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Singular Sturm comparison theorems

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ABSTRACT

Sturm's comparison theorem is discussed for differential equations whose coefficients are continuous in an open, finite or infinite interval, but the coefficients cannot necessarily be extended continuously to the boundary points of the interval.

Sturm's comparison theorem is formulated usually as follows:

"Consider the two differential equations

$$u'' + p(x)u = 0, \tag{1}$$

$$v'' + P(x)v = 0, \tag{2}$$

where $P(x)$, $p(x)$ are two continuous functions in an interval $[a,b]$, $P(x) \geqslant p(x)$ but $P(x) \not\equiv p(x)$ there. If x_1, x_2 are two zeros of a solution u of (1) then every solution v of (2) has at least one zero in (x_1, x_2)."

This classical theorem, which was conceived in 1836, continues to draw attention. See, for example, [1]. The present work attempts to show one more point of view.

What can be said when the functions $P(x)$, $p(x)$ are continuous only in an open interval but are unbounded as its endpoints are approached and the zeros of the solution u are located at the singular endpoints? Such is, for example, the case for equation

$$y'' + \frac{1}{(1-x^2)^2} y = 0 \tag{3}$$

and its solution $y(x) = (1-x^2)^{1/2}$, which played a decisive role in the development of the theory of disconjugate differential equations and univalent analytic functions [5]. Quiet surprisingly the above formulation of Sturm's theorem is not necessarily valid, even if $P(x) > p(x)$ and the difference between $P(x)$ and $p(x)$ grows near the boundary points. In [6, p. 703], Nehari applies Sturm's comparison theorem to Eq. (3) without giving any hint why it is permissible. The present article is a result of our efforts to fill this gap.

Several results about singular equations are summarized in [7, Chapter 1.4]. The aim of this discussion is to check under which other singular situations the Sturm theorem continues to hold and to demonstrate by counter-examples when it cannot be extended. The following formulation treats two singular cases: a finite interval and unbounded coefficients and an infinite interval.

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Theorem 1 *(Singular Sturm theorem). Let $P(x)$, $p(x)$ be continuous functions on the open, finite or infinite interval (a,b) (but not necessarily at its endpoints), and $P(x) \geqslant p(x)$, $P(x) \not\equiv p(x)$ on (a,b).*

(i) *Suppose that the differential equation*

$$u'' + p(x)u = 0, \quad a < x < b, \tag{4}$$

has a solution u which satisfies the boundary conditions

$$\int_a \frac{dx}{u^2(x)} = \infty, \qquad \int^b \frac{dx}{u^2(x)} = \infty. \tag{5}$$

Then every solution of the equation

$$v'' + P(x)v = 0, \quad a < x < b, \tag{6}$$

has a zero in (a,b).

(ii) *In particular, if (a,b) is a finite interval and a positive solution u satisfies*

$$u(x) \leqslant M(x-a)^{\lambda_1} \quad \text{near } x = a,$$
$$u(x) \leqslant M(b-x)^{\lambda_2} \quad \text{near } x = b,$$

with $\lambda_1, \lambda_2 \geqslant \frac{1}{2}$, then every solution of Eq. (6) *has a zero in (a,b). If either $0 < \lambda_1 < \frac{1}{2}$ or if $0 < \lambda_2 < \frac{1}{2}$, then the above claim is not necessarily true.*

In the classical case, when $p(x)$ is continuous in the closed interval $[a,b]$, the zeros of solutions are simple and are included in the case $\lambda_1 = \lambda_2 = 1$.

(iii) *If $(a,b) = (0,\infty)$ and the differential equation*

$$u'' + p(x)u = 0, \quad 0 < x < \infty, \tag{7}$$

has a solution u which is positive in $(0,\infty)$ and satisfies

$$u(x) \leqslant Mx^{\lambda_1} \quad \text{near } x = 0,$$
$$u(x) \leqslant Mx^{\lambda_2} \quad \text{near } x = \infty,$$

with $\lambda_2 \leqslant \frac{1}{2} \leqslant \lambda_1$, then every solution of the equation

$$v'' + P(x)v = 0, \quad 0 < x < \infty, \tag{8}$$

has a zero in $(0,\infty)$. If $0 < \lambda_1 < \frac{1}{2}$ or $\lambda_2 > \frac{1}{2}$, then this claim is not necessarily true.

