatment of the vicinities of $x = 1, -1$ poses special challenges as is seen from [**14, 15**]. The approximation of Jacobi polynomials outside their interval of orthogonality, namely on $[1, \infty)$, is related to a question regarding eigenvalues of some large matrices, [**13, 19**]. Our approach could supplement the studies in [**1, 11, 14, 15, 17**] in the specialized case of the Jacobi polynomials.

We shall use matrix methods of asymptotic analysis that will help us derive one definite set of asymptotic formulas (4.4)–(4.7) that approximate a fundamental set of solutions of the equation

$$y'' = p(x)y \tag{1.1}$$

and their derivatives. This set of formulas is applied to the special case of the hypergeometric equation

(1.2)
$$y'' = \left[\frac{\alpha^2 - 1}{4(x-1)^2} + \frac{\beta^2 - 1}{4(x+1)^2} + \frac{n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)/2}{x^2-1}\right] y.$$

One of the functions, (4.4), in the fundamental set of solutions is singled out. It will allow us to obtain the desired approximations for the Jacobi polynomials.

The general approach in this work is in the spirit of "WKB methods" and "L-G" (Liouville-Green) approximations methods. However, a few special features in this work as well as in [**4–6**] are noteworthy. Unlike the celebrated L-G approximation, the formulas (4.4)–(4.7) do not fail at a so called "turning point," where $p(x) = 0$. Neither do they fail at regular singular points. The formulas (4.4)–(4.7) provide the same prescription whether we approximate solutions of (1.1) at an irregular singular point, at a regular singular point or at a turning point. (It is possible to extract the celebrated L-G approximation from (4.4)–(4.7) as a particular case).

We also differ from the classical account in [**11**]. We do not treat "turning point" problems by special functions. We need not labor on an intelligent decomposition of $p(x)$ into a yet-to-be determined sum of unknown functions, $p(x) = f(x) + g(x)$, in order to obtain valid asymptotic approximations at a regular singular point.

The analysis of [**11**, Section 12] resorts to Bessel functions in order to ensure the globality of the approximation to Jacobi polynomials on $[1, \infty)$. Similarly, [**14**, formula (8.1.1)] delegates the asymptotic approximation of one type of special functions, the Jacobi polynomial, to another type of special functions, Bessel functions. Our approach differs from these techniques and we avoid asymptotic approximation by special functions. Our substitute for the special functions employed in the literature are the values of $q_{jk}(x)$ that feature in (4.4)–(4.7).

We will focus on the interval $[1, \infty)$ as either n or α tends to infinity. In particular, we derive from our approximation formula the asymptotic behavior as $x \to 1^+$ as well as $x \to \infty$. Moreover, we obtain the asymptotic behavior of $P_n^{(\alpha,\beta)}(x)$ as $n \to \infty$, uniformly on $[1+\psi(n), \infty)$, with a suitable $\psi(n) \to 0$ as $n \to \infty$. Furthermore, from one and the same asymptotic formula we obtain numerical approximations to $P_n^{(\alpha,\beta)}(x)$ on the interval $[1, \infty)$.

The building blocks of the asymptotic approximations derived from formulas (4.4)–(4.7) consist of $p^{1/4}$, $p'/4p$ or the quantity $l(x) = p'/4p^{3/2}$. The formulas (4.4)–(4.7) also contain perturbation terms $q_{jk}(x)$. These are given by certain resolvent series, that are absolutely convergent. It is a pleasant surprise that, for the range of parameters in (1.2) that is related to the Jacobi polynomials, the functions $l(x)$ and $l'(x)$ do not change sign on the interval $[1, \infty)$. This ensures that the rate of convergence of the series representing $q_{jk}(x)$ is faster than a geometric series of positive terms with the ratio of successive terms being $\pi^2/32$. It reminds us of a similar situation in the context of the modified Bessel functions. See [**6**]. One may wonder whether analogous situations occur in the theory of other special functions.

There are advantages in the set of formulas (4.4)–(4.7) over the formulas given in [**4–6**]: Complex terms will not enter the approximation of a solution of (1.1) when the solution is real valued. This and the markedly improved rate of convergence of $q_{jk}(x)$, are an edge when delegating the task of numerical approximations to a digital computer.

The order of events in this paper is as follows. In Section 2 we produce linear transformations that aid us in deriving asymptotic formulas. In

Section 3 we derive integral equations for the perturbation terms. In Section 4 we present our approximation formulas. The main results, elaborating on the approximations of the Jacobi polynomials are given in Section 5.

2. Some transformations. The second order differential equation

$$y'' = p(x)y, \quad x \in [a, b], \quad -\infty \le a < b \le +\infty, \tag{2.1}$$

has as a first order 2×2 companion system

$$Z' = AZ, \quad A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}. \tag{2.2}$$

Throughout this article we adopt the assumption that $p(x) \in C^2(a, b)$. In this work we assume throughout that $p > 0$. The case where p is negative will be discussed somewhere else. We apply to (2.2) a certain linear transformation $Z = TZ_1$ which takes the system $Z' = AZ$ into $Z_1' = (T^{-1}AT - T^{-1}T')Z_1$. One convenient choice is to take

$$T = \begin{pmatrix} p^{-1/4} & p^{-1/4} \\ p^{1/4} & -p^{1/4} \end{pmatrix} = \begin{pmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.3}$$

whose columns are the eigenvectors of A which correspond to the eigenvalues $p^{-1/2}, p^{-1/2}$, respectively. Here

$$T^{-1}AT = \begin{pmatrix} p^{1/2} & 0 \\ 0 & -p^{-1/2} \end{pmatrix}, \quad -T^{-1}T' = \begin{pmatrix} 0 & p'/4p \\ p'/4p & 0 \end{pmatrix}, \tag{2.4}$$

and our transformation takes (2.2) into the system

$$Z_1' = \begin{pmatrix} p^{1/2} & p'/4p \\ p'/4p & -p^{1/2} \end{pmatrix} Z_1. \tag{2.5}$$

Now we apply to (2.5) a second linear transformation $Z_1 = VZ_2$. The eigenvalues of the matrix in (2.5) are $\lambda(x), -\lambda(x)$, where $\lambda(x) = \big(p + (p'/4p)^2\big)^{1/2}$ and the corresponding eigenvectors may be taken, respectively, as

$$\begin{pmatrix} 1 + \sqrt{1 + l^2} \\ l \end{pmatrix}, \quad \begin{pmatrix} -l \\ 1 + \sqrt{1 + l^2} \end{pmatrix}, \quad \text{where} \quad l(x) = \frac{p'}{4p^{3/2}}. \tag{2.6}$$

Note that $l(x)$ is the ratio between the off-diagonal and the diagonal terms of the matrix in (2.5) and that the quantities λ and l are related by $\lambda^2 = p(1+l^2)$. We normalize the vectors in (2.6) to be unit vectors and choose them to be the columns of the transformation matrix V,

$$V = \frac{1}{\sqrt{2}(1+l^2+\sqrt{1+l^2})^{1/2}} \begin{pmatrix} 1+\sqrt{1+l^2} & -l \\ l & 1+\sqrt{1+l^2} \end{pmatrix}. \tag{2.7}$$

The change of variables $Z_1 = VZ_2$ transforms (2.5) into

$$Z_2' = \left[\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} - V^{-1}V' \right] Z_2. \tag{2.8}$$

It follows by some algebraic manipulation that

$$V^{-1}V' = \begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}, \quad \text{where} \quad r = \frac{1}{2}\, l' \Big/ (1+l^2).$$

Finally, equation (2.8) becomes

$$Z_2' = \begin{pmatrix} \lambda & r \\ -r & -\lambda \end{pmatrix} Z_2. \tag{2.9}$$

We turn to the next section that discusses certain perturbation terms.

