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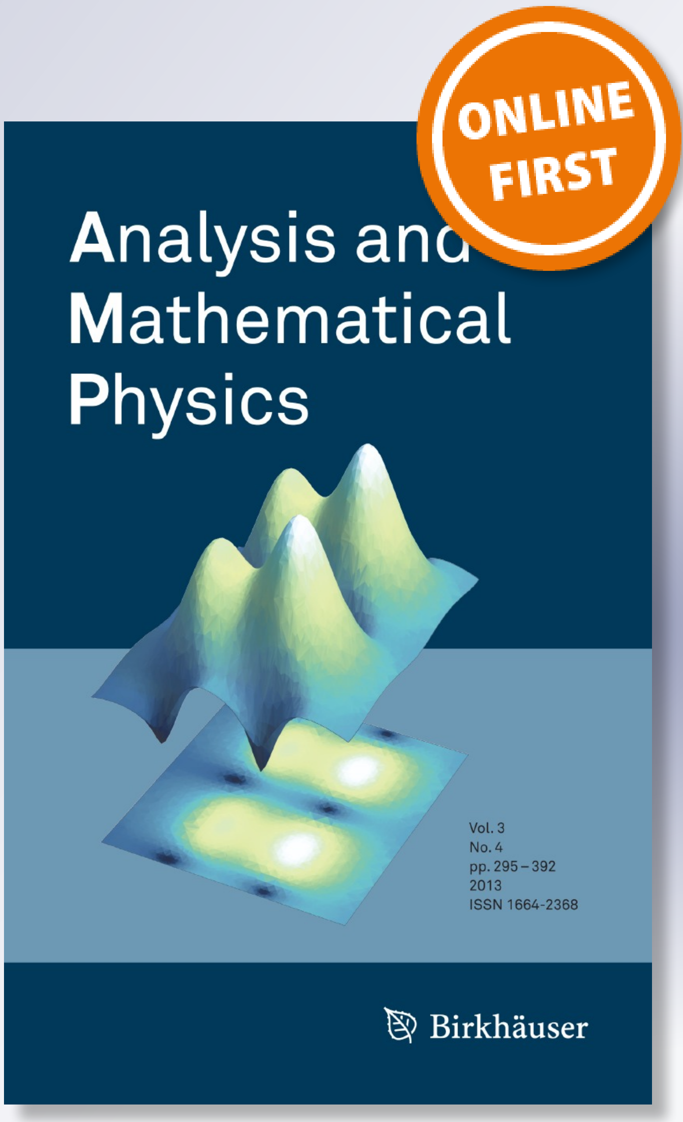

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Abstract The article discusses criteria for univalence of analytic functions in the unit disc. A unified method for creating new sets of conditions ensuring univalence is presented. Applying this method we are able to find several families of new sharp criteria for univalence.

1 Introduction

The Schwarzian derivative $Sf = \left(f''/f'\right)' - \frac{1}{2}\left(f''/f'\right)^2$ of an analytic locally univalent function plays an important role for finding sufficient conditions for univalence. Nehari [6] found conditions implying univalence expressed in terms of the Schwarzian derivative: if $|Sf| \leq 2(1-|z|^2)^{-2}$, then f is univalent in the unit disc $\Delta = \{z, |z| < 1\}$. Also if $|Sf| \leq \pi^2/2$, the same conclusion follows. For deriving his outstanding results Nehari used a useful connection between the zeros of solutions of linear second order differential equations and univalence [6]. Later Pokornyi [9] stated without proof the condition $|Sf| \leq 4(1-|z|^2)^{-1}$. Nehari then proved this condition [7]. In addition Nehari extended these results and proved a more general theorem [7,8] concerning criteria for univalence. In his theorem he also investigated the sharpness of his conditions. These pioneering works of Nehari opened a new line of research in geometric function theory.

Dedicated to Larry Zalcman on the occasion of his 70-th birthday.

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Our aim in the following article is to present a unified method, simple but useful, for finding criteria ensuring the univalence of analytic functions in the unit disc. The paper contains two theorems. In Theorem 1 we present a new (sharp) univalence criteria depending on a parameter. The proof of Theorem 1 is short and easy while the proof of Theorem 2 is technically more involved. The method of the proof of Theorem 2 is of independent interest and led us to some interesting results in approximation of trigonometric functions near their poles. See [2].

