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IMPROVED INEQUALITIES FOR TRIGONOMETRIC FUNCTIONS VIA DIRICHLET AND ZETA FUNCTIONS

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Abstract. The article discusses several improvements of well known inequalities for trigonometric functions. We utilize the monotonicity of the Riemann zeta function, as well as the Dirichlet eta, beta and lambda functions, to shorten the proofs of known inequalities for trigonometric functions, and to obtain new ones.

1. Introduction

The purpose of this work is to suggest a method which leads to short proofs for a family of trigonometric inequalities; among them, the inequality of Becker and Stark [2]

$$\frac{8/\pi^2}{1-x^2} \leqslant \frac{\tan(\pi x/2)}{\pi x/2} \leqslant \frac{1}{1-x^2}, \qquad -1<x<1, \tag{1.1}$$

the inequality of Redheffer [6]

$$\frac{1-x^2}{1+x^2} \leqslant \frac{\sin(\pi x)}{\pi x}, \qquad -1<x<1, \tag{1.2}$$

and the inequalities of Chen and Qi [4], where for all $-1<x<1$,

$$\frac{2}{\pi}\,\frac{x}{1-x^2} \leqslant \frac{1}{\pi x} - \cot(\pi x) \leqslant \frac{\pi}{3}\,\frac{x}{1-x^2}, \tag{1.3}$$

$$\frac{\pi^2}{8}\,\frac{x^2}{1-x^2} \leqslant \sec\frac{\pi x}{2} - 1 \leqslant \frac{4}{\pi}\,\frac{x^2}{1-x^2}, \tag{1.4}$$

$$\frac{\pi}{6}\,\frac{x}{1-x^2} \leqslant \operatorname{cosec}(\pi x) - \frac{1}{\pi x} \leqslant \frac{2}{\pi}\,\frac{x}{1-x^2}. \tag{1.5}$$

The main tool in doing so is to express the power series expansions of trigonometric functions in terms of the Riemann zeta function and the Dirichlet eta, beta and lambda functions,

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \eta(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}, \quad \beta(p) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^p}, \quad \lambda(p) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^p}.$$

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The expansions we use are, [7, Section 3:14],

$$\frac{\tan(\pi x/2)}{\pi x/2} = 1 + \frac{8}{\pi^2}\sum_{k=1}^{\infty}\lambda(2k+2)x^{2k}, \tag{1.6}$$

$$\sec\frac{\pi x}{2} = 1 + \frac{4}{\pi}\sum_{k=1}^{\infty}\beta(2k+1)x^{2k}, \tag{1.7}$$

$$\pi x\cot(\pi x) = 1 - 2\sum_{k=1}^{\infty}\zeta(2k)x^{2k}, \tag{1.8}$$

$$\frac{\pi x}{\sin(\pi x)} = 1 + 2\sum_{k=1}^{\infty}\eta(2k)x^{2k}, \tag{1.9}$$

$$\ln\left(\frac{\tan(\pi x/2)}{\pi x/2}\right) = \sum_{k=1}^{\infty}\frac{\eta(2k)}{k}x^{2k}, \tag{1.10}$$

$$\ln\left(\sec\frac{\pi x}{2}\right) = \sum_{k=1}^{\infty}\frac{\lambda(2k)}{k}x^{2k}. \tag{1.11}$$

$$\ln\left(\frac{\pi x}{\sin(\pi x)}\right) = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}x^{2k}, \tag{1.12}$$

Let us demonstrate a short proof of (1.6). Starting with the partial fraction expansion of the meromorphic function $\tan(\pi x/2)$, we get for $|x| < 1$ that

$$\begin{aligned}\frac{\tan(\pi x/2)}{\pi x/2} &= \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2 - x^2} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^2}\sum_{k=0}^{\infty}\left(\frac{x}{2n+1}\right)^{2k}\right)\\ &= \frac{8}{\pi^2}\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k+2}}\right)x^{2k},\end{aligned}$$

which is precisely (1.6). A similar approach may be applied to (1.7)–(1.12). Surely, expansions (1.6)–(1.12) are much simpler than the equivalent presentations which involve Bernoulli and Euler numbers. The chief benefit of these expansions lies, however, in the monotonic decreasing behavior of the coefficients $\zeta(2k)$ and $\lambda(2k)$, and the increasing behavior of $\eta(2k)$ and $\beta(2k+1)$. The decreasing of $\zeta(2k)$ and $\lambda(2k)$ is obvious. To show that $\eta(p)$ is monotone increasing, write

