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Critical points at infinity and blow up of solutions of autonomous polynomial differential systems via compactification

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Abstract

Critical points at infinity for autonomous differential systems are defined and used as an essential tool. R^n is mapped onto the unit ball by various mappings and the boundary points of the ball are used to distinguish between different directions at infinity. These mappings are special cases of compactifications. It is proved that the definition of the critical points at infinity is independent of the choice of the mapping to the unit ball.

We study the rate of blow up of solutions in autonomous polynomial differential systems of equations via compactification methods. To this end we represent each solution as a quotient of a vector valued function (which is a solution of an associated autonomous system) by a scalar function (which is a solution of a related scalar equation).

Keywords: Autonomous differential equations; Polynomial; Compactification; Critical points; Blow up

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1. Introduction

Autonomous systems of polynomial differential equations occur in numerous instances in theory and applications. Given an autonomous system of differential equations

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \tag{1.1}$$

i.e., $y_i' = f_i(y_1, \dots, y_n)$, $i = 1, \dots, n$, where $n > 1$, f_i are polynomials. The rate of blow up of solutions of (1.1) is of special interest. In this study we obtain, under appropriate conditions, the rate of blow up of solutions of (1.1) as a function of the "amount of non-linearity" present in (1.1) as manifested by the degree L of the polynomial $\mathbf{f}(\mathbf{y})$. To this end we produce global representations of (1.1) by employing compactification methods. In this setting the definition and characterization of critical points at infinity play an important role.

The idea to compactify the space R^n by addition of points at infinity and to map them into finite points is frequently used in the setting of two-dimensional differential equations. An early study of differential equations via compactification was carried out by Bendixson [1, p. 216] using the stereographic projection. However, the stereographic projection has the disadvantage of obscuring the "different directions at infinity." Poincaré overcame this difficulty by projecting R^2 onto the Poincaré hemisphere through its center. The different directions at infinity are then identified with their projections on the equator. Next the hemisphere is projected through its center to another plane which is perpendicular to the original plane and tangent to the Poincaré sphere at a point of its equator. If the original plane is identified with the points $(x, y, -1)$, the center of the Poincaré sphere is at $(0, 0, 0)$ and the final plane consists of the points $(1, u, -z)$, then our projective transformation is realized by the transformation

$$u = \frac{y}{x}, \quad z = \frac{1}{x} \qquad \text{or} \qquad x = \frac{1}{z}, \quad y = \frac{u}{z}. \tag{1.2}$$

See [1, p. 221]. This maps the points at infinity of the (x, y)-plane to the line $z = 0$ of the (u, z)-plane. (1.2) transforms the two-dimensional polynomial system

$$x' = P(x, y), \qquad y' = Q(x, y)$$

into

$$u' = -uzP(1/z, u/z) + zQ(1/z, u/z) = z^{-L} P^*(u, z),$$
$$z' = -z^2 P(1/z, u/z) = z^{-L} Q^*(u, z),$$

where L is the smallest integer so that P^*, Q^* are polynomials. Critical points at infinity are defined to be the solutions of

$$P^*(u, z) = Q^*(u, z) = 0 \quad \text{with } z = 0,$$

i.e., two equations $P^*(u, 0) = Q^*(u, 0) = 0$ for $u = y/x$. Thus u represents a direction of a point at infinity. A technical drawback of this approach is that the points in the directions $\mathbf{p} = (0, \pm 1)$ (the line $x = 0$) need a separate treatment. We can do the final projection on another perpendicular plane, for example, on $(v, 1, -z)$, which leads to the projective transformation

$$v = \frac{x}{y}, \quad z = \frac{1}{y} \qquad \text{or} \qquad x = \frac{v}{z}, \quad y = \frac{1}{z}$$

and other P^*, Q^*.

[8, p. 98] and [5, p. 201] describe another variant of the Poincaré transformation. By the projective change of variables

$$x = \frac{X}{Z}, \qquad y = \frac{Y}{Z},$$

they transform the equation

$$Q(x, y)\,dx - P(x, y)\,dy = 0$$

into

$$A(X, Y, Z)\,dX + B(X, Y, Z)\,dY + C(X, Y, Z)\,dZ = 0.$$

The critical points are defined to be the solutions of

$$A(X, Y, Z) = B(X, Y, Z) = C(X, Y, Z) = 0$$

and those with $Z = 0$ represent critical points at infinity. Compare also with [7] for some interesting aspects of Poincaré's work.

Another alternative is to project $(x, y) \in R^2$ onto the lower half of the Poincaré sphere and then project it orthogonally onto a unit disk, tangent at its pole. The point at infinity in the direction $\mathbf{p}$ is thus mapped into the point $\mathbf{p}$ of the unit circle. While this is convenient geometrically, its explicit formulas

$$u = \frac{x}{\sqrt{x^2 + y^2 + 1}}, \qquad v = \frac{y}{\sqrt{x^2 + y^2 + 1}},$$

and

$$x = \frac{u}{\sqrt{1 - u^2 - v^2}}, \qquad y = \frac{v}{\sqrt{1 - u^2 - v^2}}$$

are irrational.