3. Perturbations. Instead of equation (2.9), let us consider a more general system of a similar structure

$$Z_2' = (D+R)Z_2, \tag{3.1}$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & r_{12} \\ r_{21} & 0 \end{pmatrix}. \tag{3.2}$$

Since the unperturbed equation $Z' = DZ$ (with $R = 0$) has the solution

(3.3)

$$\Phi(x) = \exp\left(\int_{x_0}^{x} D \right) = \begin{pmatrix} \exp\left(\int_{x_0}^{x} \lambda_1(s)\, ds \right) & 0 \\ 0 & \exp\left(\int_{x_0}^{x} \lambda_2(s)\, ds \right) \end{pmatrix},$$

it is reasonable to search for solutions of (3.1) of the form

$$Z_2 = (I+Q)\Phi \tag{3.4}$$

or

$$Z_2 = \Phi(I+P), \tag{3.5}$$

where P and Q will be considered perturbation matrices. The fixed number x_0 will be chosen later in some convenient way. The comparison of these two representations $Z_2 = (I+Q)\Phi = \Phi(I+P)$ implies that

$$Q = \Phi P\,\Phi^{-1}, \tag{3.6}$$

and therefore (3.4) and (3.5) are closely related. The substitution of (3.4) into equation (3.1) implies

$$Q' = (D+R)Q - QD + R \tag{3.7}$$

while the substitution of (3.5) implies

$$P' = \Phi^{-1} R\,\Phi(I+P). \tag{3.8}$$

Since (3.8) is simpler than (3.7), we start with equation (3.8).

For 2×2 matrices, the solution process has some unique features. Component-wise,

(3.9)

$$\begin{aligned}
\Phi^{-1}R\Phi &= \begin{pmatrix} \exp\left(-\int_{x_0}^{x}\lambda_1\,ds\right) & 0 \\ 0 & \exp\left(-\int_{x_0}^{x}\lambda_2\,ds\right) \end{pmatrix}\begin{pmatrix} 0 & r_{12} \\ r_{21} & 0 \end{pmatrix} \\
&\qquad\times\begin{pmatrix} \exp\left(\int_{x_0}^{x}\lambda_1\,ds\right) & 0 \\ 0 & \exp\left(\int_{x_0}^{x}\lambda_2\,ds\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & r_{12}\exp\left(\int_{x_0}^{x}(\lambda_2-\lambda_1)\,ds\right) \\ r_{21}\exp\left(\int_{x_0}^{x}(\lambda_1-\lambda_2)\,ds\right) & 0 \end{pmatrix},
\end{aligned}$$

so (3.8) becomes

(3.10)

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}' = \begin{pmatrix} 0 & r_{12} \exp\left(\int_{x_0}^{x} (\lambda_2 - \lambda_1)\, ds\right) \\ r_{21} \exp\left(\int_{x_0}^{x} (\lambda_1 - \lambda_2)\, ds\right) & 0 \end{pmatrix} \times \begin{pmatrix} 1 + p_{11} & p_{12} \\ p_{21} & 1 + p_{22} \end{pmatrix}.$$

We integrate the (j,k)th component on some interval $[l_{jk}, x]$, discard the constants of integration, and get

$$p_{11}(x) = \int_{l_{11}}^{x} r_{12}(t) \exp\left(\int_{x_0}^{t} (\lambda_2 - \lambda_1)\, ds\right) p_{21}(t)\, dt, \tag{3.11-a}$$

$$p_{21}(x) = \int_{l_{21}}^{x} r_{21}(t) \exp\left(\int_{x_0}^{t} (\lambda_1 - \lambda_2)\, ds\right) (1 + p_{11}(t))\, dt, \tag{3.11-b}$$

and analogous equations for p_{22}, p_{12}, with the indices 1 and 2 interchanged. These integral equations clearly imply the differential system (3.10). When we substitute p_{21} in (3.11-a) by (3.11-b) and consolidate some exponential terms, the result is

(3.12-a)

$$\begin{aligned} p_{11}(x) = &\int_{l_{11}}^{x} r_{12}(t_1) \left[\int_{l_{21}}^{t_1} r_{21}(t_2) \exp\left(\int_{t_2}^{t_1} (\lambda_2 - \lambda_1)\, ds\right) dt_2 \right] dt_1 \\ &+ \int_{l_{11}}^{x} r_{12}(t_1) \left[\int_{l_{21}}^{t_1} r_{21}(t_2) \right. \\ &\qquad \left. \times \exp\left(\int_{t_2}^{t_1} (\lambda_2 - \lambda_1)\, ds\right) p_{11}(t_2)\, dt_2 \right] dt_1. \end{aligned}$$

Substitution of p_{11} in (3.11-b) by (3.11-a) yields

$$\begin{aligned} p_{21}(x) = &\int_{l_{21}}^{x} r_{21}(t_1) \exp\left(\int_{x_0}^{t} (\lambda_1 - \lambda_2)\, ds\right) dt_1 \\ &+ \int_{l_{21}}^{x} r_{21}(t_1) \left[\int_{l_{11}}^{t_1} r_{12}(t_2) \right. \\ &\qquad \left. \times \exp\left(\int_{t_2}^{t_1} (\lambda_1 - \lambda_2)\, ds\right) p_{21}(t_2)\, dt_2 \right] dt_1 \end{aligned} \tag{3.12-b}$$

and analogous integral equations for p_{21}, p_{22}. Surprisingly, these are four decoupled independent integral equations, one for each p_{jk}.

Now we return to the representation (3.4). By $Q = \Phi P \Phi^{-1}$,

$$
\begin{aligned}
&q_{11} \equiv p_{11}, \qquad q_{22} \equiv p_{22}, \\
&q_{12}(x) = p_{12}(x) \exp\left(\int_{x_0}^{x} (\lambda_1 - \lambda_2)\, ds \right), \\
&q_{21}(x) = p_{21}(x) \exp\left(\int_{x_0}^{x} (\lambda_2 - \lambda_1)\, ds \right).
\end{aligned} \tag{3.13}
$$

The integral equations for q_{11} and q_{22} are the same as those for p_{11} and p_{22}, respectively, while for q_{21} and q_{12} we reorganize the exponential terms. We get