Proof. First let us recall that a solution $u(x)$ which satisfies

$$\int^b \frac{dx}{u^2(x)} = \infty$$

is called a *principal solution* at $x = b$. See [3, Chapter XI, Section 6]. Thus (i) states that Sturm's theorem holds if the same solution u is a principal solution at both endpoints of the interval.

(i) Suppose for a moment that $u(x) > 0$ in (a,b). We take a point c in (a,b) so that $P(x) \not\equiv p(x)$ both in (a,c) and in (c,b) and consider the solution v of (6) which has at c the same initial values as u has:

$$v(c) = u(c) > 0, \qquad v'(c) = u'(c). \tag{9}$$

It will be proved that in case (i) this solution $v(x)$ has a zero in (a,c) and a zero in (c,b), i.e., at least two zeros in (a,b). Suppose, on the contrary, that $v(x) \neq 0$ in (c,b), and in fact, due to the initial conditions, $v(x) > 0$ there. From the identity $(vu' - uv')' = vu'' - uv'' = (P-p)uv$ it follows that

$$\big(vu' - uv'\big)\big|_c^x = \int_c^x (P-p)uv\,dx.$$

By the initial value conditions $(vu' - uv')(c) = 0$. Since $u, v > 0$ and since $P \geqslant p$, $P(x) \not\equiv p(x)$ in $[c,b)$, the integral on the right-hand side increases and is positive for $c < x < b$. So for some suitable value d, $c < d < b$, there exists a positive lower bound

$$\big(vu' - uv'\big)(x) \geqslant C > 0, \quad d \leqslant x < b.$$

Since $u \neq 0$ in $[c, b)$,

$$\left(-\frac{v}{u}\right)' = \frac{vu' - uv'}{u^2} \geqslant \frac{C}{u^2} > 0, \quad d \leqslant x < b.$$

Integration on $[d, x]$ yields

$$-\frac{v(x)}{u(x)} + \frac{v(d)}{u(d)} \geqslant \int_d^x \frac{C\,dx}{u^2(x)} \tag{10}$$

and by (5),

$$\lim_{x \to b^-} \left[-\frac{v(x)}{u(x)}\right] = +\infty,$$

contradicting the assumption that $u, v > 0$ on $[c, b)$. Thus v must vanish at least once in (c, b). The proof that $v(x)$ has another zero in (a, c) is analogous. Once we know that a certain solution $v(x)$ of (6) has two zeros in (a, b), it follows by the classical separation theorem that every solution of (6) has at least one zero there.

Now suppose that $u(x)$ vanishes at one point $x = x_1$ in (a, b). Then its zero at x_1 is a simple zero and so $\int^{x_1} dx/u^2 = \infty$. Hence, we may repeat the above argument either for the interval (a, x_1) or for (x_1, b). Finally, if $u(x)$ has two or more (even infinitely many) zeros in (a, b), we take two consecutive zeros x_1, x_2 in (a, b) and apply the above argument (or Sturm's classical result) for the interval $[x_1, x_2]$. Note that this possibility cannot be overlooked. Indeed, as the referee pointed out, a principal solution may have infinitely many zeros. For example, the equation $y'' + (\frac{1}{4x^2} + \frac{\lambda}{x^2 \ln^2 x})y = 0$, $\lambda > 1/4$, $0 < x < 1$, has the solution $y(x) = (x \ln(1/x))^{1/2} \cos(\sqrt{\lambda - 1/4} \ln(\ln(1/x)))$ which is both principal and oscillatory near $x = 0$ and also near $x = 1$.

After the completion of this manuscript we became aware of [2]. In fact, our result can be deduced also from the "Relative convexity lemma" and Theorem 3 of [2]. However, our proof is more direct and entirely different.