Our approach is based on the interplay between a compactification and a certain parameterization of the independent variable. For example (see Example 3.3), consider the compactification

$$\mathbf{y}(t) = \mathbf{x}(\tau)/\left(1 - \left\|\mathbf{x}(\tau)\right\|^2\right), \tag{1.3}$$

which maps $\mathbf{y} \in R^n$ into $\mathbf{x}$ in the unit ball $U \subset R^n$. The system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, where $\mathbf{f}(\mathbf{y})$ is a polynomial of order L, is transformed into the system (3.16),

$$\frac{d\mathbf{x}}{dt} = \frac{(1 - \|\mathbf{x}\|^2)^{1-L}}{1 + \|\mathbf{x}\|^2}\left[\left(1 + \|\mathbf{x}\|^2\right)\tilde{\mathbf{f}}(\mathbf{x}) - 2\langle \mathbf{x}, \tilde{\mathbf{f}}\rangle \mathbf{x}\right],$$

with a certain polynomial $\tilde{\mathbf{f}}(\mathbf{x})$, that is continuous and bounded for $\mathbf{x} \in U$. Here $\langle , \rangle$ denotes the standard inner product and $\| \ \|$ is the Euclidean norm. Along the solution $\mathbf{x}(t)$, a new independent variable $\tau = \tau(t)$ is defined by

$$\frac{dt}{d\tau} = \left(1 - \|\mathbf{x}\|^2(t)\right)^{L-1}\left(1 + \|\mathbf{x}\|^2(t)\right). \tag{1.4}$$

See (3.17). Consequently, the system (1.1) is replaced by

$$\frac{d\mathbf{x}}{d\tau} = \left(1 + \|\mathbf{x}\|^2\right)\tilde{\mathbf{f}}(\mathbf{x}) - 2\langle \mathbf{x}, \tilde{\mathbf{f}}\rangle \mathbf{x}. \tag{1.5}$$

The approach of $\mathbf{x}$ to the boundary of the unit ball is governed by the scalar differential equation (3.20),

$$\frac{d}{d\tau}\left(1 - \|\mathbf{x}\|^2\right) = -2\langle \tilde{\mathbf{f}}, \mathbf{x}\rangle \left(1 - \|\mathbf{x}\|^2\right). \tag{1.6}$$

All these turn (1.3) into a convenient representation of the solution $\mathbf{y}(t)$ of (1.1):

(a) The global behaviour of each solution of (1.1) is governed by the vector differential equation (1.5) and by the two scalar differential equations (1.4) and (1.6).
(b) The numerator $\mathbf{x}(\tau)$ of (1.3) is a solution of the polynomial differential system (1.5) and it is bounded.
(c) The denominator $1 - \|\mathbf{x}\|^2$ is a solution of the scalar equation (1.6). It governs the rate of growth of $\mathbf{y}(t)$. Moreover, the numerator and the denominator are polynomial in $\mathbf{x}$ and they do not vanish simultaneously.
(d) The ranges of the variables τ and t in (1.4) are $-\infty < \tau < \infty$ and $t_{\min} < t < t_{\max}$, respectively.

The order of topics dealt with in this paper is as follows. In Section 2 we study some families of compactifications. These are in essence bijections from the $\mathbf{y}$ space of R^n to a unit ball in an $\mathbf{x}$ space. Among their properties we note that all lead to the same critical points at infinity. In Section 3 we discuss a family of radial compactifications. It is noteworthy that on one hand the compactifications place critical points at infinity on the same footing as finite critical points. On the other hand, it turns out that a neighbourhood of a critical point infinity in the $\mathbf{x}$ space cannot be treated with same ease as a finite critical point, as will be seen in Section 4. The methods presented in Sections 2 and 3 turn out to be useful in obtaining the rate of blow up of solutions of (1.1). Under appropriate conditions we show in Section 4 that the rate of blow up of some $\mathbf{y}(t)$ at a finite time is like $(t_{\max} - t)^{-1/(L-1)}$, where L is the degree of the polynomial vector $\mathbf{f}(\mathbf{y})$.

2. Admissible compactifications

We want to distinguish between the various points of R^n at infinity according to their directions. This is done by a bijection between R^n and the unit ball $U \subset R^n$ and identification of each unit vector $\mathbf{p} = (p_1, \ldots, p_n) \in \partial U$ with a direction at infinity. Such a bijection adds to R^n a set of ideal points at infinity in any direction $\mathbf{p}$, $\mathbf{p} \in \partial U$. To be precise, we say that a sequence of points $\mathbf{y}(k) \in R^n$ *tends to infinity in the direction* $\mathbf{p}$, $\|\mathbf{p}\| = 1$, if $\|\mathbf{y}(k)\| \to \infty$ and $\mathbf{y}(k)/\|\mathbf{y}(k)\| \to \mathbf{p}$ as $k \to \infty$.