(3.14-a)

$$
\begin{aligned}
q_{11}(x) = &\int_{l_{11}}^{x} r_{12}(t_1) \left[\int_{l_{21}}^{t_1} r_{21}(t_2) \exp\left(\int_{t_2}^{t_1} (\lambda_2 - \lambda_1)\, ds \right) dt_2 \right] dt_1 \\
&+ \int_{l_{11}}^{x} r_{12}(t_1) \left[\int_{l_{21}}^{t_1} r_{21}(t_2) \right. \\
&\qquad \left. \times \exp\left(\int_{t_2}^{t_1} (\lambda_2 - \lambda_1)\, ds \right) q_{11}(t_2)\, dt_2 \right] dt_1.
\end{aligned}
$$

(3.14-b)

$$
\begin{aligned}
q_{22}(x) = &\int_{l_{22}}^{x} r_{21}(t_1) \left[\int_{l_{12}}^{t_1} r_{12}(t_2) \exp\left(\int_{t_2}^{t_1} (\lambda_1 - \lambda_2)\, ds \right) dt_2 \right] dt_1 \\
&+ \int_{l_{22}}^{x} r_{21}(t_1) \left[\int_{l_{12}}^{t_1} r_{12}(t_2) \right. \\
&\qquad \left. \times \exp\left(\int_{t_2}^{t_1} (\lambda_1 - \lambda_2)\, ds \right) q_{22}(t_2)\, dt_2 \right] dt_1.
\end{aligned}
$$

(3.14-c)

$$
\begin{aligned}
q_{12}(x) = &\int_{l_{12}}^{x} r_{12}(t_1) \exp\left(\int_{x}^{t_1} (\lambda_2 - \lambda_1)\, ds \right) dt_1 \\
&+ \int_{l_{12}}^{x} r_{12}(t_1) \exp\left(\int_{x}^{t_1} (\lambda_2 - \lambda_1)\, ds \right) \\
&\qquad \times \left[\int_{l_{22}}^{t_1} r_{21}(t_2) q_{12}(t_2)\, dt_2 \right] dt_1.
\end{aligned}
$$

(3.14-d)

$$q_{21}(x) = \int_{l_{21}}^{x} r_{21}(t_1) \exp\left(\int_x^{t_1} (\lambda_1 - \lambda_2)\, ds\right) dt_1 + \int_{l_{21}}^{x} r_{21}(t_1) \exp\left(\int_x^{t_1} (\lambda_1 - \lambda_2)\, ds\right) \times \left[\int_{l_{11}}^{t_1} r_{12}(t_2) q_{21}(t_2)\, dt_2\right] dt_1.$$

Now we return to the special system (2.9), with

$$r_{12} = r, \quad r_{21} = -\, r, \quad \lambda_1 = \lambda, \quad \lambda_2 = -\, \lambda,$$

and let for simplicity $\lambda \geq 0$. The kernel $\exp(\int_x^{t_1} (\lambda_1 - \lambda_2)\, ds)$ of (3.14-b) and (3.14-d) is replaced by $\exp(2 \int_x^{t_1} \lambda\, ds)$ while $\exp(\int_x^{t_1} (\lambda_2 - \lambda_1)\, ds)$ in (3.14-a) and (3.14-c) becomes $\exp(-2 \int_x^{t_1} \lambda\, ds)$. The lower limits l_{jk} will be chosen so that these exponential kernels will be bounded on their respective domains of integration. $\exp(-2 \int_{t_2}^{t_1} \lambda\, ds) \leq 1$ is achieved in equation (3.12-a) for q_{11} by taking $t_2 \leq t_1$. Therefore we choose

$$l_{21} = a.$$

On the other hand, in the equation for q_{22}, and p_{22}, we have $\exp(2 \int_{t_2}^{t_1} \lambda\, ds) \leq 1$, provided that $t_2 \geq t_1$. Therefore, we choose

$$l_{12} = b,$$

(so that $t_1 \leq t_2 \leq l_{12} = b$). These two choices fit the integral equations for both q_{12} and q_{21}. We are still free to choose l_{11} and l_{22}. We choose

$$l_{11} = a, \quad l_{22} = b,$$

so that in each integral equation the lower limits will be one and the same. Consequently,

(3.15-a)

$$q_{11}(x) = -\int_a^x r(t_1) \left[\int_a^{t_1} r(t_2) \exp\left(-\, 2 \int_{t_2}^{t_1} \lambda\, ds\right) dt_2\right] dt_1 - \int_a^x r(t_1) \left[\int_a^{t_1} r(t_2) \exp\left(-\, 2 \int_{t_2}^{t_1} \lambda\, ds\right) q_{11}(t_2)\, dt_2\right] dt_1,$$

(3.15-b)
$$q_{22}(x) = -\int_x^b r(t_1)\left[\int_{t_1}^b r(t_2)\exp\left(2\int_{t_2}^{t_1}\lambda\,ds\right)dt_2\right]dt_1 \\ -\int_x^b r(t_1)\left[\int_{t_1}^b r(t_2)\exp\left(2\int_{t_2}^{t_1}\lambda\,ds\right)q_{22}(t_2)\,dt_2\right]dt_1,$$

(3.15-c)
$$q_{12}(x) = -\int_x^b r(t_1)\exp\left(-2\int_x^{t_1}\lambda\,ds\right)dt_1 \\ -\int_x^b r(t_1)\exp\left(-2\int_x^{t_1}\lambda\,ds\right)\left[\int_{t_1}^b r(t_2)q_{12}(t_2)\,dt_2\right]dt_1,$$

(3.15-d)
$$q_{21}(x) = -\int_a^x r(t_1)\exp\left(2\int_x^{t_1}\lambda\,ds\right)dt_1 \\ -\int_a^x r(t_1)\exp\left(2\int_x^{t_1}\lambda\,ds\right)\left[\int_a^{t_1} r(t_2)q_{21}(t_2)\,dt_2\right]dt_1.$$

We are ready now to formulate results that guarantee the existence of solutions q_{jk} to the integral equations (3.15-a)–(3.15-d). There is a plethora of sophisticated theorems in asymptotic integration that could guarantee the existence of q_{jk} with the desired properties. We shall formulate below a simple theorem in the case that l, λ, r are real valued.

Theorem 3.1. (a) *Let* $\lambda \geq 0$ *on* $[a,b]$, *or* $\lambda \geq 0$ *on* (a,b), $-\infty \leq a < b \leq \infty$, *and let*

$$\frac{1}{2}\left[\int_a^b |r(t)|\,dt\right]^2 < 1. \tag{3.16}$$

Then the integral equations (3.15-a)–(3.15-d) *posses solutions* q_{jk} *which are continuous and bounded on* $[a,b]$ *and satisfy*

$$q_{11}(a) = q_{21}(a) = q_{22}(b) = q_{12}(b) = 0.$$

(b) *If, in addition*

$$\int_{a^+} \lambda(t)\,dt = +\infty, \tag{3.17}$$

then also $q_{12}(a) = 0$. *Moreover, if*

$$\int^{b^-} \lambda(t)\,dt = +\infty, \tag{3.18}$$

then also $q_{21}(b) = 0$.

(c) *Let* $p(x) = p(x,\mu)$ *depend also on a parameter* μ *and*

$$\int_a^b |r(t,\mu)|\,dt \to 0 \quad as \quad \mu \to \mu_0. \tag{3.19}$$

Then $q_{jk}(x) \to 0$ *uniformly on* $[a,b]$ *as* $\mu \to \mu_0$.

