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In Memory of our Professors and Friends

Jan G. Krzyż, Zdzisław Lewandowski and Wojciech Szapiel

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JAN G. KRZYŻ, ZDZISŁAW LEWANDOWSKI AND WOJCIECH SZAPIEL

Convolution identities for zeta and related Dirichlet functions

D. AHARONOV and U. ELIAS

ABSTRACT. The article presents a complete set of convolution type identities for the Riemann zeta function and the Dirichlet eta, lambda and beta functions.

1. INTRODUCTION

An elegant property of the Riemann zeta function $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is the is well known convolution-like identity

$$\sum_{j=1}^{k-1} \zeta(2j)\,\zeta(2k-2j) = (k+\frac{1}{2})\zeta(2k). \tag{1.1}$$

Riemann zeta function is not alone. Three related functions are the Dirichlet eta, lambda and beta functions

$$\eta(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}, \quad \lambda(p) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^p}, \quad \beta(p) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^p},$$

which satisfy analogous identities. For example, the convolution of the λ and η functions is

$$\sum_{j=1}^{k-1} \lambda(2j)\,\eta(2k-2j) = (k-1)\lambda(2k). \tag{1.2}$$

Convolution identities appear in the classical literature and are well known in the area of number theory. Nielsen studied them in his encyclopedic book about the Gamma function [8, Chapter 3, p. 51]. [10] quotes many convolution identities of ζ, η, λ and β as a part of many other formulas which involve products of three or more such functions. The convolution identities are proved using several different methods.

Our aim is to present simpler unified methods proving systematically a complete set of $4+3+2+1=10$ convolution-like identities for the four functions ζ, η, λ and β. We name (1.2) the "(λ, η)-convolution" and each other identity will be called an "(a, b)-convolution" where $a, b \in \{\zeta, \eta, \lambda, \beta\}$. The convolutions are

$$(1.3) \quad (\zeta,\zeta): \quad \sum_{j=1}^{k-1} \zeta(2j)\,\zeta(2k-2j) = (k+\frac{1}{2})\,\zeta(2k),$$

$$(1.4) \quad (\zeta,\lambda): \quad \sum_{j=1}^{k-1} \zeta(2j)\,\lambda(2k-2j) = k\,\lambda(2k),$$

$$(1.5) \quad (\zeta,\eta): \quad \sum_{j=1}^{k-1} \zeta(2j)\,\eta(2k-2j) = k\,\eta(2k) - \frac{1}{2}\,\zeta(2k),$$

$$(1.6) \quad (\zeta,\beta): \quad \sum_{j=1}^{k} \zeta(2j)\,\beta(2k+1-2j) = (k-\frac{1}{2})\,\beta(2k+1) - \frac{\pi}{4}\,\frac{\eta(2k)}{2^{2k}},$$

$$(1.7) \quad (\lambda,\lambda): \quad \sum_{j=1}^{k-1} \lambda(2j)\,\lambda(2k-2j) = (k-\frac{1}{2})\,\lambda(2k),$$

$$(1.8) \quad (\lambda,\eta): \quad \sum_{j=1}^{k-1} \lambda(2j)\,\eta(2k-2j) = (k-1)\lambda(2k),$$

$$(1.9) \quad (\lambda,\beta): \quad \sum_{j=1}^{k} \lambda(2j)\,\beta(2k+1-2j) = k\,\beta(2k+1),$$

$$(1.10) \quad (\eta,\eta): \quad \sum_{j=1}^{k-1} \eta(2j)\,\eta(2k-2j) + \eta(2k) = (k-\frac{1}{2})\,\zeta(2k),$$

$$(1.11) \quad (\eta,\beta): \quad \sum_{j=1}^{k} \eta(2j)\,\beta(2k+1-2j) = (k-\frac{1}{2})\,\beta(2k+1) + \frac{\pi}{4}\,\frac{\eta(2k)}{2^{2k}},$$

$$(1.12) \quad (\beta,\beta): \quad \sum_{j=0}^{k-1} \beta(2j+1)\,\beta(2k-2j-1) = (k-\frac{1}{2})\,\lambda(2k).$$

These identities are summarized, formally, by the symmetric Table 1.