One can, of course, realize such a bijection by $\mathbf{x} = \mathbf{y}/(1 + \|\mathbf{y}\|^2)^{1/2}$. We shall consider a more general direction preserving bijection from R^n to U of the form

$$\mathbf{x} = \mathbf{y}/\kappa(\mathbf{y}), \quad \kappa(\mathbf{y}) = \kappa(y_1, \ldots, y_n) > 0, \tag{2.1}$$

i.e.,

$$x_i = y_i/\kappa(y_1, \dots, y_n). \tag{2.2}$$

Some assumptions which are needed in our work are summarized in the next definition. By the notation $\mathbf{u}(\mathbf{y}) \sim \mathbf{v}(\mathbf{y})$ we mean that $\lim v_i(\mathbf{y})/u_i(\mathbf{y}) = 1$, $i = 1, \dots, n$ as $\|\mathbf{y}\| \to \infty$.

Definition 2.1. The mapping (2.1) is called an admissible compactification if it satisfies the assumptions

(A0) $\kappa(\mathbf{y}) > \|\mathbf{y}\|$,
(A1) $\kappa(\mathbf{y}) \sim \|\mathbf{y}\|$ as $\|\mathbf{y}\| \to \infty$,
(A2) $\nabla\kappa(\mathbf{y}) \sim \mathbf{y}/\|\mathbf{y}\|$ as $\|\mathbf{y}\| \to \infty$, and
(A3) $\langle \mathbf{y}, \nabla\kappa\rangle < \kappa(\mathbf{y})$.

(A0) and (A1) guarantee that R^n is mapped into U and that the points at infinity are mapped onto ∂U. Since $\mathbf{0}$ is mapped to $\mathbf{0}$, the bijectivity of (2.1) implies that R^n is mapped onto U.

(A3) is related to the bijectivity of (2.1). By (2.2), $\partial x_i/\partial y_j = \kappa^{-1}\delta_{ij} - \kappa^{-2}(\partial\kappa/\partial y_j)y_i$ and in matrix notation

$$\left(\frac{\partial x_i}{\partial y_j}\right) = \kappa^{-1}\big[I - \kappa^{-1}\mathbf{y}(\nabla\kappa)^T\big].$$

This Jacobian matrix is easily inverted. Indeed, for two column vectors $\mathbf{y}, \mathbf{z}$,

$$\begin{aligned}\big(I + \beta\mathbf{y}\mathbf{z}^T\big)\big(I + \delta\mathbf{y}\mathbf{z}^T\big) &= I + (\beta + \delta)\mathbf{y}\mathbf{z}^T + \beta\delta\mathbf{y}\mathbf{z}^T\mathbf{y}\mathbf{z}^T \\ &= I + \big(\beta + \delta + \beta\delta\langle\mathbf{z}, \mathbf{y}\rangle\big)\mathbf{y}\mathbf{z}^T,\end{aligned} \tag{2.3}$$

so $I + \delta\mathbf{y}\mathbf{z}^T = (I + \beta\mathbf{y}\mathbf{z}^T)^{-1}$ if $\delta = -\beta/(1 + \beta\langle\mathbf{z}, \mathbf{y}\rangle)$. In our case $\beta = -\kappa^{-1}$, $\mathbf{z} = \nabla\kappa$, so

$$\left(\frac{\partial y_j}{\partial x_i}\right) = \left(\frac{\partial x_i}{\partial y_j}\right)^{-1} = \kappa\left[I - \frac{1}{\kappa - \langle\mathbf{y}, \nabla\kappa\rangle}\mathbf{y}(\nabla\kappa)^T\right]. \tag{2.4}$$

Hence transformation (2.1) and its inverse are C^1 locally bijective if $\kappa - \langle\mathbf{y}, \nabla\kappa\rangle \neq 0$. By considering this quantity at $\mathbf{y} = \mathbf{0}$, we see that it must be positive, which is (A3). However (2.1) maps any one-dimensional ray $\mathbf{y} = r\mathbf{v}$, $0 \leqslant r < \infty$, $\mathbf{v}$ some fixed vector, into itself. But for a continuous mapping from R to R, local bijectivity implies global bijectivity. Consequently (A3) guarantees also the global bijectivity of (2.1). (A2) will be needed later.