2 Nehari's univalence criteria

Nehari's pioneering work appeared in [6]. This work opened a fundamental line of research. His idea was to use a connection between the number of zeros of solutions of second order linear differential equations in a given domain in the complex plane and univalence of the quotient of two independent solution of this equation: If $u(z)$, $v(z)$ are two linearly independent functions (solutions of a linear, homogeneous second order differential equation) in a domain D such that every linear combination $c_1u(z)+c_2v(z)$ has at most one zero in D, then their quotient $f(z) = v(z)/u(z)$ is univalent in D.

Quotients of solutions are naturally related to a differential equation through the well known Schwarzian derivative operator

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

due to the following property: suppose we are given the linear differential equation

$$u'' + p(z)u = 0, \tag{2.1}$$

where $p(z)$ is an analytic function in the unit disc Δ and $u(z)$, $v(z)$ are any two linearly independent solutions of (2.1). Then

$$S(v/u)(z) = 2p(z). \tag{2.2}$$

We recall some other basic properties of the Schwarzian derivative. One of them is: given a Möbius map $T = (az+b)/(cz+d)$, $ad-bc \neq 0$, we have $S(T)(z) = 0$. Another useful property is for a composition of two functions $g \circ f$:

$$S(g \circ f)(z) = (S(g) \circ f(z))f'(z)^2 + S(f)(z). \tag{2.3}$$

If the above f is in particular a Möbius map T then by $S(T) = 0$,

$$S(g \circ T)(z) = (S(g) \circ T(z))T'(z)^2. \tag{2.4}$$

Nehari made use of the Schwarzian derivative and its above properties to arrive at his sufficient conditions for univalence.

Theorem A (Nehari [7]) *Suppose that*

(i) $p(x)$ *is a positive and continuous even function for* $-1 < x < 1$,
(ii) $p(x)(1-x^2)^2$ *is nonincreasing for* $0 < x < 1$,
(iii) *the real valued differential equation*

$$y''(x) + p(x)y(x) = 0, \qquad -1 < x < 1, \tag{2.5}$$

has a solution which does not vanish in $-1 < x < 1$.

Then any analytic function $f(z)$ *in* Δ *satisfying*

$$|Sf(z)| \leq 2p(|z|) \tag{2.6}$$

is univalent in the unit disc Δ.

In what follows we use the term "Nehari's function" to denote a positive even function $p(x)$ such that $p(x)(1-x^2)^2$ is nonincreasing for $0 < x < 1$. See [11].

As Nehari pointed out already in [6], the functions

$$p(x) = (1-x^2)^{-2}, \qquad p(x) = \pi^2/4, \tag{2.7}$$

and the corresponding solutions $y(x) = (1-x^2)^{1/2}$, $y(x) = \cos(\pi x/2)$ of the respective equations (2.5) have all the needed properties to conclude the sufficient conditions for univalence in Δ. Soon after that, Hille [5] made the remarkable observation that the condition $|Sf| \leq 2(1-|z|^2)^{-2}$ is sharp.

Let us assume, in addition, that $p(z)$ is also analytic in the unit disc Δ and consider together with the real equation (2.5) also the analytic differential equation

$$u''(z) + p(z)u = 0, \qquad z \in \Delta. \tag{2.8}$$

In this case the following definition will be useful:

Definition 1 We shall say that a function $p(z)$, analytic in the open unit disc Δ, is self majorant if $|p(z)| \leq p(|z|)$ for each $z \in \Delta$.

For example, if $p(z) = \sum A_k z^k$ in Δ and $A_k \geq 0$ for all k, then $p(z)$ is self majorant.