	ζ	λ	η	β
ζ	ζ	λ	η	β
λ	λ	λ	λ	β
η	η	λ	ζ	β
β	β	β	β	λ

Table 1.

2. A TRIGONOMETRIC APPROACH

Our starting point is the observation that the power series expansions of the four functions $\cot(\pi x)$, $\operatorname{cosec}(\pi x)$, $\tan(\pi x/2)$ and $\sec(\pi x/2)$ are naturally related to the functions ζ, η, λ, β by [11, Section 3:14],

$$\frac{\tan(\pi x/2)}{\pi x/2} = 1 + \frac{8}{\pi^2}\sum_{k=1}^{\infty}\lambda(2k+2)x^{2k}, \tag{2.1}$$

$$\sec\frac{\pi x}{2} = 1 + \frac{4}{\pi}\sum_{k=1}^{\infty}\beta(2k+1)x^{2k}, \tag{2.2}$$

$$\pi x\cot(\pi x) = 1 - 2\sum_{k=1}^{\infty}\zeta(2k)x^{2k}, \tag{2.3}$$

$$\frac{\pi x}{\sin(\pi x)} = 1 + 2\sum_{k=1}^{\infty}\eta(2k)x^{2k}. \tag{2.4}$$

Let us demonstrate a short proof of (2.1). Starting with the partial fraction expansion of the meromorphic function $\tan(\pi x/2)$, we get for $|x| < 1$ that

$$\begin{aligned}\frac{\tan(\pi x/2)}{\pi x/2} &= \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2 - x^2} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^2}\sum_{k=0}^{\infty}\left(\frac{x}{2n+1}\right)^{2k}\right)\\ &= \frac{8}{\pi^2}\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k+2}}\right)x^{2k},\end{aligned}$$

which is precisely (2.1). A similar approach may be applied to (2.2)–(2.4).

In this review we verify all ten convolution-like identities by pairwise multiplying all four basic trigonometric functions (2.1)–(2.4). It is a surprising observation that each such product is closely related to the derivative of one of the four functions. When each such trigonometric identity is expanded into power series, the comparison of the coefficients leads to one of the convolutions. We present here some examples and outline the other proofs.

In order to verify the (ζ, λ) convolution (1.4), we multiply (2.3) and (2.1) which are related, respectively, to $\zeta(2k)$ and $\lambda(2k)$:

$$\begin{aligned}\pi x \cot \pi x \cdot \frac{\tan(\pi x/2)}{\pi x/2} &= 2\,\frac{2\cos^2(\pi x/2)-1}{2\sin(\pi x/2)\cos(\pi x/2)} \cdot \frac{\sin(\pi x/2)}{\cos(\pi x/2)} \\ &= 2 - \sec^2(\pi x/2) = 2 - \frac{d}{dx}\left(\frac{\tan(\pi x/2)}{\pi/2}\right)\end{aligned} \tag{2.5}$$

The corresponding power series expansions are

$$\left(\frac{8}{\pi^2}\sum_{k=1}^{\infty}\lambda(2k)x^{2k-2}\right)\left(1-2\sum_{k=1}^{\infty}\zeta(2k)x^{2k}\right) = 2-\frac{8}{\pi^2}\sum_{k=1}^{\infty}(2k-1)\lambda(2k)x^{2k-2}.$$

Some manipulation and using $\lambda(2) = \pi^2/8$ reveal the required

$$\sum_{j=1}^{k-1}\zeta(2j)\,\lambda(2k-2j) = k\,\lambda(2k). \tag{2.6}$$

The (ζ, β) convolution looks slightly different from the other ones. For this identity it is natural to multiply (2.2) and (2.3). The equality

$$\begin{aligned}\pi x \cot \pi x \cdot \sec\frac{\pi x}{2} &= \pi x\,\frac{\cos^2(\pi x/2)-\sin^2(\pi x/2)}{2\sin(\pi x/2)\cos^2(\pi x/2)} \\ &= \frac{\pi x}{2}\operatorname{cosec}\left(\frac{\pi x}{2}\right) - x\frac{d}{dx}\left(\sec\frac{\pi x}{2}\right),\end{aligned} \tag{2.7}$$