Proof. Let $||q_{jk}|| = \sup_x |q_{jk}(x)|$, $j,k = 1,2$. Note that the exponential terms in (3.15-a)–(3.15-d) are bounded from above by 1 by the nonnegativity of λ and the choices of the limits. Because of the same lower limit in the double integrals, we have in (3.15-a) the estimates

$$\begin{aligned}\left|\int_a^x r(t_1)\left[\int_a^{t_1} r(t_2)\exp\left(-2\int_{t_2}^{t_1}\lambda\,ds\right)q_{11}(t_2)\,dt_2\right]dt_1\right| \\ \le \frac{1}{2}\left[\int_a^b r(t_1)\,dt_1\right]^2 ||q_{11}||,\end{aligned}$$

and similar estimates in (3.15-b)–(3.15-d). Thus the integral equations (3.15-a), (3.15-b) imply that

$$||q_{jj}|| \le \frac{1}{2}\left[\int_a^b |r(t)|\,dt\right]^2 (1 + ||q_{jj}||), \quad j = 1,2, \tag{3.20}$$

and (3.15-c), (3.15-d) imply that

$$||q_{jk}|| \le \left[\int_a^b |r(t)|\,dt\right] + \frac{1}{2}\left[\int_a^b |r(t)|\,dt\right]^2 ||q_{jk}||, \quad j \ne k. \tag{3.21}$$

In each equation the assumption $\left[\int_a^b |r(t)|\,dt\right]^2/2 < 1$ is tantamount to saying that the integral operator is a contraction mapping. $q_{jk}(x)$ is the unique fixed point of the corresponding integral operator and it is given by a certain resolvent series. The above inequalities immediately yield uniform bounds on $||q_{jk}||$ which depend solely on the value of $\int_a^b |r(t)|\,dt$.

For the resolvent series for q_{11} and q_{22}, more delicate estimates are available provided that $r(t)$ does not change its sign. Let us define

$$L[g](x) = \int_a^x r(t_1)\left[\int_a^{t_1} r(t_2)\exp\left(-2\int_{t_2}^{t_1}\lambda\,ds\right)g(t_2)\,dt_2\right]dt_1.$$

Then equation (3.15-a) is written as

$$q_{11}(x) = -L[1] - L[q_{11}].$$

If g is positive, nondecreasing on $[a,b]$, then $L[g](x)$ is positive, nondecreasing as well, and

$$L[g](x) \le \left[\int_a^x r(t_1)\int_a^{t_1} r(t_2)\,dt_2\,dt_1\right]g(x) = \frac{1}{2}\left[\int_a^x |r(t)|\,dt\right]^2 g(x).$$

Hence, by assumption (3.16),

$$g > L[g] > L^2[g] > \cdots$$

Now define the iteration

$$f_0(x) = 1, \quad f_i(x) = -L[1] - L[f_{i-1}], \quad i = 1, 2, \ldots$$

Then

$$q_{11}(x) = -L[1] + L^2[1] - L^3[1] + - \cdots \tag{3.22}$$

is an alternating series with $L^i[1] > L^{i+1}[1] > 0$. Its convergence will be as fast at least as that of a geometric series with the ratio $\max L[1] \le \left[\int_a^b |r(t)|\,dt\right]^2/2 < 1$ and $L^i[1] \to 0$ as $i \to \infty$. Moreover, as the terms of the resolvent series have alternating signs, its convergence is accelerated and enables a convenient error estimate. The same argument is applicable for the solution q_{22} of equation (3.15-b) but now the operator L has other limits of integration. Moreover, since $q_{11}(b)$ and $q_{22}(a)$ are given by the same resolvent series, we have

$$q_{11}(b) = q_{22}(a).$$

The terminal values

$$q_{11}(a) = q_{22}(b) = q_{21}(a) = q_{12}(b) = 0$$

follow by the limits of integration in (3.15-a)–(3.15-d).

Now we show that also $q_{12}(a) = 0$, provided that (3.17) holds. q_{12} is the solution of equation (3.15-c), namely

(3.23)

$$\begin{aligned} q_{12}(x) = &- \int_x^b r(t_1) \exp\left(-2\int_x^{t_1} \lambda\, ds\right) dt_1 \\ &- \int_x^b r(t_1) \exp\left(-2\int_x^{t_1} \lambda\, ds\right) \left[\int_{t_1}^b r(t_2) q_{12}(t_2)\, dt_2\right] dt_1. \end{aligned}$$

We first prove that if (3.17) holds then the first integral in equation (3.23) tends to 0 as $x \to a$. Indeed, for an arbitrary small $\varepsilon > 0$ take a fixed c, $a < c < b$, such that $\int_a^c |r(t_1)|\, dt_1 < \varepsilon$. Then the first integral of (3.23) is less then

$$\begin{aligned} &\int_x^c \left| r(t_1) \exp\left(-2\int_x^{t_1} \lambda\, ds\right)\right| dt_1 + \int_c^b \left| r(t_1) \exp\left(-2\int_x^{t_1} \lambda\right) ds\right| dt_1 \\ &\qquad \le \int_a^c |r(t_1)|\, dt_1 + \exp\left(-2\int_x^c \lambda\, ds\right) \int_c^b |r(t_1)|\, dt_1. \end{aligned}$$

The first term is less than ε due to the choice of c and the second term tends to 0 as $x \to a$ due to (3.17). Since q_{12} and $\int_a^b |r(t_2)|\, dt_2$ are bounded, the second term of (3.23) is treated similarly and it follows that $q_{12} \to 0$ as $x \to a^+$.

The fact that $q_{21} \to 0$ as $x \to b$ follows by similar reasoning from (3.18). □

4. The asymptotic approximations. Let $p > 0$ and consequently λ, l and r are real valued. Once the existence of q_{jk} is established, say by Theorem 3.1, a fundamental solution of equation (2.9) has by (3.4) the asymptotic approximation $Z_2 = (I + Q)\Phi$, and the solution of the corresponding system (2.2) may be represented as

$$Z = T\, V\, (I + Q)\, \Phi.$$

Recall that $T = \begin{pmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} V = \begin{pmatrix} \mu_2 & \mu_1 \\ \mu_1 & -\mu_2 \end{pmatrix}$ where, due to (2.7) and some calculations, μ_1 and μ_2 are

$$\mu_1 = \frac{1+\sqrt{1+l^2}-l}{\sqrt{2}(1+l^2+\sqrt{1+l^2})^{1/2}} = \left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2}, \tag{4.1}$$

$$\mu_2 = \frac{1+\sqrt{1+l^2}+l}{\sqrt{2}(1+l^2+\sqrt{1+l^2})^{1/2}} = \left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2}. \tag{4.2}$$

So

$$\begin{aligned} Z = &\begin{pmatrix} p^{-1/4} & 0 \\ 0 & p^{1/4} \end{pmatrix} \begin{pmatrix} \mu_2 & \mu_1 \\ \mu_1 & -\mu_2 \end{pmatrix} \begin{pmatrix} 1+q_{11} & q_{12} \\ q_{21} & 1+q_{22} \end{pmatrix} \\ &\times \begin{pmatrix} \exp\left(\int_{x_0}^x \lambda\, ds\right) & 0 \\ 0 & \exp\left(-\int_{x_0}^x \lambda\, ds\right) \end{pmatrix}. \end{aligned} \tag{4.3}$$

