If we take another look at our example where

$$p(x) = (x-1)(x-2)^4(x-3)^2(x-4),$$

then we now find the bounds $2 + 4/7 \leq \eta_5 \leq 3 - 2/7$, which are an improvement over what we previously found. Furthermore, our technique is probably simple enough to allow for the computation of more complicated bounds, which take into account the separation between other roots as well.

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Qualitative Analysis of a Differential Equation of Abel

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In courses on differential equations, qualitative behaviour is frequently demonstrated via equations like $y' = y(1-y)$ or $y' = 1 - y^3$. Our aim is to show by elementary means that the naive looking

$$y' = \sin t - y^3$$

(an Abel equation in canonical form, see [**1**, p. 24]) is a useful instrument to display rich qualitative properties such as extendability and finite escape time, stability and instability, boundedness, and periodicity. We show the following:

(a) *The equation has a unique solution that is defined on* $(-\infty, \infty)$. *This solution is periodic.*

(b) *As* t *increases, each solution is attracted into the strip* $|y| < 1$ *in a uniformly bounded time interval. Any two solutions* $y_1(t)$, $y_2(t)$ *satisfy*

$$\lim_{t\to+\infty} \big(y_1(t) - y_2(t)\big) = 0.$$

(c) *As* t *decreases, each solution, except the periodic one, has a finite escape time. Namely, for each nonperiodic solution* $y(t)$ *there exists some* τ *such that* $\lim_{t\searrow\tau^+} |y(t)| = \infty$.

Let us outline the proof of these properties. See Figure 1.

In the half plane $y > 1$ each solution decreases. Let us separate the discussion into the domains $y \geq 2$ and $1 \leq y \leq 2$. Above the line $y = 2$, $y' = \sin t - y^3 \leq 1 - y^3 <$

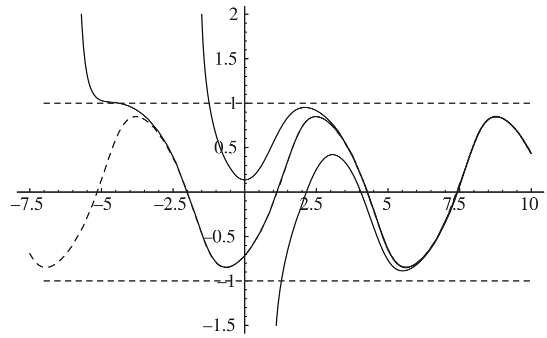

Figure 1. Four solutions, the periodic one is dashed.

$-y^3/2$, so for any $t_2 > t_1$, integrating the inequality $y^{-3}y' \le -1/2$ from t_1 to t_2 yields

$$1/y^2(t_2) - 1/y^2(t_1) \ge t_2 - t_1.$$

Therefore each solution $y(t)$ decreases from any value $y(t_1) > 2$ to $y(t_2) = 2$ during a time interval $[t_1, t_2]$ satisfying $t_2 - t_1 \le 1/4$. For $1 \le y \le 2$,

$$y' = \sin t - y^3 \le \sin t - 1,$$

so for $t_3 > t_2$ we have

$$y(t_3) - y(t_2) \le \int_{t_2}^{t_3} (\sin t - 1)\, dt = -\cos t_3 + \cos t_2 - (t_3 - t_2) \le 2 - (t_3 - t_2).$$

This rough estimate implies that $y(t)$ decreases from $y(t_2) = 2$ to $y(t_3) = 1$ during an interval $[t_2, t_3]$ with $t_3 - t_2 \le 3$. Once a solution arrives at any point $(t_3, 1)$, it must cross from $y > 1$ into $|y| < 1$. For, if $\sin t_3 < 1$ then $y'(t_3) < 0$, while if $\sin t_3 = 1$ then the conclusion follows by $y' = y'' = 0$, $y''' = -1$. The half plane $y < -1$ is treated similarly and the first part of (b) is verified.

As t decreases from an initial point t_0 with $y(t_0) > 1$, the situation is reversed and the solution is destined to blow up at some finite $\tau < t_0$. Indeed, if it does not blow up by the time t_1, $t_1 = t_0 - 3\frac{1}{4}$, then working forward from t_1, the previous paragraph implies $y(t_0) \le 1$. The same argument holds for $y(t_0) < -1$.

