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More on the identity of Chaundy and Bullard

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ABSTRACT

In what follows we present a homogeneous identity which implies a more elementary treatment of the Chaundy–Bullard identity with n variables. In a different direction we bring another ramification of the Chaundy–Bullard identity.

The identity of Chaundy and Bullard [2]

$$x^{k+1}\sum_{i=0}^{m}\binom{k+i}{k}(1-x)^i+(1-x)^{m+1}\sum_{i=0}^{k}\binom{m+i}{m}x^i=1, \tag{1}$$

and its homogeneous form

$$(x+y)^{m+k+1}=x^{k+1}\sum_{i=0}^{m}\binom{k+i}{k}y^i(x+y)^{m-i}+y^{m+1}\sum_{i=0}^{k}\binom{m+i}{m}x^i(x+y)^{k-1}, \tag{2}$$

$k,m\ge 0$ have a very long history and numerous different proofs. See the detailed accounts [5] and [6] by Koornwinder and Schlosser. For the case $m=k$, (1) is frequently called the Daubechies identity. See [9].

In [7] an n-variable generalization of (1) is suggested: *If* $x_1+\cdots+x_n=1$, *then*

$$\sum_{t=1}^{n}x_t\sum_{i_1=0}^{p_1}\frac{x_1^{i_1}}{i_1!}\sum_{i_2=0}^{p_2}\frac{x_2^{i_2}}{i_2!}\cdots\sum_{i_n=0}^{p_n}\frac{x_n^{i_n}}{i_n!}\delta_{i_t,p_t}(i_1+\cdots+i_n)!=1. \tag{3}$$

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(3) is verified in [7] both by a probabilistic argument and by a use of a generating function. In [5, (10.1)–(10.2)], (3) is reformulated in a homogeneous form as

$$(x_1 + \cdots + x_n)^{m_1+\cdots+m_n+1} = \sum_{\sigma} f_{m_{\sigma(1)},\ldots,m_{\sigma(n)}}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \tag{4}$$

where the summation is over all cyclic permutations σ of $1, \ldots, n$ and

$$f_{m_1,\ldots,m_n}(x_1, \ldots, x_n) = x_n^{m_n+1} \sum_{i_1=0}^{m_1} \cdots \sum_{i_{n-1}=0}^{m_{n-1}} \frac{(m_n+1)_{i_1+\cdots+i_{n-1}}}{i_1! \cdots i_{n-1}!} \times {x_1}^{i_1} \cdots {x_{n-1}}^{i_{n-1}} (x_1 + \cdots + x_n)^{(m_1-i_1)+\cdots+\cdots(m_{n-1}-i_{n-1})}. \tag{5}$$

We propose a homogeneous polynomial identity with n variables which is equivalent on one side to (4)–(5), while on another side it generalizes an identity given by Graham, Knuth and Patashnik [4, p. 246]: *If $xy = x + y$ then*

$$x^{m+1}y^{k+1} = \sum_{i=0}^{m} \binom{k+i}{k} x^{m-i+1} + \sum_{i=0}^{k} \binom{m+i}{m} y^{k-i+1}. \tag{6}$$

1. A homogeneous polynomial identity with n variables

Our method uses only elementary tools of analysis. We apply the differential operator $(-\frac{\partial}{\partial x_1})^{m_1} \cdots (-\frac{\partial}{\partial x_n})^{m_n}$ to the elementary identity

$$\frac{1}{x_1 x_2 \cdots x_n} = \sum_{t=1}^{n} \frac{1}{x_1 \cdots \langle {}^{x_t}_{\text{skipped}} \rangle \cdots x_n (x_1 + \ldots + x_n)}. \tag{7}$$

The result of applying the operator to the left hand side of (7) is

$$\frac{m_1! m_2! \ldots m_n!}{x_1^{m_1+1} x_2^{m_2+1} \cdots x_n^{m_n+1}}. \tag{8}$$

On the right hand side of (7) we differentiate each term separately, i.e., take a fixed t, $1 \leq t \leq n$, and calculate

$$\left(-\frac{\partial}{\partial x_1}\right)^{m_1} \cdots \left(-\frac{\partial}{\partial x_n}\right)^{m_n} \frac{1}{x_1 \cdots \langle {}^{x_t}_{\text{skipped}} \rangle \cdots x_n (x_1 + \ldots + x_n)}. \tag{9}$$