is translated to

$$\begin{aligned}&\left(\frac{4}{\pi}\sum_{k=0}^{\infty}\beta(2k+1)x^{2k}\right)\left(1-2\sum_{k=1}^{\infty}\zeta(2k)x^{2k}\right) \\ &= \left(1+2\sum_{k=1}^{\infty}\eta(2k)\left(\frac{x}{2}\right)^{2k}\right) - \left(\frac{4}{\pi}\sum_{k=0}^{\infty}2k\,\beta(2k+1)x^{2k}\right).\end{aligned}$$

This simplifies into

$$\sum_{j=1}^{k}\zeta(2j)\,\beta(2k+1-2j) = (k-\frac{1}{2})\,\beta(2k+1) - \frac{\pi}{4}\,\frac{\eta(2k)}{2^{2k}}. \tag{2.8}$$

Each of the other convolutions is obtained by a suitable trigonometric identity which is constructed from the product of two of the functions (2.3)-(2.2) and the derivative of one of them. The required identities are summarized in Table 2.

(ζ,ζ)	$\left(\pi x\cot(\pi x)\right)^2 = -\pi^2x^2 - x^2\frac{d}{dx}\left(\pi\cot(\pi x)\right)$
(ζ,λ)	$\pi x\cot\pi x\cdot\frac{\tan(\pi x/2)}{\pi x/2} = 2 - \frac{d}{dx}\left(\frac{\tan(\pi x/2)}{\pi/2}\right)$
(ζ,η)	$\pi x\cot\pi x\cdot\pi x\operatorname{cosec}\pi x = -\pi x^2\frac{d}{dx}\left(\operatorname{cosec}(\pi x)\right)$
(ζ,β)	$\pi x\cot\pi x\cdot\sec\frac{\pi x}{2} = \frac{\pi x}{2}\operatorname{cosec}(\frac{\pi x}{2}) - x\frac{d}{dx}\left(\sec\frac{\pi x}{2}\right)$
(λ,λ)	$\left(\tan(\pi x/2)\right)^2 = \frac{d}{dx}\left(\frac{\tan(\pi x/2)}{\pi/2}\right) - 1$
(λ,η)	$\frac{\tan(\pi x/2)}{\pi x/2}\cdot\pi x\operatorname{cosec}\pi x = \frac{d}{dx}\left(\frac{\tan(\pi x/2)}{\pi/2}\right)$
(λ,β)	$\frac{\tan(\pi x/2)}{\pi x/2}\cdot\sec(\frac{\pi x}{2}) = \frac{4}{\pi^2 x}\frac{d}{dx}\left(\sec(\frac{\pi x}{2})\right)$
(η,η)	$\left(x\pi\operatorname{cosec}(\pi x)\right)^2 = x\pi\cot(\pi x) - x\frac{d}{dx}\left(\pi x\cot(\pi x)\right)$
(η,β)	$\pi x\operatorname{cosec}\pi x\cdot\sec\frac{\pi x}{2} = \frac{\pi x}{2}\operatorname{cosec}(\frac{\pi x}{2}) + x\frac{d}{dx}\left(\sec(\frac{\pi x}{2})\right)$
(β,β)	$\left(\sec(\frac{\pi x}{2})\right)^2 = \frac{d}{dx}\left(\frac{\tan(\pi x/2)}{\pi/2}\right)$

Table 2.