(ii) By the assumption of (ii),

$$u^2(x) \leqslant K^2(b - x)^{2\lambda_2}, \quad d \leqslant x < b,$$

where the constant K is determined by the combination of two restrictions: the inequality $u(x) \leqslant M(b - x)^{\lambda_2}$ in some left neighborhood of $x = b$ and the boundedness of u away from $x = b$. Since in case (ii) we have $2\lambda_2 \geqslant 1$,

$$\int_d^x \frac{dx}{u^2(x)} \geqslant \frac{1}{K^2} \int_d^x \frac{dx}{(b - x)^{2\lambda_2}} \to +\infty \tag{11}$$

as $x \to b^-$, and the conclusion follows from (i). Fortunately, Nehari's equation (3) belongs to this class with $\lambda_1 = \lambda_2 = \frac{1}{2}$.

A concrete counter-example for the cases $0 < \lambda_1 < \frac{1}{2}$ or $0 < \lambda_2 < \frac{1}{2}$ is constructed as follows: $u(x) = (1 - x)^\alpha (1 + x)^\beta$ is a positive solution of the differential equation

$$u'' + p_{\alpha\beta}(x)u = 0, \quad -1 < x < 1,$$

with

$$p_{\alpha\beta}(x) = \frac{\alpha(1 - \alpha)}{(1 - x)^2} + \frac{\beta(1 - \beta)}{(1 + x)^2} + \frac{2\alpha\beta}{1 - x^2}.$$

If $0 < \alpha, \beta < \frac{1}{2}$, let us choose A, B such that $0 < \alpha < A < \frac{1}{2}$, $0 < \beta < B < \frac{1}{2}$. Since $\alpha(1 - \alpha) < A(1 - A)$, etc., we have

$$p_{AB}(x) > p_{\alpha\beta}(x) > 0 \quad \text{for } -1 < x < 1.$$

Nevertheless, the differential equation $v'' + p_{AB}(x)v = 0$ has the solution $v(x) = (1 - x)^A (1 + x)^B$ which has no zero in the interval $(-1, 1)$. Hence the Sturm comparison theorem cannot be extended to this pair of singular equations.

If only $0 < \alpha < \frac{1}{2}$ while $\frac{1}{2} \leqslant \beta < 1$, the choice $0 < \alpha < A < \frac{1}{2}$, $B = \beta$ leads to a similar counter-example.

(iii) When $(a, b) = (0, \infty)$ and $2\lambda_2 \leqslant 1$, inequality (11) is replaced by

$$\int_d^x \frac{dx}{u^2(x)} \geqslant \frac{1}{K^2} \int_d^x \frac{dx}{x^{2\lambda_2}} \to +\infty \tag{12}$$

as $x \to +\infty$, and the conclusion follows. Note that in the case $b = \infty$, $u(x)$ need not tend to zero as $x \to \infty$ and the singular version of Sturm's theorem may be valid without two "consecutive" zeros.

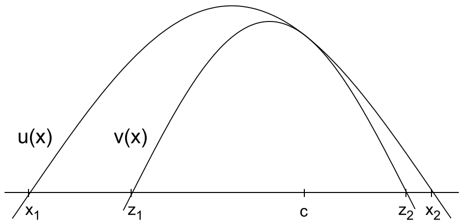

Fig. 1.

For counter-examples to case (iii) we utilize $u(x) = x^\alpha(1+x)^{\beta-\alpha}$. This function is a positive solution of the differential equation

$$u'' + p_{\alpha\beta}(x)u = 0, \quad 0 < x < \infty,$$

with

$$p_{\alpha b}(x) = \frac{\beta(1-\beta)x^2 + 2\alpha(1-\beta)x + \alpha(1-\alpha)}{x^2(1+x)^2} \tag{13}$$

and it satisfies $0 < u(x) \leqslant Mx^\alpha$ near $x = 0^+$, $0 < u(x) \leqslant Mx^\beta$ near $x = \infty$.

If $0 < \alpha < \frac{1}{2}$ and $0 \leqslant \beta \leqslant 1$, we choose A, B such that $\alpha < A < \frac{1}{2}$, $B = \beta$. Then

$$p_{AB}(x) > p_{\alpha\beta}(x) > 0 \quad \text{for } 0 < x < \infty,$$

but the differential equation $v'' + p_{AB}(x)v = 0$ has the solution $v(x) = x^A(1+x)^{B-A}$, which has no zero in $(0,\infty)$. Hence the Sturm comparison theorem cannot be extended to this pair of singular equations.