If, in addition to the assumptions of Theorem A, $p(z)$ is self majorant, then $f_0(z) = v(z)/u(z)$ satisfies

$$|Sf_0(z)| = 2|p(z)| \leq 2p(|z|) \tag{2.9}$$

and $f_0(z)$ itself is univalent. Namely, equation (2.8) naturally generates a univalent function. If by reduction of order of the differential equation we take it's second solution as $v = u\int u^{-2}$, then we conclude that equation (2.8) generates a univalent function

$$f_0(z) = \int_0^z \frac{dt}{u^2(t)}. \tag{2.10}$$

Note, for example, Nehari's more general condition [8] for univalence in Δ,

$$|Sf(z)| \le 2(1-\mu^2)(1-|z|^2)^{-2} + 2\mu(2+\mu)(1+|z|^2)^{-2}, \quad 0 \le \mu \le 1, \tag{2.11}$$

which which is generated by the function $y(x) = (1-x^2)^{(\mu+1)/2}(1+x^2)^{-\mu/2}$, corresponds to a function $p(z)$ which is not self majorant for μ close to 1. On the other hand, Nehari's other condition in [8],

$$|Sf(z)| \le 2(1+\mu)(1-\mu|z|^2)(1-|z|^2)^{-2}, \quad 0 \le \mu \le 1, \tag{2.12}$$

which is generated by the function $y(x) = (1-x^2)^{(\mu+1)/2}$, corresponds to a function $p(z)$ which is self majorant.

In the spirit of Steinmetz [11] we define

Definition 2 We shall say that the univalence criteria (2.6) is sharp if for an analytic function $g(z)$, the conditions $Sg(x) \ge 2p(x)$ for $-1 < x < 1$, $Sg(z) \not\equiv 2p(z)$ in Δ imply that $g(z)$ is not univalent in Δ.

We claim that if the solution $y(x)$ of the real valued differential equation (2.5) in Nehari's Theorem A satisfies

$$\int^{1} \frac{dt}{y^2(t)} = \infty, \qquad \int_{-1} \frac{dt}{y^2(t)} = \infty, \tag{2.13}$$

then the corresponding univalence criterion (2.6) is sharp. For this purpose recall Theorem 1 from [1], where a singular Sturm comparison theorem is presented:

Let $P(x),\ p(x)$ be continuous functions on the open, finite or infinite interval (a,b) (but not necessarily at its endpoints), and $P(x) \ge p(x),\ P(x) \not\equiv p(x)$ on (a,b). If the differential equation

$$u'' + p(x)u = 0, \qquad a < x < b,$$

has a solution $u(x)$ which satisfies the boundary conditions

$$\int_{a} \frac{dx}{u^2(x)} = \infty, \qquad \int^{b} \frac{dx}{u^2(x)} = \infty,$$

then every solution of the equation

$$v'' + P(x)v = 0, \qquad a < x < b,$$

has a zero in (a,b). In particular, there exists a solution $v(x)$ which has two zeros in (a,b).

Let $g(z)$ be an analytic function such that $Sg(x) \geq 2p(x)$ for $-1 < x < 1$, $Sg(z) \not\equiv 2p(z)$ in Δ. Consider the differential equation

$$v'' + \frac{1}{2}Sg(z)v = 0, \qquad z \in \Delta \tag{2.14}$$

Due to (2.13) and the singular Sturm comparison theorem, the corresponding real differential equation

$$v'' + \frac{1}{2}Sg(x)v = 0$$

has a solution $v(x)$ with (at least) two zeros in $(-1,1)$. These are, of course, also zeros of the analytic solution $v(z)$ of (2.14). Hence any quotient $v(z)/u(z)$ of two linearly independent solutions of (2.14) has two zeros in Δ and is not univalent there. Our $g(z)$ is also a quotient of two certain solutions of (2.14) and is related to $v(z)/u(z)$ by a Möbius map, so it follows that also $g(z)$ is not univalent in Δ, as claimed.

It is worth noting that we may use a different approach to prove sharpness. One can use [4, Thm. 3], which is based on the "relative convexity lemma" . Also in [11], Corollary 5, it is proved that (2.13) implies sharpness for a more restricted case, namely for "Nehari's functions".

We now outline our method of finding families of conditions for univalence. The classical Theorem A of Nehari is the main tool in what follows. Our main idea is to consider a family of differential equations depending on parameters. Let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a vector of n free real parameters. Let $u = u(z, \Lambda)$ be a family of analytic functions in Δ depending on these n free parameters. We now generate for each vector Λ, through

$$p = p(z, \Lambda) = -u''(z, \Lambda)/u(z, \Lambda) \tag{2.15}$$

a differential equation

$$u'' + p(z, \Lambda)u = 0, \qquad z \in \Delta. \tag{2.16}$$

In addition we assume that the restriction of u to the real axis, $u(x, \Lambda)$, is the solution of the real valued differential equation

$$y''(x) + p(x, \Lambda)y(x) = 0, \qquad -1 < x < 1, \tag{2.17}$$

which does not vanish in $-1 < x < 1$.