Since the variable x_t appears only in one factor of the denominator of (9), while each other x_j, $j \neq t$ appears in two factors, we apply $(-\partial/\partial x_t)^{m_t}$ first and get

$$\frac{m_t!}{x_1 \cdots \langle {}^{x_t}_{\text{skipped}} \rangle \cdots x_n (x_1 + \ldots + x_n)^{m_t+1}}. \tag{10}$$

Next we apply $\prod_{j \neq t} (-\partial/\partial x_j)^{m_j}$ to (10). By the Leibnitz formula $(fg)^{(m)} = \sum_{i=0}^{m} \binom{m}{i} f^{(m-i)} g^{(i)}$, we get that (9) equals

$$\prod_{\substack{1 \leq j \leq n \\ j \neq t}} \left(-\frac{\partial}{\partial x_j}\right)^{m_j} \frac{m_t!}{x_1 \cdots \langle {}^{x_t}_{\text{skipped}} \rangle \cdots x_n (x_1 + \ldots + x_n)^{m_t+1}}$$

$$= \sum_{\substack{0\le i_j\le m_j\\ j\ne t}} \cdots \sum \binom{m_1}{i_1} \frac{(m_1-i_1)!}{x_1^{m_1-i_1+1}} \cdots \left\langle \begin{matrix} i_t, x_t \\ \text{skipped} \end{matrix} \right\rangle \cdots \binom{m_n}{i_n} \frac{(m_n-i_n)!}{x_n^{m_n-i_n+1}}$$
$$\times \frac{(m_t+i_1+\ldots+i_{t-1}+i_{t+1}+\ldots+i_n)!}{(x_1+\ldots+x_n)^{m_t+i_1+\ldots+i_{t-1}+i_{t+1}+\ldots+i_n+1}}. \tag{11}$$

Summing (11) for $t = 1, \ldots, n$ and comparing with (8), results

$$\frac{1}{x_1^{m_1+1} x_2^{m_2+1} \cdots x_n^{m_n+1}}$$
$$= \sum_{t=1}^{n} \Bigg[\sum_{\substack{0\le i_j\le m_j\\ j\ne t}} \cdots \sum \frac{(i_1+\ldots+i_{t-1}+m_t+i_{t+1}+\ldots+i_n)!}{i_1!\ldots i_{t-1}! m_t! i_{t+1}!\ldots i_n!}$$
$$\times \frac{1}{x_1^{m_1-i_1+1} \cdots \langle {x_t \atop \text{skipped}} \rangle \cdots x_n^{m_n-i_n+1} (x_1+\ldots+x_n)^{i_1+\ldots+i_{t-1}+m_t+i_{t+1}+\ldots+i_n+1}} \Bigg]. \tag{12}$$

Finally we replace x_i by x_i^{-1} and use for two of the basic symmetric polynomials in n variables the notation

$$S_{n,n}(x_1, \ldots, x_n) = x_1 \cdots x_n, \qquad S_{n-1,n}(x_1, \ldots, x_n) = \sum_{t=1}^{n} x_1 \cdots \left\langle \begin{matrix} x_t \\ \text{skipped} \end{matrix} \right\rangle \cdots x_n.$$

Then (12) becomes our main homogeneous identity:

Theorem 1.

$$x_1^{m_1+1} x_2^{m_2+1} \cdots x_n^{m_n+1}$$
$$= \sum_{t=1}^{n} \Bigg[\sum_{\substack{0\le i_j\le m_j\\ j\ne t}} \cdots \sum \frac{(i_1+\cdots+i_{t-1}+m_t+i_{t+1}+\cdots i_n)!}{i_1!\cdots i_{t-1}!\, m_t!\, i_{t+1}!\cdots i_n!}$$
$$\times x_1^{m_1-i_1+1} \cdots \left\langle \begin{matrix} x_t \\ \text{skipped} \end{matrix} \right\rangle \cdots x_n^{m_n-i_n+1} \left(\frac{S_{n,n}}{S_{n-1,n}} \right)^{i_1+\cdots i_{t-1}+m_t+i_{t+1}+\cdots i_n+1} \Bigg]. \tag{13}$$

Examples. For $n = 2$, (13) is

$$x_1^{m_1+1} x_2^{m_2+1} = \sum_{i_2=0}^{m_2} \binom{m_1+i_2}{m_1} x_2^{m_2-i_2+1} \left(\frac{x_1x_2}{x_1+x_2} \right)^{m_1+i_2+1} + \sum_{i_1=0}^{m_1} \binom{i_1+m_2}{m_2} x_1^{m_1-i_1+1} \left(\frac{x_1x_2}{x_1+x_2} \right)^{i_1+m_2+1}. \tag{14}$$

This leads directly to the identity (6). If we take $x_1 + x_2 = 1$, (1) follows.