Example 2.2. Suppose that the transformation (2.1) has a radial symmetry, i.e., $\kappa = \kappa(r) = \kappa(\sqrt{y_1^2 + \cdots + y_n^2})$, $\kappa(r) > r$. Then the assumptions (A1)–(A3) become

$$\kappa(r) \sim r \quad \text{as } r \to \infty, \tag{2.5}$$

$$\kappa'(r) \sim 1 \quad \text{as } r \to \infty, \tag{2.6}$$

$$r\kappa'(r) < \kappa \quad \text{for all } r > 0, \tag{2.7}$$

respectively. Of course, (2.6) implies (2.5). Squaring (2.2) and summing it, we get

$$R = \frac{r}{\kappa(r)}, \quad 0 \leqslant r < \infty, \quad \text{with} \quad r^2 = \sum y_i^2, \quad R^2 = \sum x_i^2, \tag{2.8}$$

and (2.7) simply means that R is an increasing function of r.

What is the form of the transformed differential equation (1.1) under an admissible compactification (2.1)? Applying (2.1) to (1.1), we obtain

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \frac{d}{dt}\big(\mathbf{y}/\kappa(\mathbf{y})\big) = \frac{d\mathbf{y}}{dt}\Big/\kappa - \mathbf{y}\left\langle \nabla\kappa(\mathbf{y}), \frac{d\mathbf{y}}{dt}\right\rangle\Big/\kappa^2 \\ &= \kappa^{-1}\big[\mathbf{f}(\mathbf{y}) - \big\langle \nabla\kappa, \mathbf{f}(\mathbf{y})\big\rangle(\mathbf{y}/\kappa)\big] \\ &= \kappa^{-1}\big[\mathbf{f}(\kappa\mathbf{x}) - \big\langle \nabla\kappa, \mathbf{f}(\kappa\mathbf{x})\big\rangle\mathbf{x}\big], \end{aligned} \tag{2.9}$$

i.e.,

$$\frac{d\mathbf{x}}{dt} = \kappa^{-1}\big(\mathbf{y}(\mathbf{x})\big)\big[\mathbf{f}(\kappa\mathbf{x}) - \big\langle \nabla\kappa, \mathbf{f}(\kappa\mathbf{x})\big\rangle\mathbf{x}\big]. \tag{2.10}$$

Proposition 2.3. *The transformation* (2.1) *maps finite critical points of* (1.1) *in* R^n *into finite critical points of* (2.10) *in* U *and vice versa.*

Proof. If $\mathbf{y}$ is a critical point of (1.1), i.e., $\mathbf{f}(\mathbf{y}) = 0$, then the right-hand side of (2.10) obviously vanishes at the corresponding $\mathbf{x}$. Conversely, suppose that the right-hand side of (2.10) vanishes for some $\mathbf{x} \in U$, $\|\mathbf{x}\| < 1$:

$$\mathbf{f}(\kappa\mathbf{x}) - \big\langle \nabla\kappa, \mathbf{f}(\kappa\mathbf{x})\big\rangle\mathbf{x} = 0. \tag{2.11}$$

Multiplying by $\nabla\kappa$, we get

$$\langle \nabla\kappa, \mathbf{f}\rangle\big(1 - \langle \nabla\kappa, \mathbf{x}\rangle\big) = 0.$$

However, due to (A3) we have $|\langle \nabla\kappa, \mathbf{x}\rangle| = |\langle \nabla\kappa(\mathbf{y}), \mathbf{y}/\kappa(\mathbf{y})\rangle| < 1$, so $\langle \nabla\kappa, \mathbf{f}\rangle = 0$ and consequently, by (2.11), $\mathbf{f}(\mathbf{y}) = \mathbf{f}(\kappa\mathbf{x}) = 0$. □

We focus now on points at infinity. They are mapped by (2.1) to the boundary of U whose points represent the points at infinity in various directions. On ∂U, $\kappa = \kappa(\mathbf{y}(\mathbf{x})) \to \infty$ and Eq. (2.10) is obviously singular. Let us extract from (2.10) its singular scalar part and its continuous multidimensional part.

Let L be the degree of the polynomial $\mathbf{f}(\mathbf{y})$ and let $\mathbf{f}(\mathbf{y})$ be written in the form

$$\mathbf{f}(\mathbf{y}) = \mathbf{p}_0(\mathbf{y}) + \mathbf{p}_1(\mathbf{y}) + \cdots + \mathbf{p}_L(\mathbf{y}),$$

where $\mathbf{p}_j(\mathbf{y})$, $j = 0, 1, \ldots, L$, are homogeneous polynomials of degree j, respectively. We want to extract the highest order homogeneous polynomial $\mathbf{p}_L(x)$ from $\mathbf{f}(\mathbf{y})$, since we expect that $\mathbf{p}_L(\mathbf{y})$ and consequently $\mathbf{p}_L(\mathbf{x})$ will play the major part in the behaviour of our system at infinity. Let us put $\mathbf{y} = \kappa\mathbf{x}$ and define

$$\tilde{\mathbf{f}}(\mathbf{x}, \kappa) = \kappa^{-L}\mathbf{f}(\kappa\mathbf{x}) = \kappa^{-L}\mathbf{p}_0(\mathbf{x}) + \kappa^{-L+1}\mathbf{p}_1(\mathbf{x}) + \cdots + \mathbf{p}_L(\mathbf{x}), \tag{2.12}$$

where $\kappa = \kappa(\mathbf{y}(\mathbf{x}))$. According to (2.12), Eq. (2.10) may be written as

$$\frac{d\mathbf{x}}{dt} = \kappa^{L-1}\big[\tilde{\mathbf{f}}(\mathbf{x}, \kappa) - \langle \nabla\kappa, \tilde{\mathbf{f}}\rangle \mathbf{x}\big]. \tag{2.13}$$