Suppose we can find a range for Λ such that $p(x, \Lambda)(1 - x^2)^2$ is non increasing for $-1 < x < 1$. If this is done—we may apply Theorem A in order to find a family of univalence criteria depending on the vector Λ.

Since we are mainly interested in sharp conditions for univalence, it will be useful, due to the previous claim, to deal only with cases where $u = u(z, \Lambda)$ vanish at $z = \pm 1$.

In the next two sections we suggest two univalence criteria, the first with one parameter and the other with two parameters.

3 Univalence criteria depending on one parameter

We have

Theorem 1 *Let*

$$p(x,\lambda) = \frac{2(1+\lambda) - 12\lambda x^2}{(1-x^2)(1-\lambda x^2)}. \tag{3.1}$$

If $f(z)$ is an analytic function in Δ satisfying

$$|Sf(z)| \le 2p(|z|), \tag{3.2}$$

with

$$0 \le \lambda \le 1/5, \tag{3.3}$$

then $f(z)$ is univalent in Δ. Moreover the theorem is sharp.
Also

$$f(z) = \int_0^z \frac{dt}{(1-t^2)^2(1-\lambda t^2)^2}$$

is an odd univalent function in Δ for $0 \le \lambda \le 1/5$.

Proof Consider the following family of functions depending on the real parameter λ,

$$u = u(x,\lambda) = (1-x^2)(1-\lambda x^2) \tag{3.4}$$

which are positive on (-1, 1) for $\lambda \le 1$. For the sake of simlicity we restrict ourself to values of $\lambda \ge 0$. By a straight forward calculation this u is a solution of the differential equation $u'' + p(x,\lambda)u = 0$, where $p = -u''/u$ is given by (3.1). In order to apply Theorem A, we have to show that

$$p(x)(1-x^2)^2 = \frac{(2(1+\lambda) - 12\lambda x^2)(1-x^2)}{1-\lambda x^2}$$

is positive and non increasing for $0 \le x \le 1$. With $y = x^2$ we have to verify that

$$G(y) = \frac{(2(1+\lambda) - 12\lambda y)(1-y)}{1-\lambda y}$$

is non increasing, i.e., that

$$(1-\lambda y)^2 G'(y) = -12\lambda^2 y^2 + 24\lambda y + 2(\lambda^2 - 6\lambda - 1) \le 0$$

for $0 \le y \le 1$. A simple calculation shows that this condition holds if $0 \le \lambda \le 1/5$.

By another elementary computation we have that all Taylor coefficients appearing in the expansion of $p(z)$ around zero are nonegative for λ satisfying (3.3). Indeed, with $y = x^2$,

$$p(x,\lambda) = \frac{2(1+\lambda) - 12\lambda x^2}{(1-x^2)(1-\lambda x^2)} = \frac{2(1+\lambda) - 12\lambda y}{(1-y)(1-\lambda y)} = \frac{2(1+\lambda)}{1-\lambda y} + \frac{2(1-5\lambda)y}{(1-y)(1-\lambda y)}$$

and all Taylor coefficients are positive for $0 \le \lambda \le 1/5$. As a corollary of the Taylor coefficients being nonnegative, we conclude that

$$|p(z)| \le p(|z|). \tag{3.5}$$

i.e., $p(z)$ is self majorant. Consequently, by (2.10), the function

$$f(z,\Lambda) = \int^{z} \frac{dz}{(1-z^2)^2(1-\lambda z^2)^2} \tag{3.6}$$

is univalent in Δ for $0 \le \lambda \le 1/5$.