To verify the second sentence of (b), consider any two different solutions $y_1(t)$ and $y_2(t)$. Since solutions do not intersect, we may assume without loss of generality that $y_2(t) > y_1(t)$ for all t. By the inequality $u^3 - v^3 \ge (u - v)^3/4$ for $\infty > u > v > -\infty$ (which follows from $\min_{(-\infty,\infty)}(x^3 - 1)/(x - 1)^3 = 1/4$), we have

$$(y_2 - y_1)' = -(y_2^3 - y_1^3) \le -(y_2 - y_1)^3/4 < 0.$$

Integrating the inequality $(y_2 - y_1)^{-3}(y_2 - y_1)' \le -1/4$ from t_0 to t leads to

$$0 \le y_2(t) - y_1(t) \le \left[(t - t_0)/2 + (y_2(t_0) - y_1(t_0))^{-2}\right]^{-1/2} \to 0 \qquad \text{as } t \to \infty.$$

Finally we show that between the continuum of solutions that escape in finite time to $+\infty$ and those that escape to $-\infty$, there hides precisely one solution which is defined on the whole $(-\infty, \infty)$. Let $y_{\alpha\beta}$ be the solution which is defined by the initial value $y(\alpha) = \beta$, and let U (respectively L) be the set of points (α, β) such that the solution $y_{\alpha\beta}$ escapes at some time $\tau < \alpha$ to $+\infty$ (to $-\infty$). U is the basin of attraction of $+\infty$.

Then

(i) as shown above, U contains the half plane $y > 1$,

(ii) by the definition of U, $(\alpha, \beta) \in U$ implies that $(t, y_{\alpha\beta}(t)) \in U$ for every admissible t,

(iii) $(\alpha, \beta) \in U$ implies $(\alpha, \gamma) \in U$ for $\gamma > \beta$, since solutions do not intersect.

U is an open set: if $(\alpha, \beta) \in U$, there exists $t_1 < \alpha$ such that $y_{\alpha\beta}(t_1) > 10$. Due to the continuous dependence of a solution on the initial value condition, $y_{\gamma\delta}(t_1) > 9$ provided that (γ, δ) is sufficiently close to (α, β). By (i), the solution $y_{\gamma\delta}$ too escapes to $+\infty$ and consequently $(\gamma, \delta) \in U$. Similar arguments apply to L. Thus U, L are two open, disjoint sets.

Take any point (α, β) not in $U \cup L$. By (ii), the graph of the solution $y_{\alpha\beta}$ cannot enter either U or L. Hence $y_{\alpha\beta}$ is bounded, $-1 \le y_{\alpha\beta} \le 1$. If, in particular, $(\alpha, \beta) \in \partial U$, the boundary of U, then the graph $(t, y_{\alpha\beta}(t))$ of the corresponding solution must coincide with ∂U. Otherwise, if the graph leaves ∂U (and, of course, does not enter U), there exists t_1 such that $(t_1, y_{\alpha\beta}(t_1)) \notin \partial U$. Take some $\eta > y_{\alpha\beta}(t_1)$ such that $(t_1, \eta) \notin U$. Then the graph of the corresponding solution $y_{t_1\eta}(t)$ stays out of U. Since solutions do not intersect, $y_{t_1\eta}(t) > y_{\alpha\beta}(t)$ for every t. This contradicts the fact that the graph $(t, y_{\alpha\beta}(t))$ meets ∂U at (α, β).

Let us denote the solution whose graph coincides with ∂U by $y_u(t)$. Obviously $y_u(t + 2\pi)$ is a solution, bounded as well. If $y_u(t) \not\equiv y_u(t + 2\pi)$ then they never intersect and, by the previous argument, they must satisfy $y_u(t + 2\pi) < y_u(t)$ for every t. However the same can be said about $y_u(t - 2\pi)$ and we conclude that also $y_u(t - 2\pi) < y_u(t)$ for every t. This inequality, with t replaced by $t + 2\pi$, contradicts the previous one. Hence $y_u(t + 2\pi) \equiv y_u(t)$ and $y_u(t)$ is 2π-periodic.

The same analysis can be carried out for the solution $y_\ell(t)$ whose graph is the boundary of L, and it turns out that y_ℓ is periodic as well. On the other hand, by (b), $\lim_{t\to+\infty} \big(y_u(t) - y_\ell(t)\big) = 0$ and for two periodic solutions this may happen only if $y_u(t) \equiv y_\ell(t)$. So y_u is the only bounded solution and its graph is the boundary of both U and L. By (b) and (c), $y_u(t)$ is globally asymptotic stable as $t \to \infty$ and unstable as $t \to -\infty$.

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Closed Curves on Spheres

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Let S^2 denote the unit sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. By a *path* on S^2 we mean a continuous function $f : [0, 1] \to S^2$. The length $\lambda(f)$ of the path is defined by the formula