For the scalar equation (2.1), this yields the asymptotic approximations

(4.4)
$$\begin{aligned} y_1 = &\left[(1+q_{11})\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2} + q_{21}\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right]p^{-1/4} \\ &\times \exp\left(\int_{x_0}^x \sqrt{p+(p'/4p)^2}\, ds\right), \end{aligned}$$

(4.5)
$$\begin{aligned} y_2 = &\left[q_{12}\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2} + (1+q_{22})\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right]p^{-1/4} \\ &\times \exp\left(-\int_{x_0}^x \sqrt{p+(p'/4p)^2}\, ds\right), \end{aligned}$$

(4.6)

$$y_1' = \left[(1+q_{11})\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2} - q_{21}\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right]p^{1/4} \times \exp\left(\int_{x_0}^{x}\sqrt{p+(p'/4p)^2}\,ds\right),$$

(4.7)

$$y_2' = \left[q_{12}\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2} - (1+q_{22})\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right]p^{1/4} \times \exp\left(-\int_{x_0}^{x}\sqrt{p+(p'/4p)^2}\,ds\right).$$

The transformations in the previous sections hold formally even for complex valued p, provided that branches of the multivalued complex valued functions are properly chosen. This subject will not be pursued here further.

5. Approximation of the Jacobi polynomials. Jacobi polynomials $u = P_n^{(\alpha,\beta)}(x)$ are solutions of the differential equation

(5.1) $$\left((1-x)^{\alpha+1}(1+x)^{\beta+1}u'\right)' + n(n+\alpha+\beta+1)(1-x)^{\alpha}(1+x)^{\beta}u = 0,$$

and $y = (x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2}u$ satisfies the differential equation

(5.2)

$$y'' = \left[\frac{\alpha^2-1}{4(x-1)^2} + \frac{\beta^2-1}{4(x+1)^2} + \frac{n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)/2}{x^2-1}\right]y.$$

In this section it will be assumed that $\alpha, \beta > 1$.

We will extract from one and the same formula, (4.4), several distinct approximations. From (4.4) we derive a uniform approximation for $P_n^{(\alpha,\beta)}(x)$ on $(1,\infty)$,

(5.3) $$P_n^{(\alpha,\beta)}(x) = (x-1)^{-(\alpha+1)/2}(x+1)^{-(\beta+1)/2}p^{-1/4}\exp\left(\int_2^x\sqrt{p+(p'/4p)^2}\,ds\right) \times \left[(1+q_{11})\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2} + q_{21}\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right],$$

where q_{11}, q_{21} are given by resolvent series whose convergence is faster than that of a geometric series with ratio $\pi^2/32$ and $l(x)$ is given by (5.5). This is essentially a numerical rather than an asymptotic statement.

We will also extract now from (4.4) the following asymptotic values:

1. The well-known behavior of $P_n^{(\alpha,\beta)}(x)$ as $x \to 1^+$ for fixed values of the parameters α, β and n.

2. The asymptotic approximation of $P_n^{(\alpha,\beta)}(x)$ as $x \to \infty$ for fixed values of α, β and n.

3. The uniform asymptotic approximations of $P_n^{(\alpha,\beta)}(x)$ for α, β fixed and $n \to \infty$ for $x \in [1+\psi(n), \infty)$ with a suitable ψ, $\psi(n) \to 0$ as $n \to \infty$. On the interval $[1, 1+\psi(n)]$ we obtain only pointwise approximations to the Jacobi polynomials. This differs from some of the asymptotic approximations in [**14**, Chapter 8] which, while holding in the complex plane, are uniform as $n \to \infty$ only on sets which are of a positive distance away from the segment $[-1, 1]$.

4. The uniform asymptotic approximation of the Jacobi polynomials on $[1, \infty)$ as $\alpha \to \infty$. This includes the approximation of $P_n^{(\alpha+an,\beta+bn)}(x)$ as $n \to \infty$.

5. Darboux's asymptotic formula.

We turn now to the details and proceed with an informal discussion of equation (5.2) rather than formulate results in the form of theorems. For short we denote

$$p(x) = \frac{A}{(x-1)^2} + \frac{B}{(x+1)^2} + \frac{C}{x^2-1}, \tag{5.4}$$

where $A = (\alpha^2 - 1)/4$, $B = (\beta^2 - 1)/4$, $C = n(n + \alpha + \beta + 1) + (\alpha + 1)(\beta + 1)/2$. Clearly, $p(x) > 0$ on $(1, \infty)$. A straightforward calculation leads to

$$l(x) = \frac{p'}{4p^{3/2}} = -\frac{A(x+1)^3 + B(x-1)^3 + Cx(x^2-1)}{2[A(x+1)^2 + B(x-1)^2 + C(x^2-1)]^{3/2}} < 0, \tag{5.5}$$

$$l'(x) = \frac{12AB(x^2-1) + 4AC(x+1)^2 + 4BC(x-1)^2 + C^2(x^2-1)}{2[A(x+1)^2 + B(x-1)^2 + C(x^2-1)]^{5/2}} > 0 \quad \text{on} \quad [1, \infty). \tag{5.6}$$

The fact that $l < 0$ and $l' > 0$ on the entire interval $[1, \infty)$ is central to the globality of the approximations in (4.4)–(4.7) and the fast rate of convergence of the perturbation terms q_{jk}. Indeed, $r = l'/2(1 + l^2)$ is positive and

(5.7)
$$\begin{aligned}\int_1^\infty |r(t)|\,dt &= \frac{1}{2}\int_1^\infty \frac{l'}{1+l^2}\,dt \\ &= \frac{1}{2}\Big[\arctan l(\infty) - \arctan l(1)\Big] \\ &= \frac{1}{2}\left[-\arctan\left(\frac{1}{2\sqrt{A+B+C}}\right) + \arctan\left(\frac{1}{2\sqrt{A}}\right)\right] \\ &< \pi/4.\end{aligned}$$

As seen in Theorem 3.1, the rate of convergence for the resolvent series for $q_{jk}(x)$ is determined by the norm of the contraction operators in the integral equations (3.15-a)–(3.15-d). Due to $l < 0$ and $l' > 0$, we have in our case

$$\frac{1}{2}\left[\int_1^\infty |r(t)|\,dt\right]^2 < \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{32}. \tag{5.8}$$

Thus, the first part of Theorem 3.1 applies to equation (5.2). Moreover, since $\pi^2/32 \ll 1$, the convergence in (5.3) is quite fast.

A crucial role in the global and uniform approximation (4.4) is played by the quantities $\int_{x_0}^x \lambda\,ds$. Since $\lambda = \big(p + (p'/4p)^2\big)^{1/2}$ is not integrable neither at $x = 1$ nor at $x = \infty$, we take in $\int_{x_0}^x \lambda$ the lower limit $x_0 = 2$. Since both (3.17) and (3.18) hold for $a = 1$, $b = \infty$, we conclude by Theorem 3.1 that in (4.4)–(4.7)

$$\begin{aligned} q_{11}(1) &= 0, \\ q_{12}(1) &= q_{12}(\infty) = 0, \\ q_{21}(1) &= q_{21}(\infty) = 0, \\ q_{22}(\infty) &= 0. \end{aligned} \tag{5.9}$$

The approximations will be elaborated on near the regular-singular point $x = 1$ and near $x = \infty$ for any $\alpha, \beta > 1$ and n.