The sharpness of (3.2) follows from the divergence of $\int u^{-2}$ at $x = \pm 1$. □

4 Univalence criteria depending on two parameters

Now we consider the two-parametric family of functions $u(x) = (1-x^2)^{\lambda}\cos^{\mu}(\pi x/2)$ and $p(x) = -u''/u$ which it generates. By our general method we have

Theorem 2 *Let*

$$\begin{aligned} p(x) =& 4\lambda(1-\lambda)x^2(1-x^2)^{-2} + 2\lambda(1-x^2)^{-1} + \mu\pi^2/4 \\ &+ \mu(1-\mu)\pi^2\tan^2(\pi x/2)/4 - 2\mu\lambda\pi x\tan(\pi x/2)(1-x^2)^{-1} \end{aligned} \tag{4.1}$$

and let λ, μ *satisfy*

$$\lambda \ge 0, \quad \mu \ge 0, \quad 1/2 \le \lambda + \mu \le 1, \quad 2\lambda + \mu \ge 1. \tag{4.2}$$

Then if $f(z)$ *is an analytic function in* Δ *satisfying*

$$|Sf(z)| \le 2p(|z|), \qquad z \in \Delta, \tag{4.3}$$

it follows that $f(z)$ *is univalent in* Δ.
Also

$$f(z) = \int_0^z \frac{dt}{u^2(t)} = \int_0^z \frac{dt}{(1-t^2)^{2\lambda}\cos^{2\mu}(\pi t/2)} \tag{4.4}$$

is an odd univalent function in Δ. *Moreover the condition (4.3) is sharp.*

Proof of Theorem 2 We first note that for the special case $\lambda+\mu = 1, \lambda > 0, \mu > 0$, the corresponding $p(x)$ was mentioned by Beesack in [3, p. 217] in a different connection.

In order to use Theorem A we proceed to find restrictions on λ, μ that will ensure that for $p(x)$ in (4.1) the function $\varphi(x) = p(x)(1-x^2)^2$ is positive and nonincreasing for $0 < x < 1$. It will be convenient to denote

$$G(x) = (1 - x^2)\tan(\pi x/2). \tag{4.5}$$

With this notation,

$$\varphi(x) = 4\lambda(1-\lambda)x^2 + 2\lambda(1-x^2) + \mu\pi^2(1-x^2)^2/4 + \mu(1-\mu)\pi^2 G^2(x)/4 - 2\mu\lambda\pi x G(x) \tag{4.6}$$

and

$$\begin{aligned}\varphi'(x) = 8\lambda(1-\lambda)x - 4\lambda x - \mu\pi^2 x(1-x^2)\\ +\mu(1-\mu)(\pi^2/4)2G(x)G'(x) - 2\mu\lambda\pi(xG'(x) + G(x)).\end{aligned} \tag{4.7}$$

We start with some elementary considerations. To ensure that $\varphi(x)$ is positive and nonincreasing for $0 < x < 1$, we must have in particular that $\varphi(1) \geq 0$ and $\varphi'(1) \leq 0$. By direct calculation,

$$G\Big|_{x=1^-} = 4/\pi, \qquad G'\Big|_{x=1^-} = 2/\pi, \tag{4.8}$$

After some more calculations we require that

$$\varphi(1) = 4(\lambda + \mu)(1 - \lambda - \mu) \geq 0 \tag{4.9}$$

and

$$\varphi'(1) = 4(\lambda + \mu)(1 - 2\lambda - \mu) \leq 0. \tag{4.10}$$

From (4.9) it follows that $0 \leq \lambda + \mu \leq 1$. If $\lambda + \mu = 0$, then $u = (1-x^2)^\lambda \cos^\mu(\pi x/2)$ does not vanish at $x = \pm 1$, contradicting our assumptions. Hence let $0 < \lambda + \mu \leq 1$. Summing up, we get from (4.9) and (4.10) the conditions

$$1 - \lambda - \mu \geq 0, \quad 1 - 2\lambda - \mu \leq 0. \tag{4.11}$$

Consequently $\lambda \geq 0$ and $\mu \leq 1$ are necessary for $\varphi(x)$ to be nonincreasing.

From now and on we assume for sake of simplicity that $\mu \geq 0$. From (4.11) it follows that $\lambda \geq (1 - \mu)/2$ which implies $\lambda + \mu \geq (1 + \mu)/2$. Since we assume that $\mu \geq 0$, it follows that

$$1/2 \leq \lambda + \mu \leq 1. \tag{4.12}$$

We start to show that under the restrictions above, $p(z)$ is self majorant. It will be enough to verify that all coefficients in the expansion of p around zero are non negative. For this purpose we rewrite p of (4.1) as

$$p(x) = \frac{4\lambda(1-\lambda-\mu)}{1-\mu}\frac{x^2}{(1-x^2)^2} + \frac{2\lambda}{1-x^2} + \frac{\mu\pi^2}{4} + \mu(1-\mu)\left(\frac{\pi}{2}\tan(\frac{\pi x}{2}) - \frac{2\lambda}{1-\mu}\frac{x}{1-x^2}\right)^2$$

