A Binomial Identity via Differential Equations

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Abstract. In the following we discuss a well-known binomial identity. Many proofs by different methods are known for this identity. Here we present another proof, which uses linear ordinary differential equations of the first order.

Several proofs of the well-known identity

$$\sum_{k=0}^{n} \binom{n+k}{n} 2^{-k} = 2^n \tag{1}$$

[**4**, (1.79)] appear in the literature. In [**3**, Equation (5.20)], it is proved using partial sums of binomial series. In [**6**], (1) is verified by a probabilistic argument. Another short probabilistic proof follows from [**9**] by taking $p = 1/2$. In [**2**, p. 64, Ex. 2], the sum in (1) is calculated by means of contour integration in the complex plane. In the electronic manuscript [**8**, p. 62], the equivalent

$$\sum_{k=0}^{n} \binom{2n-k}{n-k} 2^{k} = 2^{2n}$$

is shown by means of Riordan arrays and the Lagrange Inversion Formula. In addition, one referee of this manuscript suggested another proof by means of generating functions and the Lagrange Inversion Formula. Another referee was kind enough to point out the similarity between (1) and the identity

$$\sum_{k=0}^{n} \binom{n+k}{2k} 2^{-k} + \sum_{k=0}^{n-1} \binom{n+k}{2k+1} 2^{-(k+1)} = 2^n$$

(see [**5**]).

We suggest another proof of (1) by methods of ordinary differential equations. The use of elementary analytic methods to prove combinatorial identities goes back at least to Euler's period. In the course of years, these basic tools were replaced by more sophisticated modern ones. Our main purpose in the following is to point out the efficiency of elementary methods which are easily accessible, but still powerful.

Consider the polynomial

$$V(x) = \sum_{k=0}^{n} \binom{n+k}{n} x^k.$$

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With this notation, the sum in (1) is $V(\frac{1}{2})$. Observe that

$$
\begin{aligned}
n!V(x) &= \sum_{k=0}^{n}(n+k)(n+k-1)\cdots(k+1)x^k \\
&= \left(\sum_{k=0}^{n} x^{n+k}\right)^{(n)} \\
&= \left(\frac{x^n - x^{2n+1}}{1-x}\right)^{(n)}. \qquad (2)
\end{aligned}
$$

Let us denote $T(x) = \big(x^n - x^{2n+1}/1-x\big)$ and differentiate $n+1$ times the equality $(1-x)T(x) = x^n - x^{2n+1}$. We obtain

$$(1-x)T^{(n+1)}(x) - (n+1)T^{(n)}(x) = -(2n+1)(2n)\cdots(n+1)x^n.$$

Due to $V(x) = T^{(n)}(x)/n!$, this is equivalent to the differential equation

$$(1-x)V'(x) - (n+1)V(x) = -\frac{(2n+1)!}{(n!)^2}x^n. \qquad (3)$$

Multiplying (3) by $(1-x)^n$ yields

$$
\begin{aligned}
\big((1-x)^{n+1}V(x)\big)' &= (1-x)^{n+1}V'(x) - (n+1)(1-x)^n V(x) \\
&= -\frac{(2n+1)!}{(n!)^2}x^n(1-x)^n.
\end{aligned}
$$

Integration of this polynomial equality on $[x, 1]$ results in

$$(1-x)^{n+1}V(x) = \frac{(2n+1)!}{(n!)^2}\int_x^1 t^n(1-t)^n\,dt. \qquad (4)$$

Using the beta function $B(n+1, m+1) = \int_0^1 t^n(1-t)^m\,dt = n!m!/(n+m+1)!$, the polynomial $V(x)$ is

$$V(x) = \frac{1}{(1-x)^{n+1}}\,\frac{\int_x^1 t^n(1-t)^n\,dt}{B(n+1, n+1)}.$$

For $x = \frac{1}{2}$, we have

$$\int_0^{1/2} t^n(1-t)^n\,dt = \int_{1/2}^1 t^n(1-t)^n\,dt = \frac{1}{2}B(n+1, n+1),$$

hence the required identity (1) follows. Note that when we let x approach 1 from the left, the l'Hopital rule leads to the identity

$$V(1) = \sum_{k=0}^{n}\binom{n+k}{n} = \binom{2n+1}{n}.$$

The same approach may be applied to more general sums of the form

$$W(x) = W(n,x) = \sum_{k=0}^{n} \binom{p+k}{r} x^{k+p-r} = \frac{1}{r!}\left(\sum_{k=0}^{n} x^{p+k}\right)^{(r)} = \frac{1}{r!}\left(\frac{x^p - x^{n+p+1}}{1-x}\right)^{(r)},$$

with integers n, p, r, where $p \geq r$. We denote $W(x) = T^{(r)}(x)/r!$ with $T(x) := (x^p - x^{n+p+1})/(1-x)$, and differentiate $r+1$ times the equality $(1-x)T(x) = x^p - x^{n+p+1}$. We obtain

$$(1-x)T^{(r+1)}(x) - (r+1)T^{(r)}(x) = \frac{p!}{(p-r-1)!}x^{p-r-1} - \frac{(n+p+1)!}{(n+p-r)!}x^{n+p-r}. \quad (5)$$

Note that when $p = r$, the first term on the right-hand side of (5) is obviously absent. This occurs, as well, in the upcoming equations.