$$\eta(p) = 1 - \left[\left(\frac{1}{2^p} - \frac{1}{3^p}\right) + \left(\frac{1}{4^p} - \frac{1}{5^p}\right) + \ldots\right],$$

and note that each term in the square brackets can be written as

$$\begin{aligned}\frac{1}{n^p} - \frac{1}{(n+1)^p} &= \frac{(n+1)^p - n^p}{n^p(n+1)^p} = \frac{\big[(n+1) - n\big]\big[(n+1)^{p-1} + (n+1)^{p-2}n + \ldots + n^{p-1}\big]}{n^p(n+1)^p}\\ &= \frac{1}{n^p(n+1)} + \frac{1}{n^{p-1}(n+1)^2} + \ldots + \frac{1}{n(n+1)^p},\end{aligned}$$

which is a decreasing function of p. A similar argument shows that $\beta(p)$ is increasing as well.

In Section 2 we use the monotonic behavior of the coefficients to give short proofs to inequalities (1.1)–(1.5).

In Section 3 we apply the same idea to the three one-sided inequalities, [4],

$$\ln\left(\frac{\tan(\pi x/2)}{\pi x/2}\right) \leqslant \frac{\pi^2}{12}\,\frac{x^2}{1-x^2}, \tag{1.13}$$

$$\ln\left(\sec\frac{\pi x}{2}\right) \leqslant \frac{\pi^2}{8}\,\frac{x^2}{1-x^2}, \tag{1.14}$$

$$\ln\left(\frac{\pi x}{\sin(\pi x)}\right) \leqslant \frac{\pi^2}{6}\,\frac{x^2}{1-x^2}. \tag{1.15}$$

We shall give lower bounds to the functions in the left-hand sides of (1.13)–(1.14), as well as improve the given upper bounds.

Section 4 is devoted to the two-sided inequality for the remainder of the power series of the tangent function,

$$\frac{2^{2(n+1)}\left(2^{2(n+1)}-1\right)B_{n+1}}{(2n+2)!}\,x^{2n}\tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n} x^{2n}\tan x, \tag{1.16}$$

proved by Chen and Qi in [3]. Using the power series (1.6), we provide a short proof to these inequalities.

All the above inequalities, as well as many others, can be found in the extensive review of Qi et al, [5]. Associated inequalities for Bessel functions are discussed by Baricz [1, Chapter 3].

2. Inequalities (1.1)–(1.5)

We begin by proving the Becker-Stark inequality (1.1). Since

$$1 < \lambda(2k+2) \leqslant \lambda(4) = \pi^4/96$$

for all $k \geqslant 1$, we use expansion (1.6) to obtain for $|x| < 1$,

$$1+\frac{(8/\pi^2)x^2}{1-x^2} = 1+\frac{8}{\pi^2}\sum_{k=1}^{\infty} x^{2k} \leqslant \frac{\tan(\pi x/2)}{\pi x/2} \leqslant 1+\frac{8}{\pi^2}\lambda(4)\sum_{k=1}^{\infty} x^{2k} = 1+\frac{(\pi^2/12)x^2}{1-x^2}. \tag{2.1}$$

Rewriting (2.1) as

$$\frac{(8/\pi^2)+(1-8/\pi^2)(1-x^2)}{1-x^2} \leqslant \frac{\tan(\pi x/2)}{\pi x/2} \leqslant \frac{1-(1-\pi^2/12)x^2}{1-x^2}, \tag{2.2}$$

it becomes clear that (2.2) implies (1.1).

Next, since $1 < \zeta(2k) \leqslant \zeta(2) = \pi^2/6$ for all $k \geqslant 1$, we deduce from (1.8) that for $|x| < 1$,

$$\frac{2}{\pi}\,\frac{x}{1-x^2} = \frac{2}{\pi}\sum_{k=1}^{\infty} x^{2k-1} \leqslant \frac{1}{\pi x} - \cot(\pi x) \leqslant \frac{2}{\pi}\,\zeta(2)\sum_{k=1}^{\infty} x^{2k-1} = \frac{\pi}{3}\,\frac{x}{1-x^2}\,,$$

which is (1.3).