Since $4\lambda(1-\lambda-\mu)/(1-\mu)$, 2λ, $\mu\pi^2/4$ and $\mu(1-\mu)$ are all non negative, it is sufficient to prove that the Taylor coefficients of $\frac{\pi}{2}\tan(\frac{\pi x}{2}) - \frac{2\lambda}{1-\mu}\frac{x}{1-x^2}$ are non negative. For this aim we recall the formula [10, Section 3.14]

$$\frac{\tan \pi x/2}{\pi x/2} = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\lambda(2k+2)x^{2k}, \qquad \lambda(p) = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^p}. \tag{4.13}$$

According to (4.13),

$$\left(\frac{\pi}{2}\right)\tan\left(\frac{\pi x}{2}\right) - \frac{2\lambda}{1-\mu}\frac{x}{1-x^2} = \sum_{k=0}^{\infty}\left(2\lambda(2k+2) - \frac{2\lambda}{1-\mu}\right)x^{2k+1}. \tag{4.14}$$

Since $\lambda(p) > 1$, the coefficients of the power series of (4.14) are

$$2\lambda(2k+2) - \frac{2\lambda}{1-\mu} > 2 - \frac{2\lambda}{1-\mu} = \frac{2(1-\mu-\lambda)}{1-\mu} \geq 0.$$

Consequently $p(z)$ is self majorant.

The univalence criterion (4.3) will be proved when we show that $\varphi(x) = p(x)(1-x^2)^2$ is indeed positive and nonincreasing for $0 \leq x \leq 1$. With the notation

$$L(x) = (\pi/2)\tan(\pi x/2) - \frac{2\lambda}{1-\mu}\frac{x}{1-x^2}. \tag{4.15}$$

we have

$$\varphi(x) = p(x)(1-x^2)^2 = \frac{4\lambda(1-\lambda-\mu)}{1-\mu}x^2 + 2\lambda(1-x^2) + (\mu\pi^2/4)(1-x^2)^2 + \mu(1-\mu)\left[(1-x^2)L(x)\right]^2$$

and

$$\varphi'(x) = \frac{2\lambda(1-2\lambda-\mu)}{1-\mu}2x - \mu\pi^2 x(1-x^2) + 2\mu(1-\mu)\left[(1-x^2)L(x)\right]\left[(1-x^2)L(x)\right]'.$$

Since we saw that $p(z)$ is self majorant, it is obvious by the same argument that $p(x) \geq 0$ and $\varphi(x) \geq 0$ for $0 \leq x \leq 1$. So, it remains to establish $\varphi'(x) \leq 0$. Since $1-2\lambda-\mu \leq 0$ by (4.11), it is left to show only that

$$2\mu(1-\mu)\Big[(1-x^2)L(x)\Big]\Big[(1-x^2)L(x)\Big]' \le \mu\pi^2 x(1-x^2) \tag{4.16}$$

We postpone the proof of (4.16) to a later stage (Lemma 1). Once (4.16) will be verified, the univalence criterion (4.3) will follow. The univalence of the function (4.4) follows from the self majorance of $p(z)$, as in Theorem 1.

Finally consider the issue of sharpness of (4.3). Note that the multiplicity of the zeros of $u(x) = (1-x^2)^\lambda \cos^\mu(\pi x/2)$ at $x = 1, -1$ is $\lambda + \mu$ and by (4.12), $\lambda + \mu \ge 1/2$. Consequently,

$$\int^1 \frac{dx}{(1-x^2)^{2\lambda}\cos^{2\mu}(\pi x/2)} = \infty$$

and the sharpness of (4.3) follows.

Thus the proof of the theorem is complete, except of inequality (4.16). □

The next lemma provides some inequalities about the function $G(x) = (1 - x^2)\tan(\pi x/2)$ which are needed to prove (4.16). More upper and lower bounds for $\tan(\pi x/2)$ which are very precise even at the poles at $x = \pm 1$ are discussed in [2].