Near $x = 1^+$. $P_n^{(\alpha,\beta)}(x)$ is easily calculated near $x = 1$, say by the explicit hypergeometric function representation $\binom{n+\alpha}{n} F(-n, -n+\alpha+\beta-1, \alpha+1, (x-1)/2)$. We demonstrate that our (4.4) is compatible with the known behavior even near the regular-singular $x = 1$.

Since $l(1) = -(\alpha^2-1)^{-1/2}$, $\lambda(x) = \alpha/2(x-1)(1+O(x-1))$, we rewrite $\int_2^x \lambda\, ds$ as

(5.10)
$$\int_2^x \lambda = \int_1^x \left(\lambda - \frac{\alpha}{2(s-1)}\right) ds - \int_1^2 \left(\lambda - \frac{\alpha}{2(s-1)}\right) ds + \int_2^x \frac{\alpha}{2(s-1)}\, ds.$$

Here $\int_1^x (\lambda - (\alpha/2(s-1)))\, ds \to 0$ as $x \to 1$ and $\gamma_1 = -\int_1^2 (\lambda - (\alpha/2(s-1)))\, ds$ is a regular integral. Consequently, $\exp\left(\int_2^x \lambda\, ds\right) \sim (x-1)^{\alpha/2} e^{\gamma_1}$ near $x = 1$. Since $q_{11}(1) = q_{21}(1) = 0$, (4.4) becomes near $x = 1$

$$\begin{aligned} y_1 &\sim \left(1 + \frac{l}{\sqrt{1+l^2}}\right)^{1/2} p^{-1/4}(x) \exp\left(\int_2^x \lambda\, ds\right) \\ &\sim \left(\frac{2}{\alpha}\right)^{1/2} \left(\frac{\alpha-1}{\alpha+1}\right)^{1/4} e^{\gamma_1} (x-1)^{(\alpha+1)/2}. \end{aligned} \tag{5.11}$$

Thus y_1 of (4.4) yields the solution $(x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2} P_n^{(\alpha,\beta)}(x)$ of (5.2) up to a known numerical factor which may be determined by $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$.

Near $x = \infty$. Now we investigate the asymptotic approximation (4.4) near $x = \infty$. By (5.4) and (5.5),

$$\begin{aligned} p(x) &\sim (A+B+C)x^{-2}, \\ l(x) &= -(1/2)(A+B+C)^{-1/2} + O(x^{-1}). \end{aligned}$$

Since $A+B+C+1/4 = (n+((\alpha+\beta+1)/2))^2$, we get

$$\lambda(x) = \Big(p(x)\big(1+\ell^2(x)\big)\Big)^{1/2} \sim \left(n + \frac{\alpha+\beta+1}{2}\right) x^{-1} + O(x^{-2})$$

and

$$\left(1 + \frac{l(\infty)}{\sqrt{1+l^2(\infty)}}\right)^{1/2} = \left(\frac{n+(\alpha+\beta)/2}{n+(\alpha+\beta+1)/2}\right)^{1/2}.$$

Now we adopt the decomposition

$$\int_2^x \lambda\,ds = \int_\infty^x \Big(\lambda - \Big(n + \frac{\alpha+\beta+1}{2}\Big)s^{-1}\Big)\,ds \\ - \int_\infty^2 \Big(\lambda - \Big(n+ \frac{\alpha+\beta+1}{2}\Big)s^{-1}\Big)\,ds + \int_2^x \Big(n+ \frac{\alpha+\beta+1}{2}\Big)s^{-1}\,ds.$$

Accordingly, near $x = \infty$,

$$\exp\Big(\int_2^x \lambda\,ds\Big) \\ = \Big(\frac{x}{2}\Big)^{n+(\alpha+\beta+1)/2} e^{\gamma_2}\Big(-\int_x^\infty \Big(\lambda - \Big(n + \frac{\alpha+\beta+1}{2}\Big)s^{-1}\Big)\,ds\Big), \tag{5.12}$$

with the last exponential $\exp\big(\int_x^\infty O(s^{-2})\,ds\big)$ converging to 1 as $x \to \infty$ and

$$\gamma_2 = \int_2^\infty \Big(\lambda - \Big(n + \frac{\alpha+\beta+1}{2}\Big)s^{-1}\Big)\,ds.$$

Finally recall that, according to (5.9), $q_{21}(\infty) = q_{12}(\infty) = 0$.

Summarizing all of these details, we see that the same $y_1(x)$ of (4.4) which is given near $x = 1$ by (5.11), is given near $x = \infty$ by

(5.13)

$$\begin{aligned} y_1(x) &\sim (1 + q_{11}(\infty))\Big(1 + \frac{l(\infty)}{\sqrt{1 + l^2(\infty)}}\Big)^{1/2} p^{-1/4}(x)\,\exp\Big(\int_2^x \lambda\,ds\Big) \\ &\sim (1 + q_{11}(\infty))\Big(\frac{n + (\alpha+\beta)/2}{n + (\alpha+\beta+1)/2}\Big)^{1/2} \\ &\quad \times (A + B + C)^{-1/4}\, x^{1/2}\Big(\frac{x}{2}\Big)^{n+(\alpha+\beta+1)/2} e^{\gamma_2} \\ &= (1 + q_{11}(\infty))\Big(\frac{n + (\alpha+\beta)/2}{n + (\alpha+\beta)/2 + 1}\Big)^{1/4} \\ &\quad \times \frac{\sqrt{2}}{\big(n + (\alpha+\beta+1)/2\big)^{1/2}} \Big(\frac{x}{2}\Big)^{n+(\alpha+\beta)/2+1} e^{\gamma_2}. \end{aligned}$$

On the other hand, recall the well-known asymptotic approximation [**14**, 4.21.6]

$$P_n^{(\alpha,\beta)}(x) \sim \Big(\frac{x}{2}\Big)^n \binom{2n+\alpha+\beta}{n} \quad \text{as} \quad x \to \infty.$$

Once e^{γ_2} is known, this and (5.13) enable us to calculate the value of $q_{11}(\infty)$. $\square$

The case $n \to \infty$. Until now, α, β and n were fixed constants. We now demonstrate how (4.4) is used to approximate Jacobi polynomials on $[1, \infty)$ as $n \to \infty$. Here the situation is completely different since when both $x \to 1$ and $n \to \infty$, then

$$l(x) \approx -\frac{8A + 2C(x-1)}{2[4A + 2C(x-1)]^{3/2}} \quad \text{with} \quad C \sim n^2.$$

This does not converge uniformly to a limit in any neighborhood of $x = 1$ as $n \to \infty$. Moreover, for fixed α, β and large n, we have $r(1) = l'(1)/2(1 + l^2(1)) \approx n^2$. Hence uniform estimates are available only if we keep sufficiently away from the endpoint $x = 1$.