The obvious linear relation

$$\zeta(p) + \eta(p) = 2\lambda(p) \tag{2.9}$$

hints that the ten convolution identities (1.3)-(1.12) are not independent. Moreover, by $\zeta(p) = \sum(2n)^{-p} + \sum(2n+1)^{-p} = 2^{-p}\zeta(p) + \lambda(p)$ and by (2.9), it follows that

$$\lambda(p) = \left(1 - 2^{-p}\right)\zeta(p), \qquad \eta(p) = \left(1 - 2^{1-p}\right)\zeta(p). \tag{2.10}$$

Six of the ten convolutions involve only ζ, λ and η and (2.9) reduces the number of essentially different convolutions of ζ, λ and η from six to three. For example, the (ζ,ζ) and (ζ,λ) relations imply the (ζ,η) identity; (ζ,λ) plus (λ,λ) imply the $(\eta,\lambda)\equiv(\lambda,\eta)$ identity; and the above deduced (ζ,η) and (λ,η) imply the (η,η) identity. In fact, every three identities out of the six involving ζ, λ and η, which are not located on one row or one column of

Table 1, imply the other three. Similarly, among the four identities involving β, only three are essential since (ζ, β) plus (η, β) imply (λ, β).

A remark. The decision to work with $\cot(\pi x)$, $\tan(\pi x/2)$, $\operatorname{cosec}(\pi x)$ and $\sec(\pi x/2)$ not only leads to elegant formulas but it has an additional bonus in other areas. Let us mention one example:

$\zeta(p)$ and $\lambda(p)$ monotonically decrease with p, while $\eta(p)$ and $\beta(p)$ increase. Consequently, all coefficients in the expansion

$$(2.11)\quad (1-x^2)\frac{\tan(\pi x/2)}{\pi x/2} = 1 + \frac{8}{\pi^2}\sum_{k=1}^{\infty}(\lambda(2k+2)-\lambda(2k))x^{2k}, \quad |x|<3,$$

except the first one, are negative and the function is decreasing and concave for $0 \le x < 3$. $\lim_{x\to 1}(1-x^2)\tan(\pi x/2)/(\pi x/2)$ is easily calculated to be $8/\pi^2$ and consequently the inequality of Becker and Stark [2],

$$(2.12)\qquad \frac{8}{\pi^2} \le (1-x^2)\frac{\tan(\pi x/2)}{\pi x/2} \le 1, \qquad |x|<1,$$

follows.

The left hand side of (2.12) and the monotonicity property mentioned above are surprisingly connected to the theory of univalent functions. Recall that by Hayman's regularity theorem [5, Thms. 1.5, 1.12], [9, p. 141], for every normalized univalent function $f(z) = z + a_2 z^2 + \dots$, $|z| < 1$, the quantity $(1-r)^2 M(r,f)/r$ decreases to α as $r \to 1$ and $|a_n|/n \to \alpha$ as $n \to \infty$, for some α, $0 \le \alpha \le 1$. Any odd univalent function in $|z|<1$, $h(z) = z + \sum c_{2n+1} z^{2n+1}$, is a transform $h(z) = \sqrt{f(z^2)}$ of some univalent $f(z)$, so

$$(1-r^2)M(r,h)/r \quad \text{decreases to} \quad \sqrt{\alpha} \quad \text{as} \quad r\to 1$$

and

$$\lim_{n\to\infty} |c_{2n+1}| = \sqrt{\alpha},$$

with the corresponding $\alpha \le 1$ [4, Theorem 5.8]. In our case the function $h(z) = \tan(\pi z/2)/(\pi/2)$ is such an odd, univalent function, and according to our previous observations, $(1-x^2)\dfrac{\tan(\pi x/2)}{\pi x/2}$ decreases to $8/\pi^2$ as $x \to 1^-$.

3. A logarithmic approach

Some, but not all, of the convolution identities can be proved also by the following well known claim about the logarithm of power series [7, (2.28)]:

Lemma 3.1. *If* $\ln\left(\sum_{k=0}^{\infty} d_k z^k\right) = \sum_{k=0}^{\infty} a_k z^k$, $d_0 \neq 0$, *then*

$$(3.1)\qquad \sum_{j=1}^{k} j\, a_j\, d_{k-j} = k\, d_k.$$

Indeed, the derivative of the logarithm is

$$\frac{\sum_{k=1}^{\infty} k d_k z^{k-1}}{\sum_{k=0}^{\infty} d_k z^k} = \sum_{k=1}^{\infty} k a_k z^{k-1}$$

and (3.1) follows by multiplication and comparison of coefficients. (3.1) was important for the derivation of the Lebedev-Milin inequality [6, p. 581]. This inequality plays an essential role in the proof of the Milin conjecture [6, p. 584] by de Branges that led to the solution of the famous Bieberbach conjecture [6, p. 610].