Lemma 1 *For the interval (0,1) we have:*

(i) The function $G(x) = (1-x^2)\tan(\pi x/2)$ *of (4.5) is convex.*

(ii) The function $\dfrac{(1-x^2)\tan(\pi x/2)}{\pi x/2}$ *decreases for* $0 \le x \le 1$ *and its maximal value at* $x = 0$ *is 1.*

(iii)

$$G(x) \le \frac{2}{\pi}(1+x).$$

(iv)

$$\frac{\pi}{2}G'(x) \le 1 + \frac{\pi^2}{4}(1-x^2).$$

Proof Convexity of $G(x)$ in the interval (0,1) follows from (4.13). Indeed,

$$G(x) = \frac{\pi}{2}x - \frac{4}{\pi}\sum_{k=1}^{\infty}(\lambda(2k) - \lambda(2k+2))\, x^{2k+1} \tag{4.17}$$

and $G''(x) < 0$ due to monotonicity of $\lambda(p)$. This proves (i).

Since $\left(\dfrac{G(x)}{\pi x/2}\right)' \le 0$ by (4.17), it follows that $\dfrac{G(x)}{\pi x/2}$ is decreasing and thus its maximum is attained at $x = 0$. This proves (ii).

For the proof of (iii) we recall that $G(1) = 4/\pi$, $G'(1) = 2/\pi$ as calculated in (4.8). The equation of the tangent line through the point $(1, 4/\pi)$ is $y = \frac{4}{\pi} - \frac{2}{\pi}(1-x) = \frac{2}{\pi}(1+x)$. Now (iii) follows using the convexity proved in (i).

In order to establish (iv) we use (iii) twice. Indeed,

$$G'(x) = -2x\tan(\pi x/2) + \frac{\pi}{2}(1-x^2)\left(1+\tan^2(\pi x/2)\right),$$

hence

$$\begin{aligned} G'(x) - \frac{\pi}{2}(1-x^2) &= -2x\tan(\pi x/2) + \tan(\pi x/2)\frac{\pi}{2}G(x) \\ &\le -2x\tan(\pi x/2) + (1+x)\tan(\pi x/2) \\ &= \frac{1-x^2}{1+x}\tan(\pi x/2) = \frac{G(x)}{1+x} \le \frac{2}{\pi}. \end{aligned}$$

This ends the proof of (iv). □

Now we are ready to complete the proof of (4.16). First recall that $L(x) = (\pi/2)\tan(\pi x/2) - \frac{2\lambda}{1-\mu}\frac{x}{1-x^2} \ge 0$ for $0 \le x \le 1$. This follows from the fact that the coefficients of the power series for $L(x)$ in (4.14) are positive. Second, $\dfrac{-2\lambda}{1-\mu} \le -1$ by (4.11). In addition, by (ii) we have

$$\frac{\pi}{2}(1-x^2)\tan\left(\frac{\pi x}{2}\right) = \left(\frac{\pi}{2}\right)^2 \frac{(1-x^2)\tan(\pi x/2)}{\pi x/2} x \le \left(\frac{\pi}{2}\right)^2 x.$$

Moreover, by (iv),

$$\frac{\pi}{2}\left((1-x^2)\tan\frac{\pi x}{2}\right)' = \left(\frac{\pi}{2}\right)G'(x) \le 1 + \frac{\pi^2}{4}(1-x^2).$$

Combining these facts together, we finally have

$$\begin{aligned} &2\left[(1-x^2)L(x)\right]\left[(1-x^2)L(x)\right]' \\ &\le 2\left(\frac{\pi}{2}(1-x^2)\tan\left(\frac{\pi x}{2}\right) - \frac{2\lambda}{1-\mu}x\right)\left(\frac{\pi}{2}\left((1-x^2)\tan\left(\frac{\pi x}{2}\right)\right)' - \frac{2\lambda}{1-\mu}\right) \\ &\le 2\left(\left(\frac{\pi}{2}\right)^2 x - x\right)\left(1 + \frac{\pi^2}{4}(1-x^2) - 1\right) \\ &= 2\left(\left(\frac{\pi}{2}\right)^2 - 1\right)\frac{\pi^2}{4}x(1-x^2) \le \pi^2 x(1-x^2). \end{aligned}$$

Since $0 < \mu \le 1$, the last inequality implies (4.16).

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