Proposition 5.1. *Let ψ be a function such that $\psi(n) > 0$, $\psi(n) \to 0$, $n^2\psi(n) \to \infty$ as $n \to \infty$. Then, as $n \to \infty$,*

$$y_1(x) \sim \big(1 + q_{11}(x)\big)p^{-1/4}(x) \exp\left(\int_{x_0}^{x} \sqrt{p + (p'/4p)^2}\, ds\right) \tag{5.14}$$

uniformly on $[1 + \psi(n), \infty)$. $q_{11}(x)$ is bounded on $[1, \infty)$ for any n by $-\pi^2/32 < q_{11}(x) \le 0$.

Proof. In order to conclude this from the asymptotic approximation (4.4), we have to show that, as $n \to \infty$, $q_{21}(x)$ and $l(x)$ converge uniformly to 0 on $[1 + \psi(n), \infty)$. The claim of Theorem 3.1 (c) does not apply to equation (5.2), since by (5.7) the integral $\int_1^\infty |r(t)|\, dt$ does not tend to 0 as $n \to \infty$. Therefore we need more delicate estimates.

The estimate of $q_{21}(x)$. q_{21} is the solution of the Volterra equation (3.15-d) with $a = 1$,

$$\begin{aligned} q_{21}(x) = &-\int_1^x r(t_1) \exp\left(-2\int_{t_1}^x \lambda\, ds\right) dt_1 \\ &- \int_1^x r(t_1) \exp\left(-2\int_{t_1}^x \lambda\, ds\right)\left[\int_1^{t_1} r(t_2) q_{21}(t_2)\, dt_2\right] dt_1, \end{aligned}$$

where l is given by (5.5), $l', r > 0$. Our purpose is to show that the first integral, namely

$$I(x,n) = \int_1^x r(t_1) \exp\left(-2\int_{t_1}^x \lambda\, ds\right) dt_1 \tag{5.15}$$

converges uniformly to 0 on $[1+\psi(n), \infty)$ as $n \to \infty$. Once this is verified, the second integral

$$\int_1^x r(t_1) \exp\left(-2\int_{t_1}^x \lambda\, ds\right) \left[\int_1^{t_1} r(t_2) q_{21}(t_2)\, dt_2\right] dt_1 \tag{5.16}$$

is estimated similarly. Indeed, $\int_1^\infty |r(t_2)|\, dt_2 < \pi/4 < 1$ and, according to the proof of Theorem 3.1, $q_{21}(x)$ is bounded by a uniform bound, independent of n. Hence, (5.1) too tends uniformly to 0 as $n \to \infty$ and the estimate of q_{21} will be completed.

We break the integral $I(x,n)$ into the sum $I_1(x,n) + I_2(x,n)$, where

$$\begin{aligned} I_1(x,n) &= \int_1^{1+\delta(n)} r(t_1) \exp\left(-2\int_{t_1}^x \lambda\, ds\right) dt_1, \\ I_2(x,n) &= \int_{1+\delta(n)}^x r(t_1) \exp\left(-2\int_{t_1}^x \lambda\, ds\right) dt_1 \end{aligned}$$

with $\delta(n) = \psi(n)/2$. $I_2(x,n)$ is easily estimated for $x \geq 1 + \psi(n) > 1 + \delta(n)$ since

$$\begin{aligned} I_2(x,n) &\leq \int_{1+\delta(n)}^\infty |r(t)|\, dt \\ &= \frac{1}{2}\int_{1+\delta(n)}^\infty l'/(1+l^2)|\, dt \\ &= \frac{1}{2}\Big[\arctan l\,(\infty) - \arctan l\,(1+\delta)\Big] \\ &= \frac{1}{2}\left[-\arctan\left(\frac{1}{2\sqrt{A+B+C}}\right) + \arctan\left(l\,(1+\delta)\right)\right]. \end{aligned} \tag{5.17}$$

The first term approaches 0 when $n \to \infty$ because $C \sim n^2$. Since $\delta = \delta(n) = \psi(n)/2 \to 0$ (and in particular $\delta < 1$) and since

$C\delta \sim n^2\delta(n) \to \infty$ when $n \to \infty$,

$$
\begin{aligned}
|\,l\,(1+\delta)| &= \left| -\frac{A(2+\delta)^3 + B\delta^3 + C\delta(1+\delta)(2+\delta)}{2[A(2+\delta)^2 + B\delta^2 + C\delta(2+\delta)]^{3/2}} \right| \\
&\le \frac{27A + B + 6C\delta}{2[2C\delta]^{3/2}} \\
&= O\big((n^2\delta)^{-3/2}\big) + O\big((n^2\delta)^{-1/2}\big) \\
&= O\left(\frac{1}{n\sqrt{\psi(n)}}\right) \longrightarrow 0.
\end{aligned} \tag{5.18}
$$

$I_1(x,n)$ needs a more delicate estimate. We bound $\int_{t_1}^{x} \lambda(s)\,ds$ from below for $1 \le t_1 \le 1+\delta(n) = 1+\psi(n)/2$ and $1+\psi(n) \le x < \infty$. Let first $1+\psi(n) \le x \le 2$. Then, since $A > 0$, $B > 0$ and $C > n^2$,

$$
\lambda^2 = p + \left(\frac{p'}{4p}\right)^2 \ge p(x) = \frac{A}{(x-1)^2} + \frac{B}{(x+1)^2} + \frac{C}{x^2-1} \ge \frac{C}{3(x-1)},
$$

so

$$
\begin{aligned}
\int_{t_1}^{x} \lambda(s)\,ds &\ge (1/\sqrt{3}) \int_{t_1}^{x} \frac{n}{(s-1)^{1/2}}\,ds \\
&= (2/\sqrt{3})\,n\,[\sqrt{x-1} - \sqrt{t_1-1}] \\
&\ge n[\sqrt{\psi(n)} - \sqrt{\psi(n)/2}] \ge n\sqrt{\psi(n)}/4 \to \infty,
\end{aligned} \tag{5.19}
$$

Consequently,

$$
\begin{aligned}
I_1(x,n) &\le \int_1^{1+\delta(n)} |r(t_1)| \exp\left(-2\int_{t_1}^{x} \lambda\,ds\right) dt_1 \\
&\le e^{-n\sqrt{\psi(n)}/2} \int_1^{1+\delta(n)} |r(t_1)|\,dt_1 \\
&= \exp(-n\sqrt{\psi(n)}/2)[\arctan\,l\,(1+\delta) - \arctan\,l\,(1)] \\
&= \exp(-n\sqrt{\psi(n)}/2)[-O\big(1/n\sqrt{\psi(n)}\big) + \arctan(1/2\sqrt{A})] \longrightarrow 0
\end{aligned}
$$

as $n \to \infty$.