κ^{L-1} is positive and unbounded on ∂U while the rest of Eq. (2.13) is continuous on the closed ball $\overline{U}$. If we define along a trajectory $\mathbf{x}(t)$ a new independent variable τ by

$$\frac{d\tau}{dt} = \kappa^{L-1}\big(\mathbf{y}(t)\big), \tag{2.14}$$

i.e.,

$$\tau = \int^{t} \kappa^{L-1}\big(\mathbf{y}(t)\big)\,dt,$$

then Eq. (2.13) becomes

$$\frac{d\mathbf{x}}{d\tau} = \tilde{\mathbf{f}}(\mathbf{x}, \kappa) - \langle \nabla\kappa, \tilde{\mathbf{f}}\rangle \mathbf{x}. \tag{2.15}$$

The trajectories of (1.1) in R^n and those of (2.15) in U have the same topology and, according to Proposition 2.3, the same (finite) critical points. In order to put the critical points of (1.1) at finite points and its critical points at infinity on equal footing, we propose the following definition:

Definition 2.4. We say that Eq. (1.1) has a critical point at infinity in the direction $\mathbf{x}$, $\|\mathbf{x}\| = 1$, if $\mathbf{x}$ is a critical point of Eq. (2.15) in ∂U, i.e., the right-hand side of (2.15) vanishes there.

Proposition 2.5. *If a solution $\mathbf{y}(t)$ of (1.1) has a maximal interval of existence (a, b) and $\mathbf{y}(t)$ tends to infinity in the direction $\mathbf{p}$ as $t \to b^-$ (or as $t \to a^+$), then $\mathbf{p}$ is a critical point of (2.15) in ∂U.*

Proof. The maximality of $t = b^-$ for $\mathbf{y}(t)$ implies that the corresponding τ for $\mathbf{x}(\tau)$ must be $+\infty$. For, suppose on the contrary that $\tau \to \tau_0 < \infty$ and $\lim_{\tau\to\tau_0-} \mathbf{x}(\tau) = \mathbf{p}$. Then $\mathbf{x}(\tau_0) = \mathbf{p}$ is a regular initial value condition at a point of continuity $\mathbf{p}$ for Eq. (2.15), so its solution $\mathbf{x}(\tau)$ is defined on a whole neighborhood of τ_0. This contradicts the maximality of $t = b$. Thus $\tau \to +\infty$.

Now, if $\lim_{\tau\to+\infty} \mathbf{x}(\tau) = \mathbf{p}$ for a solution of (2.15), then it is well known that $\mathbf{p}$ is a critical point of (2.15). □

As $\|\mathbf{x}\| \to 1$, we have $\|\mathbf{y}\| \to \infty$, and by assumptions (A1), (A2), $\kappa(\mathbf{y}(\mathbf{x})) \to \infty$ and $\nabla\kappa(\mathbf{y}) \sim \mathbf{y}/\|\mathbf{y}\| \sim \mathbf{y}/\kappa(\mathbf{y}) = \mathbf{x}$. Thus, the vanishing of the right-hand side of Eq. (2.15),

$$\tilde{\mathbf{f}}(\mathbf{x}, \kappa) - \langle \nabla\kappa, \tilde{\mathbf{f}}\rangle \mathbf{x} = 0, \tag{2.16}$$

on $\|\mathbf{x}\| = 1$ is equivalent to

$$\tilde{\mathbf{f}}(\mathbf{x}, \infty) - \big\langle \mathbf{x}, \tilde{\mathbf{f}}(\mathbf{x}, \infty)\big\rangle \mathbf{x} = 0. \tag{2.17}$$

We shall refer to (2.17) as the (nonlinear) characteristic equation of Eq. (1.1) at infinity. Note that as $\mathbf{x} \to \partial U$, we have $\kappa \to \infty$, so

$$\lim_{\mathbf{x}\to\partial U} \tilde{\mathbf{f}}(x,\kappa) = \mathbf{p}_L(\mathbf{x}). \tag{2.18}$$

Consequently, the critical points at infinity are influenced only by the highest order powers in the right-hand side of Eq. (1.1). We summarize this in the following proposition.

Proposition 2.6.