If $0 < \beta < \frac{1}{2}$ while $0 \leqslant \alpha \leqslant 1$, the choice $A = \alpha$, $\beta < B < \frac{1}{2}$ leads to a similar counter-example. □

Example. $u(x) = (1-x^2)^\lambda$ is a solution of the equation

$$u'' + \left[\frac{4\lambda(1-\lambda)}{(1-x^2)^2} + \frac{2\lambda(2\lambda-1)}{1-x^2}\right]u = 0,$$

while $v(x) = (1-x^2)^\lambda C_n^{(2\lambda-1/2)}(x)$ (where $C_n^{(\mu)}(x)$ denote the Gegenbauer polynomials) satisfies the equation

$$v'' + \left[\frac{4\lambda(1-\lambda)}{(1-x^2)^2} + \frac{(2\lambda+n)(2\lambda-1+n)}{1-x^2}\right]v = 0.$$

It is known that for $2\lambda - \frac{1}{2} = \mu > -\frac{1}{2}$, i.e., for $\lambda > 0$, the Gegenbauer polynomials $C_n^{(2\lambda-1/2)}(x)$ are orthogonal in $(-1,1)$ and all their n zeros are in $(-1,1)$. See [4, p. 41]. Hence for these two equations the consequences of Sturm's theorem hold when $\lambda > 0$. Thus the conditions $\lambda_1, \lambda_2 \geqslant \frac{1}{2}$ of (ii) are sufficient but not necessary.

The utilization of the initial values (9) enables another twist in the formulation of Sturm's theorem even in the classical case of continuous coefficients. The initial value conditions (9) mean that the graphs of $u(x)$ and $v(x)$ are tangent. The following contains an additional geometric variant of Sturm's theorem.

Theorem 2. *Let $P(x)$, $p(x)$ be continuous, $P(x) > p(x)$ on an interval $[a,b]$ and let $u(x)$ be a solution of* (4) *with two zeros, x_1, x_2. Then every solution $v(x)$ of* (6) *which is tangent to the solution $u(x)$ of* (4) *at some point $x = c$, $x_1 < c < x_2$, has at least two zeros in (x_1,x_2), one on each side of the tangency point $x = c$. Between c and the two adjacent zeros z_1, z_2 of $v(x)$ on the two sides of c, the graph of $v(x)$ lies strictly on one side of the graph of $u(x)$ (Fig. 1).*

Proof. Suppose, for simplicity, that $u(x) > 0$ for $x_1 < x < x_2$. In addition to (9),

$$v(c) = u(c) > 0, \qquad v'(c) = u'(c),$$

we have, by $P(x) > p(x)$, that

$$v''(c) = -P(c)v(c) < -p(c)u(c) = u''(c).$$

Hence $0 < v(x) < u(x)$ on some small intervals on both sides of $x = c$. It is known from Theorem 1(i) that $v(x)$ has a zero in (x_1,c) and a zero in (c,x_2). Suppose that $v(x)$ meets $u(x)$ again at some point $x = \bar{c}$, $c < \bar{c} < x_2$, without vanishing first, i.e., $0 < v(x) < u(x)$, $c < x < \bar{c}$. As $x \to \bar{c}$ from the left side, $v(x)$ approaches $u(x)$ from below, so the slope $u'(\bar{c})$ cannot be bigger than the slope $v'(\bar{c})$. Consequently

$$v(\bar{c}) = u(\bar{c}) > 0, \qquad v'(\bar{c}) \geqslant u'(\bar{c}). \tag{14}$$

But then the identity

$$vu' - uv'\Big|_c^{\bar c} = \int_c^{\bar c} (P - p)uv\,dx$$

together with the initial values (9), (14), yield

$$0 \geqslant u(\bar c)\big[u'(\bar c) - v'(\bar c)\big] - 0 = \int_c^{\bar c} (P - p)uv\,dx > 0,$$

a contradiction. So the graph of $v(x)$ must be below the graph of $u(x)$ for (z_1, z_2), as in Fig. 1.

An analogous claim may be formulated also for a singular equation on an open interval (a, b). $\square$

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