The proof of inequality (1.4) is a similar one-liner. Since $\pi^3/32 = \beta(3) \leqslant \beta(2k+1) < 1$ for all $k \geqslant 1$, we get from (1.7),

$$1 + \frac{\pi^2}{8}\,\frac{x^2}{1-x^2} = 1 + \frac{4}{\pi}\cdot\beta(3)\sum_{k=1}^{\infty} x^{2k} \leqslant \sec\frac{\pi x}{2} \leqslant 1 + \frac{4}{\pi}\sum_{k=1}^{\infty} x^{2k} = 1 + \frac{4}{\pi}\,\frac{x^2}{1-x^2}\,, \quad (2.3)$$

and (1.4) follows.

Finally, the proof of inequality (1.5) is similarly short. Since $\pi^2/12 = \eta(2) \leqslant \eta(2k) < 1$ for all $k \geqslant 1$, we have from (1.9) that for $|x| < 1$,

$$1 + \frac{\pi^2}{6}\,\frac{x^2}{1-x^2} = 1 + 2\eta(2)\sum_{k=1}^{\infty} x^{2k} \leqslant \frac{\pi x}{\sin(\pi x)} \leqslant 1 + 2\sum_{k=1}^{\infty} x^{2k} = 1 + \frac{2x^2}{1-x^2} = \frac{1+x^2}{1-x^2}. \quad (2.4)$$

Hence,

$$\frac{1-x^2}{1+x^2} \leqslant \frac{\sin(\pi x)}{\pi x} \leqslant \frac{1-x^2}{1+\left(\frac{\pi^2}{6}-1\right)x^2}\,, \quad (2.5)$$

which yields (1.5) by a simple algebraic manipulation.

Note that the left hand side of (2.5) coincides with Redheffer's inequality in (1.2) for $|x| < 1$. While numerous proofs were suggested for this inequality, ours is by far the shortest. We remind the reader that Redheffer's inequality holds also for $|x| \geqslant 1$, but for $x \geqslant 1$ it follows at once from $\left|\frac{\sin \pi x}{\pi x}\right| = \left|\frac{\sin \pi(x-1)}{\pi x}\right| \leqslant \frac{x-1}{x} \leqslant \frac{x^2-1}{x^2+1}$.

We observe that (2.3) can be written as

$$\frac{1-x^2}{1+\left(\frac{4}{\pi}-1\right)x^2} \leqslant \cos\frac{\pi x}{2} \leqslant \frac{1-x^2}{1+\left(\frac{\pi^2}{8}-1\right)x^2}\,, \qquad |x| < 1\,. \quad (2.6)$$

We claim that the coefficients of x^2 in the respective denominators of (2.5) and (2.6), namely $1, \frac{\pi^2}{6}-1, \frac{4}{\pi}-1$ and $\frac{\pi^2}{8}-1$, are the best possible ones. Dealing with (2.6), let us begin by showing that in the left hand side denominator of (2.6) the constant $\frac{4}{\pi}-1$ cannot be replaced by a smaller number. Indeed, as $x \to 1^-$ we have, putting $x = 1-z$,

$$\begin{aligned}\cos\frac{\pi x}{2} - \frac{1-x^2}{1+ax^2} &= \cos\left(\frac{\pi}{2} - \frac{\pi z}{2}\right) - \frac{2z-z^2}{(1+a)-2az+az^2}\\ &= \sin\frac{\pi z}{2} - \frac{2}{1+a}z + \mathscr{O}(z^2) = \left(\frac{\pi}{2} - \frac{2}{1+a}\right)z + \mathscr{O}(z^2).\end{aligned}$$

This is positive for small $z>0$ only if $\frac{\pi}{2}-\frac{2}{1+a}\geqslant 0$, hence $a\geqslant \frac{4}{\pi}-1$ is necessary. Next, in the right hand side denominator of (2.6) the constant $\frac{\pi^2}{8}-1$ cannot be replaced by any larger number, since near $x=0$, $\cos\frac{\pi x}{2}-\frac{1-x^2}{1+ax^2}=\left(-\frac{\pi^2}{8}+1+a\right)x^2+\mathscr{O}(x^4)$ is negative only if $a\leqslant\frac{\pi^2}{8}-1$. Proving the optimality of the constants 1 and $\frac{\pi^2}{6}-1$ in the denominator of (2.5) is left to the reader.