(a) *The definition of a critical point of* (1.1) *at infinity in the direction* $\mathbf{x}$ *is independent of the choice of the admissible compactification* (2.1).
(b) *The critical points at infinity in the direction* $\mathbf{x}$ *depend only on the highest order homogeneous terms of* $\mathbf{f}$*, namely,* $\mathbf{p}_L$ *and the nonlinear characteristic equation* (2.17) *is equivalent to*

$$\begin{aligned} &\mathbf{p}_L(\mathbf{x}) - \langle \mathbf{x}, \mathbf{p}_L(\mathbf{x})\rangle \mathbf{x} = 0, \\ &\|\mathbf{x}\| = 1. \end{aligned} \tag{2.19}$$

(c) *If* (1.1) *has a critical point at infinity in the direction* $\mathbf{p}$*, then it also has a critical point at infinity in the direction* $-\mathbf{p}$.

(c) is an immediate consequence of (2.19) and the homogeneity of $\mathbf{p}_L$. (2.19) can be written in an equivalent way as a homogeneous equation

$$\mathbf{p}_L(\mathbf{x}) - \frac{\langle \mathbf{x}, \mathbf{p}_L(\mathbf{x})\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} \mathbf{x} = 0. \tag{2.20}$$

For example, the linear system $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$, $A \in R^{n\times n}$, has a critical point at infinity in the direction $\mathbf{p}$ if and only if $\mathbf{p}$ is a real valued unit eigenvector of A. This follows from (2.20), since $\mathbf{p}_L(\mathbf{x}) = A\mathbf{x}$ with $L = 1$.

Example 2.7. The system

$$\begin{aligned} y_1' &= y_1^2 + y_2^2 - 1, \\ y_2' &= 5(y_1 y_2 - 1), \end{aligned}$$

was discussed by Poincaré and several authors [5, p. 204], [8, p. 103] and [6, p. 272] choose it as an illustration. It can be treated by our procedure as well. Here $L = 2$, $\mathbf{p}_2(\mathbf{y}) = (y_1^2 + y_2^2, 5y_1y_2)$, and Eq. (2.19) becomes

$$\begin{aligned} &x_1^2 + x_2^2 - \bigl(x_1\bigl(x_1^2 + x_2^2\bigr) + x_2(5x_1x_2)\bigr)x_1 = 0, \\ &5x_1x_2 - \bigl(x_1\bigl(x_1^2 + x_2^2\bigr) + x_2(5x_1x_2)\bigr)x_2 = 0. \end{aligned}$$

With $x_1^2 + x_2^2 = 1$, this simplifies to

$$\begin{aligned} 1 &= x_1\bigl(x_1 + 5x_1x_2^2\bigr), \\ 5x_1x_2 &= x_2\bigl(x_1 + 5x_1x_2^2\bigr), \end{aligned}$$

and yields the expected six solutions $(\pm 1, 0)$, $(\pm 1/\sqrt{5}, \pm 2/\sqrt{5})$.

The next proposition shows that our definition of a critical point agrees with the definitions used in the literature for $n = 2$. Compare with [6, p. 268].

Proposition 2.8. *The point at infinity in the direction* $\mathbf{x}$ *is a critical point for Eq.* (1.1) *if and only if*

$$\begin{aligned} &x_i \tilde{f}_j(\mathbf{x}, \infty) = x_j \tilde{f}_i(\mathbf{x}, \infty), \quad i, j = 1, \dots, n,\ i < j, \\ &\|\mathbf{x}\| = 1. \end{aligned} \tag{2.21}$$

The proof is obtained by writing (2.17) in scalar form, namely,

$$\tilde{f}_i(\mathbf{x}, \infty) = \big\langle \mathbf{x}, \tilde{\mathbf{f}}(\mathbf{x}, \infty)\big\rangle x_i, \quad i = 1, \dots, n, \tag{2.22}$$

which obviously implies (2.21). Conversely, assume that (2.21) holds. Notice that at least one component of $\mathbf{x}$ is nonzero, say $x_r \neq 0$. Hence

$$\tilde{f}_i = x_i \frac{\tilde{f}_r}{x_r}, \quad i = 1, \dots, n, \tag{2.23}$$

and

$$\langle \mathbf{x}, \tilde{\mathbf{f}} \rangle = \sum x_i \tilde{f}_i = \frac{\tilde{f}_r}{x_r} \sum x_i^2 = \frac{\tilde{f}_r}{x_r}.$$

Substituting $\tilde{f}_r / x_r = \langle \mathbf{x}, \tilde{\mathbf{f}} \rangle$ into (2.23) leads to the required (2.22). The $n(n-1)/2$ equations of (2.21) are, of course, not independent.