Assuming the equality $S_{n,n} = S_{n-1,n}$, identity (13) implies an n-variable analogue to (6). For $n = 3$ it is: *If* $xyz = xy + yz + zx$, *then*

$$x^{m_1+1} y^{m_2+1} z^{m_3+1} = \sum_{j\le m_2,\, k\le m_3} \frac{(m_1+j+k)!}{m_1!\, j!\, k!} y^{m_2-j+1} z^{m_3-k+1}$$
$$+ \sum_{k\le m_3,\, i\le m_1} \frac{(i+m_2+k)!}{i!\, m_2!\, k!} z^{m_3-k+1} x^{m_1-i+1}$$
$$+ \sum_{i\le m_1,\, j\le m_2} \frac{(i+j+m_3)!}{i!\, j!\, m_3!} x^{m_1-i+1} y^{m_2-j+1}. \tag{15}$$

If we divide (13) by $x_1^{m_1+1}\cdots x_n^{m_n+1}$ and take $S_{n-1,n}=1$, we get an n-variable analogue of (1). For $n=3$ it is: *If* $xy+yz+zx=1$, *then*

$$(yz)^{m_1+1}\sum_{j\le m_2,\,k\le m_3}\frac{(m_1+j+k)!}{m_1!\,j!\,k!}y^kz^jx^{j+k}+(zx)^{m_2+1}\sum_{k\le m_3,\,i\le m_1}\frac{(i+m_2+k)!}{i!\,m_2!\,k!}z^ix^ky^{i+k}$$
$$+(xy)^{m_3+1}\sum_{i\le m_1,\,j\le m_2}\frac{(i+j+m_3)!}{i!\,j!\,m_3!}x^jy^iz^{i+j}=1. \tag{16}$$

It may be interesting to note that the case $n=2$, $m_1=m_2=1$ of (12), namely

$$\frac{1}{x^2y^2}=\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\frac{1}{(x+y)^2}+\left(\frac{1}{x}+\frac{1}{y}\right)\frac{2}{(x+y)^3},$$

played a central role in the development of the theory of G. Eisenstein about periodic functions. See [3, p. 252]. □

The change of variables $u_t=x_1\cdots x_{t-1}x_{t+1}\cdots x_n$, $t=1,\dots,n$, and the inverse transformation

$$x_t=\frac{(u_1\cdots u_n)^{1/(n-1)}}{u_t},\quad t=1,\dots,n,$$

yield $S_{n-1,n}(x_1,\dots,x_n)=u_1+\ldots+u_n$, $S_{n,n}(x_1,\dots,x_n)=(u_1\ldots u_n)^{1/(n-1)}$. After some elementary calculation this transforms identity (13) into

$$(u_1+\cdots+u_n)^{m_1+\cdots+m_n+1}=\sum_{t=1}^{n}\Bigg[u_t^{m_t+1}\sum_{\substack{0\le i_j\le m_j\\ j\ne t}}\cdots\sum\frac{(i_1+\cdots+i_{t-1}+m_t+i_{t+1}+\cdots i_n)!}{i_1!\cdots i_{t-1}!\,m_t!\,i_{t+1}!\cdots i_n!}$$
$$\times\, {u_1}^{i_1}\cdots\left\langle\begin{matrix}u_t\\ \text{skipped}\end{matrix}\right\rangle\cdots {u_n}^{i_n}(u_1+\cdots+u_n)^{\sum_{j\ne t}(m_j-i_j)}\Bigg]. \tag{17}$$

(17) is evidently identical with (4)–(5).

2. Another generalization of CB

Using ideas presented in [1], we now prove another generalization of (1) which depends on three independent integer parameters:

Theorem 2. *Let* $m-r+k-\ell=0$, m,r,k,ℓ *positive integers. Then*

$$(1-x)^{r+1}\sum_{i=0}^{k}\binom{m+i}{r}x^{i+m-r}+x^{\ell+1}\sum_{i=0}^{m}\binom{k+i}{\ell}(1-x)^{i+k-\ell}$$
$$=\begin{cases}1-\sum_{i=0}^{m-r-1}\binom{m}{i}x^i(1-x)^{m-i} & \text{if } m-r>0,\\ 1-\sum_{i=0}^{k-\ell-1}\binom{k}{i}(1-x)^ix^{k-i} & \text{if } k-\ell>0,\\ 1 & \text{if } m=r,\ \ k=l.\end{cases} \tag{18}$$