Since $W(x) = T^{(r)}(x)/r!$, the polynomial $W(x)$ satisfies the differential equation

$$(1-x)W'(x) - (r+1)W(x) = \frac{p!}{(p-r-1)!r!}x^{p-r-1} - \frac{(n+p+1)!}{(n+p-r)!r!}x^{n+p-r}. \quad (6)$$

We multiply (6) by $(1-x)^r$, integrate on $[x, 1]$, and use the beta function notation to obtain that the polynomial $W(x)$ satisfies

$$-(1-x)^{r+1}W(x) = \frac{\int_x^1 t^{p-r-1}(1-t)^r\,dt}{B(p-r, r+1)} - \frac{\int_x^1 t^{n+p-r}(1-t)^r\,dt}{B(n+p-r+1, r+1)}.$$

In terms of the "regularized incomplete beta function"

$$I_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt \Big/ \int_0^1 t^{a-1}(1-t)^{b-1}\,dt,$$

the last equation is written as

$$(1-x)^{r+1}W(x) = (1 - I_x(n+p-r+1, r+1)) - (1 - I_x(p-r, r+1)). \quad (7)$$

We rephrase (7) by using [**1**, p. 944, 26.5.7]),

$$1 - I_x(a,b) = \sum_{i=0}^{a-1} \binom{a+b-1}{i} x^i (1-x)^{a+b-1-i}. \quad (8)$$

We prove (8) by induction. First we confirm (8) for $b = 1$ and any integer a. Then we apply the recurrence relation $I_x(a,b) = I_x(a+1, b-1) + \binom{a+b-1}{a}x^a(1-x)^{b-1}$, which is verified by using integration by parts, to complete the proof of (8). Due to (8), equality (7) is transformed into

$$\sum_{k=0}^{n} \binom{p+k}{r} x^{k+p-r} = \sum_{i=0}^{n+p-r} \binom{n+p+1}{i} x^i (1-x)^{n+p-r-i} - \sum_{i=0}^{p-r-1} \binom{p}{i} x^i (1-x)^{p-r-i-1}. \qquad (9)$$

It is worth pointing out that a nice symmetric form of (9) is attained by considering the difference $W(n,x) - W(m,x)$ for $n \geq m$, $p+m+1 \geq r$. We get

$$\sum_{k=m+1}^{n} \binom{p+k}{r} x^{k+p-r} = \sum_{i=0}^{n+p-r} \binom{n+p+1}{i} x^i (1-x)^{n+p-r-i} - \sum_{i=0}^{m+p-r} \binom{m+p+1}{i} x^i (1-x)^{m+p-r-i}. \qquad (10)$$

Note that when $p+m+1 = r$, the second sum on the right-hand side of (10) is obviously empty.

For $x = 1/2$, identity (10) yields

$$\sum_{k=m+1}^{n} \binom{p+k}{r} 2^{-k} = 2^{-n} \sum_{i=0}^{n+p-r} \binom{n+p+1}{i} - 2^{-m} \sum_{i=0}^{m+p-r} \binom{m+p+1}{i}, \qquad (11)$$

which reduces to (1) when $m = -1$ and $n = p = r$. Finally, putting $w = \frac{1-x}{x}$, we get

$$\sum_{k=m+1}^{n} \binom{p+k}{r} (1+w)^{-k} = (1+w)^{-n} \sum_{i=0}^{n+p-r} \binom{n+p+1}{i} w^{p-r+n-i} - (1+w)^{-m} \sum_{i=0}^{m+p-r} \binom{m+p+1}{i} w^{p-r+m-i}. \qquad (12)$$

In the particular case when $m = -1$ and $p = r$, identity (12) reduces to

$$\sum_{k=0}^{n} \binom{p+k}{p} (1+w)^{n-k} = \sum_{i=0}^{n} \binom{n+p+1}{i} w^{n-i},$$

which is equivalent to an identity of [**7**, p. 47].

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Another Proof for Non-Supercyclicity in Finite Dimensional Complex Banach Spaces

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Abstract. We give an elementary proof, based on linear algebra and on a simple and well-known technique from the theory of dynamical systems, for the non-existence of supercyclic linear operators defined on a finite dimensional complex Banach space with dimension greater than or equal to two.

1. INTRODUCTION. Consider X to be a real or complex Banach space X, and denote by $\mathcal{L}(X)$ the set consisting of all bounded linear operators going from X into X. An operator $T \in \mathcal{L}(X)$ is said to be *supercyclic* if there is some $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{K}, n = 0, 1, \dots\}$ is dense in X. In this case, x is called a supercyclic vector for T. Supercyclicity is an intermediate property between cyclicity and hypercyclicity [**1**].

Recall that any linear operator defined on a finite dimensional Banach space is bounded. So, if $\dim X = 1$, we have that any nonzero linear operator $T : X \to X$ is supercyclic. When X is a real Banach space with $\dim X = 2$, then X also has supercyclic operators. For example, in $X := \mathbb{R}^2$ a rotation (with respect to the origin) by an irrational number defines a supercyclic operator. G. Herzog established in 1992 that separable infinite dimensional Banach spaces always have supercyclic operators and, on the other side, that finite dimensional complex Banach spaces with dimension greater than 2 do not have supercyclic operators [**2**]. He also proved that in the real case, this holds when $\dim X \geq 3$. The proof we present for the complex case will follow readily from two simple results. The first is based on linear algebra and the second uses a simple and well-known technique from the theory of dynamical systems [**3**, p. 6–8].)

2. THE PROOF.

Lemma 1. *Let X be a two-dimensional complex Banach space. If $T : X \to X$ is a linear operator, then T is not supercyclic.*

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