Inequalities (2.5) and (2.6) may be improved if we take additional exact terms of the corresponding power series. For example, if we note that $\eta(2)<\eta(4)\leqslant\eta(2k)<1,\ k\geqslant 2$, we get

$$1+2\left[\eta(2)x^2+\eta(4)\sum_{k=2}^{\infty}x^{2k}\right]\leqslant\frac{\pi x}{\sin(\pi x)}\leqslant 1+2\left[\eta(2)x^2+\sum_{k=2}^{\infty}x^{2k}\right], \tag{2.7}$$

with $\eta(2)=\pi^2/12<\eta(4)=7\pi^4/720$. These inequalities may be presented also as

$$\frac{1-x^2}{1+x^2-2\big(1-\eta(2)\big)x^2(1-x^2)}\leqslant\frac{\sin(\pi x)}{\pi x}\leqslant\frac{1-x^2}{1+\left(\frac{\pi^2}{6}-1\right)x^2+2\big(\eta(4)-\eta(2)\big)x^4},$$

which is obviously an improvement of (2.5).

Similarly, since $\beta(3)<\beta(5)\leqslant\beta(2k+1)<1$ for all $k\geqslant 2$, then

$$1+\frac{4}{\pi}\left[\beta(3)x^2+\beta(5)\sum_{k=2}^{\infty}x^{2k}\right]\leqslant\sec\frac{\pi x}{2}\leqslant 1+\frac{4}{\pi}\left[\beta(3)x^2+\sum_{k=2}^{\infty}x^{2k}\right], \tag{2.8}$$

i.e.,

$$\frac{1-x^2+\frac{4}{\pi}\beta(3)x^2(1-x^2)+\frac{4}{\pi}\beta(5)x^4}{1-x^2}\leqslant\sec\frac{\pi x}{2}\leqslant\frac{1-x^2+\frac{4}{\pi}\beta(3)x^2(1-x^2)+\frac{4}{\pi}x^4}{1-x^2},$$

where $\beta(3)=\pi^3/32<\beta(5)=5\pi^5/1536$. These last inequalities may take the form

$$\frac{1-x^2}{1+\left(\frac{4}{\pi}-1\right)x^2-\frac{4}{\pi}\big(1-\beta(3)\big)x^2(1-x^2)}\leqslant\cos\frac{\pi x}{2}\leqslant\frac{1-x^2}{1+\left(\frac{\pi^2}{8}-1\right)x^2+\frac{4}{\pi}\big(\beta(5)-\beta(3)\big)x^4},$$

which improves (2.6).

Evidently, one can further improve the above inequalities by considering further terms of the corresponding power series.

3. Inequalities (1.13)–(1.15)

The following claim generalizes and improves inequalities (1.13)–(1.15).

THEOREM 3.1. *For all* $|x| < 1$,

$$\frac{\pi^2}{12}\ln\left(1+\frac{x^2}{1-x^2}\right) \leqslant \ln\left(\frac{\tan(\pi x/2)}{\pi x/2}\right) \leqslant \ln\left(1+\frac{\pi^2}{12}\frac{x^2}{1-x^2}\right), \tag{3.1}$$

$$\ln\left(1+\frac{\pi^2}{8}\frac{x^2}{1-x^2}\right) \leqslant \ln\left(\sec\frac{\pi x}{2}\right) \leqslant \frac{\pi^2}{8}\ln\left(1+\frac{x^2}{1-x^2}\right), \tag{3.2}$$

$$\ln\left(1+\frac{\pi^2}{6}\frac{x^2}{1-x^2}\right) \leqslant \ln\left(\frac{\pi x}{\sin(\pi x)}\right) \leqslant \frac{\pi^2}{6}\ln\left(1+\frac{x^2}{1-x^2}\right). \tag{3.3}$$

Proof. The right hand sides in inequalities (3.1)–(3.3) are, respectively, the logarithms of the right hand sides of (2.1), (2.3) and (2.4).

We now turn to the left hand side inequalities in (3.1)–(3.3). Since $\eta(2k)$ increases and $\eta(2) = \pi^2/12$, we have by (1.10),

$$\ln\frac{\tan(\pi x/2)}{\pi x/2} = \sum_{k=1}^{\infty}\frac{\eta(2k)}{k}x^{2k} \geqslant \eta(2)\sum_{k=1}^{\infty}\frac{x^{2k}}{k} = -\frac{\pi^2}{12}\ln\left(1-x^2\right) = \frac{\pi^2}{12}\ln\left(1+\frac{x^2}{1-x^2}\right)$$

for $0 < |x| < 1$. Thus, the lower bound in (3.1) is verified.