If $x \geq 2$ (and $1 \leq t_1 \leq 1 + \delta(n)$, $\delta(n) \to 0$), then $\lambda^2 \geq p(x) \geq C/(x^2-1) > Cx^{-2}$, so $\int_{t_1}^{x} \lambda(s)\, ds > \sqrt{C} \int_{t_1}^{x} ds/s \geq n \log 2$ and

$$\begin{aligned} I_1(x,n) &= \int_1^{1+\delta(n)} r(t_1) \exp\left(-2\int_{t_1}^{x} \lambda\, ds\right) dt_1 \\ &\leq e^{-2n\log 2} \int_1^{1+\delta(n)} |r(t_1)|\, dt_1 \\ &= 4^{-n}[\arctan l\,(1+\delta) - \arctan l\,(1)] \\ &= 4^{-n}[-O\big(1/n\sqrt{\psi(n)}\big) + \arctan(1/2\sqrt{A})] \longrightarrow 0 \end{aligned}$$

as $n \to \infty$. This completes the claim that $q_{21}(x)$ converges uniformly to 0 on $[1+\psi(n), \infty)$ as $n \to \infty$.

An analogous claim can be proved for $q_{12}(x)$. It is a solution of the integral equation (3.15-c) with $b = \infty$,

(5.20)
$$\begin{aligned} q_{12}(x) = &-\int_x^{\infty} r(t_1) \exp\left(-2\int_x^{t_1} \lambda\, ds\right) dt_1 \\ &- \int_x^{\infty} r(t_1) \exp\left(-2\int_x^{t_1} \lambda\, ds\right)\left[\int_{t_1}^{\infty} r(t_2) q_{12}(t_2)\, dt_2\right] dt_1. \end{aligned}$$

When $1+\psi(n) \leq x < \infty$, the first integral of (5.20) is estimated by

$$\begin{aligned} &\int_x^{\infty} r(t_1) \exp\left(-2\int_x^{t_1} \lambda\, ds\right) dt_1 \\ &\quad \leq \int_x^{\infty} r(t_1)\, dt_1 \leq \frac{1}{2}\Big[\arctan l(\infty) - \arctan l(1+\psi(n))\Big] \longrightarrow 0 \end{aligned}$$

when $n \to \infty$, as done in (5.17)–(5.18). Next, since $q_{12}(x)$ is bounded by a bound which is independent of n, the second term of (5.20) tends, too, uniformly to 0 as $n \to \infty$.

The estimate of $l(x)$. As observed above, $l(x)$ is negative and increasing and, as in (5.18),

$$|\,l(x)| \leq |\,l\,(1+\psi(n))| = O\big((n^2\psi)^{-1/2}\big) \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

for x in $[1+\psi(n), \infty)$.

The estimate of $q_{11}(x)$. $q_{11}(x)$ is given by the alternating Leibnitz series (3.22), hence $-L[1] < q_{11}(x) \le 0$. By (5.8), $L[1] < \pi^2/32$, so we have

$$-\pi^2/32 < q_{11}(x) \le 0. \qquad \square$$

The case $\alpha \to \infty$. As $A = (\alpha^2 - 1)/2$,

$$\begin{aligned}\int_1^\infty |r(t)|\,dt &= \frac{1}{2}\left[-\arctan\left(\frac{1}{2\sqrt{A+B+C}}\right) + \arctan\left(\frac{1}{2\sqrt{A}}\right)\right] \\ &= O\big(\alpha^{-1}\big).\end{aligned}$$

This informs us by Theorem 3.1(c) that the error terms are $q_{jk} = O(1/\alpha)$ and $q_{jk}(x) \to 0$ uniformly on $[1,\infty)$ as $\alpha \to \infty$.

Our analysis also applies to the quantities $P_n^{(\alpha+an,\beta+bn)}(x)$, provided that $a \ne 0$, $b \ne 0$ and $1 + a + b + ab/2 > 0$. See [**2, 3**]. It can be easily verified that then $l(x) \to 0$ uniformly on $[1,\infty)$ as $n \to \infty$. Consequently, (5.3) simplifies to

$$\begin{aligned}P_n^{(\alpha+an,\beta+bn)}(x) &\sim (x-1)^{-(\alpha+an+1)/2}\,(x+1)^{-(\beta+bn+1)/2} p^{-1/4} \\ &\qquad \times \exp\left(\int_2^x \sqrt{p + (p'/4p)^2}\,ds\right)\end{aligned}$$

uniformly on $[1,\infty)$ as $n \to \infty$. $\square$

Darboux's formula. Finally, for real values of x in $[a,b]$ away from $[-1,1]$ and $n \to \infty$, our asymptotic approximation (4.4) also yields Darboux's asymptotic formula for Jacobi polynomials [**14**, Theorem 8.21.7]. For such an x, $l(x) = O(C^{-1/2}) = O(n^{-1})$, $\int_a^b |r(t)|\,dt = O(n^{-1})$, $p^{-1/4}(x) \sim n^{-1/2}(x^2-1)^{1/4}$ and

$$\lambda(x) = \Big(p(x)\big(1 + \ell^2(x)\big)\Big)^{1/2} = \frac{n + (\alpha+\beta+1)/2}{\sqrt{x^2-1}} + O(n^{-1}).$$

If we take in equation (4.4), $x_0 = 1$, then

$$\begin{aligned}\exp\left(\int_1^x \lambda\,ds\right) &\sim \exp\left(\int_1^x \frac{n + (\alpha+\beta+1)/2}{\sqrt{s^2-1}}\,ds\right) \\ &= \big(x + \sqrt{x^2-1}\,\big)^{n+(\alpha+\beta+1)/2}.\end{aligned}$$

On $[a,b]$ obviously $q_{jk}(x) \to 0$ uniformly as $n \to \infty$, so equation (4.4) yields

$$y_1(x) \sim n^{-1/2}\,(x^2-1)^{1/4}\,(x+\sqrt{x^2-1})^{n+(\alpha+\beta+1)/2}. \tag{5.21}$$

Since, for $x \geq 1$, $(x+\sqrt{x^2-1})^{1/2} \equiv [(x+1)^{1/2}+(x-1)^{1/2}]/\sqrt{2}$, we have

$$\begin{aligned} &y_1(x)(x-1)^{-(\alpha+1)/2}\,(x+1)^{-(\beta+1)/2} \\ &\qquad\sim n^{-1/2}\,(x+\sqrt{x^2-1})^{(n+1)/2}\,[(x+1)^{1/2}+(x-1)^{1/2}]^{\alpha+\beta}\,2^{-(\alpha+\beta)/2} \\ &\qquad\quad\times (x^2-1)^{-1/4}\,(x-1)^{-\alpha/2}\,(x+1)^{-\beta/2}, \end{aligned}$$

which is Darboux's formula for $P_n^{\alpha,\beta}(x)$ multiplied by $(2\pi)^{-1/2}2^{-(\alpha+\beta)/2}$.

Remarks. 1. An analogous analysis can be carried out on the interval $(-\infty,-1]$.

2. If we choose, in formulas (4.4)–(4.7), varying lower bounds in the integral equations for q_{jk} to be $1+\psi(n)$ rather than 1, q_{jk} will tend to zero as n goes to ∞, uniformly on $[1+\psi(n),\infty)$ for all j,k.

3. From formulas (4.4)–(4.7), one could get the asymptotic approximations of Jacobi functions of the second kind, namely, the asymptotic approximations for the second linearly independent solution of the given hypergeometric equation.

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