Example 2.9. Consider the nth order equation

$$y^{(n)} = h\big(y, y', \dots, y^{(n-1)}\big), \tag{2.24}$$

where h is a polynomial. The substitution $y_i = y^{(i-1)}$, $i = 1, \dots, n$, leads to

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_2 \\ \vdots \\ h(y_1, \dots, y_n) \end{pmatrix} \equiv \mathbf{f}(y_1, \dots, y_n).$$

Let $L > 1$ denote the degree of the polynomial $h(y_1, \dots, y_n)$ and p_L its terms of the maximal degree. Then

$$\tilde{\mathbf{f}}(x, \kappa) = \kappa^{-L} \begin{pmatrix} \kappa x_2 \\ \vdots \\ h(\kappa x_1, \dots, \kappa x_n) \end{pmatrix}$$

and as we let $\kappa \to \infty$, Eq. (2.19) becomes

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_L(\mathbf{x}) \end{pmatrix} = x_n p_L(\mathbf{x}) \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix},$$

with the solution $(x_1, \dots, x_n) = (0, \dots, 0, 1)$. This leads to the following proposition.

Proposition 2.10. *The companion system of an nth order polynomial differential equation* (2.24) *possesses at least one critical point at infinity in the direction* $(0, \dots, 0, 1)$*. If the solution* $\mathbf{y}(t)$ *converges to infinity in the direction* $(0, \dots, 0, 1)$*, then the corresponding scalar solution $y(t)$ of* (2.24) *must satisfy $y^{(i)}(t)/y^{(n-1)}(t) \to 0$ as $t \to t_{\max}$.*

3. Radial compactifications

Consider the rotational surface

$$x_{n+1} = q\big(\big(x_1^2 + \cdots + x_n^2\big)^{1/2}\big), \tag{3.1}$$

where q is sufficiently smooth and $n > 1$. We project the point $\mathbf{y} = (y_1, \dots, y_n)$ through the point $(0, \dots, 0, q(1))$ on the surface (3.1) and assume that there is a unique intersection point between $(y_1, \dots, y_n)$ and $(0, \dots, 0, q(1))$. Next, project this intersection point $(x_1, \dots, x_n, x_{n+1})$ orthogonally on $(x_1, \dots, x_n)$. See Fig. 1.

By similarity of triangles

$$\frac{y_j}{x_j} = \frac{q(1)}{q(1) - q(R)}, \quad R = \Big(\sum x_k^2\Big)^{1/2},$$

that is,

$$y_j = \frac{x_j}{1 - q(R)/q(1)}.$$

Let $g(R) = 1 - q(R)/q(1)$. Then, our radial compactification is written as

$$y_j = \frac{x_j}{g(R)}, \qquad R = \Big(\sum x_k^2\Big)^{1/2} \tag{3.2}$$

or

$$\mathbf{y} = \mathbf{x}/g(R), \tag{3.3}$$

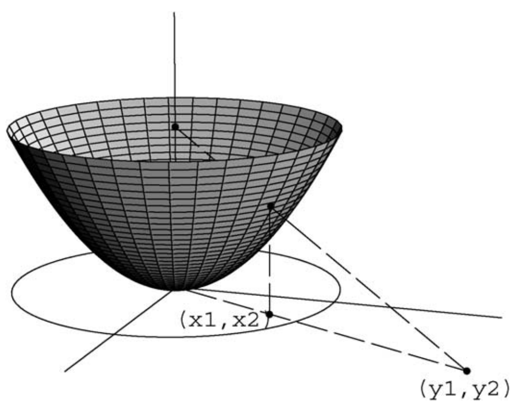

Fig. 1.

i.e., from $\mathbf{x} \in U$ to $\mathbf{y} \in R^n$, the inverse direction to (2.1). Of course, $g(1) = 0$. Squaring (3.2) and summing it yields

$$r = \frac{R}{g(R)}, \quad 0 \leqslant R < 0, \quad \text{with} \quad r^2 = \sum y_k^2, \quad R^2 = \sum x_k^2. \tag{3.4}$$

(3.4) is the inverse of (2.8) provided that $R/g(R)$ is monotone increasing, namely,

$$g(R) - Rg'(R) > 0 \quad \text{on } [0, 1]. \tag{3.5}$$

Under this condition assumptions (2.5)–(2.7) are satisfied. (2.5) holds since $r/\kappa(r) = R \to 1$ as $r \to \infty$. Next, $R'(r) = 1/r'(R) = g^2(R)/(g(R) - Rg'(R))$, so

$$\frac{d\kappa}{dr} = \frac{d}{dr}\left(\frac{r}{R}\right) = \frac{R - rR'(r)}{R^2(r)} = \frac{g'(R)}{Rg'(R) - g(R)}.$$

As $g(1) = 0$, it follows that

$$\lim_{r\to\infty} \kappa'(r) = \lim_{R\to 1} \frac{g'(R)}{Rg'(R) - g(R)} = 1, \tag{3.6}$$

this is (2.6). Finally (2.7) expresses the monotone behaviour of the inverse equation (2.8).