Next, as $\lambda(2k)$ decreases and $\lambda(2) = \pi^2/8$, it follows by (1.11) that

$$\ln\sec\frac{\pi x}{2} = \sum_{k=1}^{\infty}\frac{\lambda(2k)}{k}x^{2k} \leqslant \lambda(2)\sum_{k=1}^{\infty}\frac{x^{2k}}{k} = \frac{\pi^2}{8}\ln\left(1+\frac{x^2}{1-x^2}\right),$$

for $0 < |x| < 1$, as claimed in (3.2).

Finally, as for inequality (3.3), we observe that $\zeta(2k)$ decreases and $\zeta(2) = \pi^2/6$; so by (1.12),

$$\ln\frac{\pi x}{\sin(\pi x)} = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}x^{2k} \leqslant \zeta(2)\sum_{k=1}^{\infty}\frac{x^{2k}}{k} = \frac{\pi^2}{6}\ln\left(1+\frac{x^2}{1-x^2}\right)$$

for $0 < |x| < 1$, which is the lower bound in (3.3). □

We note that the upper bounds in (3.1)–(3.3) are better for all $|x| < 1$ than those in (1.13)–(1.15). This is so since $\ln(1+t) < t$ for $t > 0$. Further, as $x \to 1^-$, the upper bounds of (3.1)–(3.3) are of a strictly smaller order of magnitude than those in (1.13)–(1.15), since $\ln(1+t) \ll t$ as $t \to +\infty$.

Moreover, near $x = 0$ all the upper and the lower bounds of (3.1)–(3.3) are asymptotically strict, since their respective ratios satisfy $\lim_{t\to 0}\ln(1+at)/a\ln(1+t) = 1$.

An equivalent way to write (3.1)–(3.3) is

$$\left(1-x^2\right)^{-\pi^2/12} \leqslant \frac{\tan(\pi x/2)}{\pi x/2} \leqslant \frac{1-\left(1-\frac{\pi^2}{12}\right)x^2}{1-x^2}, \tag{3.4}$$

$$\left(1-x^2\right)^{\pi^2/8} \leqslant \cos\frac{\pi x}{2} \leqslant \frac{1-x^2}{1+\left(\frac{\pi^2}{8}-1\right)x^2}, \tag{3.5}$$

$$\left(1-x^2\right)^{\pi^2/6} \leqslant \frac{\sin(\pi x)}{\pi x} \leqslant \frac{1-x^2}{1+\left(\frac{\pi^2}{6}-1\right)x^2}. \tag{3.6}$$

The inequalities (3.5) and (3.6) bring to mind the following results of Zhu and Sun [8],

$$\frac{1-x^2}{1+x^2} \leqslant \cos\frac{\pi x}{2} \leqslant \left(\frac{1-x^2}{1+x^2}\right)^{\pi^2/16}, \tag{3.7}$$

$$\frac{1-x^2}{1+x^2} \leqslant \frac{\sin(\pi x)}{\pi x} \leqslant \left(\frac{1-x^2}{1+x^2}\right)^{\pi^2/12}, \tag{3.8}$$

which were achieved by different methods. The main drawback of (3.4)–(3.8) are the powers $-\pi^2/12$, $\pi^2/6$, $\pi^2/8$, $\pi^2/12$ and $\pi^2/16$ which provide poor bounds near $x=1$. In fact, the simple rational bounds (2.2), (2.5) and (2.6) are better near $x=1$.