Proof. A straightforward differentiation of $Q_{r,k,m}(x)=(1-x)^{r+1}\sum_{i=0}^{k}\binom{m+i}{r}x^{i+m-r}$ leads after some elementary rearrangements and using $\binom{m+i+1}{r}(i+m-r+1)=\binom{m+i}{r}(i+m+1)$, to

$$\frac{d}{dx}Q_{r,k,m}(x)=(1-x)^r\left[\frac{m!}{r!(m-r-1)!}x^{m-r-1}-\frac{(m+k+1)!}{r!(m+k-r)!}x^{m+k-r}\right].$$

Since $\ell = k + m - r$, we rewrite this as

$$\frac{d}{dx}Q_{r,k,m}(x) = \frac{m!}{r!(m-r-1)!}(1-x)^r x^{m-r-1} - \frac{(m+k-1)!}{r!\ell!}(1-x)^r x^\ell. \tag{19}$$

We interchange now m and k, r and ℓ, x and $1-x$ and consider $R_{\ell,m,k}(x) = x^{\ell+1}\sum_{i=0}^{m}\binom{k+i}{\ell}(1-x)^{i+k-\ell}$. Since $\frac{d}{d(1-x)} = -\frac{d}{dx}$, we get analogously

$$-\frac{d}{dx}R_{\ell,m,k}(x) = \frac{k!}{\ell!(k-\ell-1)!}x^\ell(1-x)^{k-\ell-1} - \frac{(k+m-1)!}{\ell!r!}x^\ell(1-x)^r. \tag{20}$$

(19) and (20) yield

$$\frac{d}{dx}\left[Q_{r,k,m}(x) + R_{\ell,m,k}(x)\right] = \frac{m!}{r!(m-r-1)!}(1-x)^r x^{m-r-1} - \frac{k!}{\ell!(k-\ell-1)!}(1-x)^{k-\ell-1}x^\ell. \tag{21}$$

Consider, for example, the case $m-r = \ell-k > 0$. In this case the terms in the sum $R_{\ell,m,k}(x)$ are nonzero only for $i \geq \ell - k = m - r$, the last term of (21) is absent and $Q_{r,k,m}(0) = R_{\ell,m,k}(0) = 0$. Consequently

$$Q_{r,k,m}(x) + R_{\ell,m,k}(x) = \frac{m!}{r!(m-r-1)!}\int_0^x t^{m-r-1}(1-t)^r\,dt = I_x(m-r, r+1), \tag{22}$$

where I_x denotes the normalized incomplete beta function. By the identity

$$I_x(m-r, r+1) = \sum_{i=m-r}^{m}\binom{m}{i}x^i(1-x)^{m-i} = 1 - \sum_{i=0}^{m-r-1}\binom{m}{i}x^i(1-x)^{m-i},$$

[8, Eq. 8.17.5], equality (22) yields the first case of (18). If $\ell - k = m - r < 0$, the corresponding equality is verified similarly. □

T. Koornwinder pointed out to us another proof to our Theorem 2. Indeed, he suggested to change the summation index in the first sum on the left of (18) to $j = i + (m - r)$ and in the second sum, where the first $m - r = \ell - k > 0$ terms are 0, to change to $j = i - (m - r)$. After these changes the left side is written as

$$(1-x)^{r+1}\sum_{j=m-r}^{k+m-r}\binom{j+r}{r}x^j + x^{\ell+1}\sum_{j=0}^{r}\binom{j+\ell}{\ell}(1-x)^j.$$

Let us rearrange it to

$$\left[(1-x)^{r+1}\sum_{j=0}^{\ell}\binom{j+r}{r}x^j + x^{\ell+1}\sum_{j=0}^{r}\binom{j+\ell}{\ell}(1-x)^j\right] - (1-x)^{r+1}\sum_{j=0}^{m-r-1}\binom{j+r}{r}x^j.$$

The first two sums total to 1 by the original Chaundy–Bullard identity, so (18) will follow if one shows that

$$1 - (1-x)^{r+1}\sum_{j=0}^{m-r-1}\binom{j+r}{r}x^j = 1 - \sum_{i=0}^{m-r-1}\binom{m}{i}x^i(1-x)^{m-i}.$$

But the required

$$\sum_{j=0}^{m-r-1} \binom{j+r}{r} x^j = \sum_{i=0}^{m-r-1} \binom{m}{i} x^i (1-x)^{m-r-i-1}$$

is precisely Eq. (2.7) of [5], hence (18) follows.

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