We plan to apply the lemma to four functions:

$$\ln\left(1+2\sum_{k=1}^{\infty}\eta(2k)x^{2k}\right) = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}x^{2k}, \tag{3.2}$$

$$\ln\left(\frac{4}{\pi}\sum_{k=0}^{\infty}\beta(2k+1)x^{2k}\right) = \sum_{k=1}^{\infty}\frac{\lambda(2k)}{k}x^{2k}, \tag{3.3}$$

$$\ln\left(\frac{8}{\pi^2}\sum_{k=0}^{\infty}\lambda(2k+2)x^{2k}\right) = \sum_{k=1}^{\infty}\frac{\eta(2k)}{k}x^{2k}, \tag{3.4}$$

and

$$\ln\left(1-2\sum_{k=1}^{\infty}\zeta(2k)\left(\frac{x}{2}\right)^{2k}\right) = -\sum_{k=1}^{\infty}\frac{\eta(2k)}{k}x^{2k}. \tag{3.5}$$

To prove (3.2)-(3.5), we start from some well known expansions of the logarithms of our basic trigonometric functions [1, p. 75], [11, Section 3:14]:

$$\ln(\pi x \operatorname{cosec} \pi x) = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}x^{2k}, \tag{3.6}$$

$$\ln\left(\sec\frac{\pi x}{2}\right) = \sum_{k=1}^{\infty}\frac{\lambda(2k)}{k}x^{2k}, \tag{3.7}$$

$$\ln\left(\frac{\tan(\pi x/2)}{\pi x/2}\right) = \sum_{k=1}^{\infty}\frac{\eta(2k)}{k}x^{2k} \tag{3.8}$$

and the negative of (3.8),

$$\ln\left(\frac{\pi x}{2}\cdot\cot\frac{\pi x}{2}\right) = -\sum_{k=1}^{\infty}\frac{\eta(2k)}{k}x^{2k}. \tag{3.9}$$

By the way, each of (3.6)-(3.9) follows from the integration of one of our basic trigonometric functions. For example, (3.6) is verified by integration of the expansion $\pi\cot\pi x = x^{-1} - 2\sum_{k=1}^{\infty}\zeta(2k)x^{2k-1}$. To this end, (3.2) is the combination of (2.4) and (3.6), (3.3) follows from (2.2) and (3.7), (2.1) and (3.8) produce identity (3.4) and (3.5) is the result of (2.3) and (3.9).

Now we are ready to use Lemma 3.1. Applying Lemma 3.1 to (3.2) results in

$$\sum_{j=1}^{k-1} 2j\,\frac{\zeta(2j)}{2j}\cdot 2\eta(2k-2j)+2k\,\frac{\zeta(2k)}{2k}\cdot 1 = 2k\,\eta(2k),$$

which is the (ζ,η) identity (1.5). With the aid of Lemma 3.1, we achieve from (3.3) the (λ,β) identity. Finally, Lemma 3.1 and (3.4) may used to obtain another proof to the (λ,η) identity.

Of course, not all the convolution-like identities can be deduced by Lemma 3.1, since (3.1) contains d_n terms on both of its sides.

A similar procedure applied to (3.5) leads to a result of a different type. Lemma 3.1 translates (3.5) into an identity whose raw form is

$$\sum_{j=1}^{k-1} 2j\,\frac{-2\eta(2j)}{j}\cdot\frac{-2\zeta(2k-2j)}{2^{2k-2j}}+2k\,\frac{-\eta(2k)}{k}\cdot 1 = 2k\cdot\frac{-2\zeta(2k)}{2^{2k}}.$$

After some simplification and use of (2.10), it becomes

$$\sum_{j=1}^{k-1} 2^{2j}\,\eta(2j)\zeta(2k-2j) = \left(2^{2k-1}-k-1\right)\zeta(2k). \tag{3.10}$$

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