4. Inequality (1.16)

Inequality (1.16) is proved in [3] by a complicated calculation which shows that the quotient

$$h(x) = \frac{\tan x - S_n(x)}{x^{2N}\tan x}$$

is monotone. We propose to provide a short proof of (1.16) by establishing the following equivalent inequality:

THEOREM 4.1. *For all* $|x|<1$,

$$\frac{8}{\pi^2}\lambda(2N+2)x^{2N}\tan\frac{\pi x}{2} \leqslant \tan\frac{\pi x}{2} - \frac{4}{\pi}\sum_{k=1}^{N}\lambda(2k)x^{2k-1} \leqslant x^{2N}\tan\frac{\pi x}{2}. \tag{4.1}$$

Proof. The essence of our proof hinges on the monotonicity of the coefficients $\lambda(2k)$. By (1.6),

$$\tan\frac{\pi x}{2} - \frac{4}{\pi}\sum_{k=1}^{N}\lambda(2k)x^{2k-1} = \frac{4}{\pi}\sum_{k=N+1}^{\infty}\lambda(2k)x^{2k-1} = \frac{4}{\pi}\sum_{k=1}^{\infty}\lambda(2k+2N)x^{2k+2N-1}.$$

Since $\lambda(2k+2N) < \lambda(2k)$, we have

$$\tan\frac{\pi x}{2} - \frac{4}{\pi}\sum_{k=1}^{N}\lambda(2k)x^{2k-1} \leqslant x^{2N}\frac{4}{\pi}\sum_{k=1}^{\infty}\lambda(2k)x^{2k-1} = x^{2N}\tan\frac{\pi x}{2},$$

and the right hand side of (4.1) follows.

In order to prove the left hand side of (4.1), we begin by showing that

$$\lambda(2k+2N) \geqslant \frac{8}{\pi^2}\lambda(2N+2)\lambda(2k), \qquad k,N = 1,2,3,\dots \tag{4.2}$$

and that the inequality is strict for all $k > 1$. Indeed, for $k = 1$, (4.2) is an equality, since $\lambda(2) = \pi^2/8$. For $k \geqslant 2$, we recall that $\lambda(2k)$ is decreasing, so

$$\frac{8}{\pi^2}\lambda(2k)\lambda(2N+2) \leqslant \frac{8}{\pi^2}\lambda(4)\lambda(4) = \frac{8}{\pi^2}\frac{\pi^4}{96}\frac{\pi^4}{96} < 1.$$

On the other hand, we obviously have $\lambda(2k+2N) > 1$, hence (4.2) is verified. Consequently,

$$\begin{aligned}\tan\frac{\pi x}{2} - \frac{4}{\pi}\sum_{k=1}^{N}\lambda(2k)x^{2k-1} &= \frac{4}{\pi}\sum_{k=1}^{\infty}\lambda(2k+2N)x^{2k+2N-1}\\ &\geqslant \frac{8}{\pi^2}\lambda(2N+2)x^{2N}\frac{4}{\pi}\sum_{k=1}^{\infty}\lambda(2k)x^{2k-1} = \frac{8}{\pi^2}\lambda(2N+2)x^{2N}\tan\frac{\pi x}{2},\end{aligned}$$

and the proof is complete. □

Since inequality (4.2) is strict for all $k \geqslant 2$, the left hand side of (4.1) may be improved in the spirit of Section 2 by taking additional exact terms of the power series.

We conclude by noting that if we replace the inequalities

$$\frac{8}{\pi^2}\lambda(2N+2)\lambda(2k) \leqslant \lambda(2k+2N) \leqslant \lambda(2k), \qquad k \geqslant 1,\ N \geqslant 1,$$

which were used in the proof of (4.1), by the obvious inequalities

$$1 < \lambda(2k+2N) \leqslant \lambda(2k+2), \qquad N \geqslant 1,$$

we get simple rational bounds for the remainder term

$$\frac{4}{\pi}\frac{x^{2N+1}}{1-x^2} \leqslant \tan\frac{\pi x}{2} - \frac{4}{\pi}\sum_{k=1}^{N}\lambda(2k)x^{2k-1} \leqslant \frac{4}{\pi}\lambda(2N+2)\frac{x^{2N+1}}{1-x^2}, \qquad |x| < 1. \tag{4.3}$$

For example, by $\lambda(2) = \pi^2/8$, $\lambda(4) = \pi^4/96$, and $\lambda(6) = \pi^6/960$, it follows that

$$1 + \frac{\pi^2 x^2}{12} + \frac{8}{\pi^2}\frac{x^4}{1-x^2} \leqslant \frac{\tan(\pi x/2)}{\pi x/2} \leqslant 1 + \frac{\pi^2 x^2}{12} + \frac{\pi^4}{120}\frac{x^4}{1-x^2}.$$

Here $8/\pi^2 \approx 0.8105$, $\pi^4/120 \approx 0.8117$.

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