What is the shape of the transformed differential equation under the compactification (3.3)? Since we do not have the mapping (2.1) explicitly but rather its inverse (3.3), some inversion process is required. The equation $d\mathbf{y}/dt = \mathbf{f}(\mathbf{y})$ becomes

$$\frac{d\mathbf{x}}{dt} = \left(\frac{\partial x_i}{\partial y_j}\right)\frac{d\mathbf{y}}{dt} = \left(\frac{\partial y_j}{\partial x_i}\right)^{-1} \mathbf{f}(\mathbf{y}). \tag{3.7}$$

The elements $\frac{\partial y_j}{\partial x_i}$ of the Jacobian matrix are calculated in a straightforward manner from (3.2) as follows:

$$\frac{\partial y_j}{\partial x_i} = \frac{1}{g(R)}\delta_{ij} + x_j \frac{-g'(R)}{g^2(R)} \frac{x_i}{R}$$

and in matrix notation

$$\left(\frac{\partial y_j}{\partial x_i}\right) = \frac{1}{g(R)}\left[I - \frac{g'(R)}{g(R)R}\mathbf{x}\mathbf{x}^T\right]. \tag{3.8}$$

According to (2.3) the inverse of the matrix (3.8) is

$$\left(\frac{\partial y_j}{\partial x_i}\right)^{-1} = g(R)\left[I + \frac{g'}{R(g - Rg')}\mathbf{x}\mathbf{x}^T\right]$$

and the transformed equation (3.7) is

$$\frac{d\mathbf{x}}{dt} = g(R)\left[\mathbf{f}(\mathbf{y}) - \frac{g'(R)\langle \mathbf{x}, \mathbf{f}(\mathbf{y})\rangle}{R(Rg' - g)}\mathbf{x}\right]. \tag{3.9}$$

Let $\mathbf{f}(\mathbf{y})$ be again a polynomial of degree L. When $\mathbf{y}$ is replaced by $\mathbf{x}/g(R)$, and

$$\tilde{\mathbf{f}}(\mathbf{x}, 1/g) = g^L(R)\mathbf{f}(\mathbf{x}/g) = g^L \mathbf{p}_0(\mathbf{x}) + g^{L-1}\mathbf{p}_1(\mathbf{x}) + \cdots + \mathbf{p}_L(\mathbf{x}),$$

we get the analogue of (2.13),

$$\frac{d\mathbf{x}}{dt} = g^{1-L}(R)\left[\tilde{\mathbf{f}}(\mathbf{x}, 1/g) - \frac{g'(R)\langle \mathbf{x}, \tilde{\mathbf{f}}\rangle}{R(Rg'-g)}\mathbf{x}\right]. \tag{3.10}$$

The change of variable $\tau = \int^t g^{1-L}(R(t))\,dt$ leads to the renormalized equation

$$\frac{d\mathbf{x}}{d\tau} = \tilde{\mathbf{f}}(\mathbf{x}, 1/g) - \frac{g'(R)\langle \mathbf{x}, \tilde{\mathbf{f}}\rangle}{R(Rg'-g)}\mathbf{x}, \quad g = g(R). \tag{3.11}$$

Recall that (3.11) corresponds to a critical point at infinity if its right-hand side vanishes at $R = 1$. By the limit (3.6) and since $g(1) = 0$, this leads again to the nonlinear eigenvalue problem (2.17), namely $\tilde{\mathbf{f}}(\mathbf{x}, \infty) - \langle \mathbf{x}, \tilde{\mathbf{f}}\rangle\mathbf{x} = 0$.

Proposition 3.1. *If* $\|\mathbf{x}_0\| < 1$, *the solution of Eq.* (3.11) *with an initial value condition* $\mathbf{x}(0) = \mathbf{x}_0$, *can be continued for the entire infinite interval* $-\infty < \tau < \infty$ *and it stays in the open unit ball* U. *If* $\|\mathbf{x}_0\| = 1$ *and* $\mathbf{x}_0$ *is not a critical point of* (3.11), *then the whole corresponding trajectory of* (3.11) *lies in* ∂U.

Proof. Multiply both sides of (3.11) by $\mathbf{x}$. We obtain

$$-\frac{1}{2}\frac{d}{d\tau}\left(1 - R^2\right) = \mathbf{x}\frac{d\mathbf{x}}{d\tau} = \langle \mathbf{x}, \tilde{\mathbf{f}}\rangle\left[1 - \frac{g'(R)\langle \mathbf{x}, \mathbf{x}\rangle}{R(Rg'-g)}\right] = \langle \mathbf{x}, \tilde{\mathbf{f}}\rangle\frac{-g(R)}{Rg'-g}.$$

Since $g(1) = 0$, we may write $g(R) = (1 - R)h(R)$, $0 \leqslant R \leqslant 1$, with a smooth function h. Thus

$$\frac{d}{d\tau}\left(1 - R^2\right) = \left(1 - R^2\right)\frac{2\langle \mathbf{x}, \tilde{\mathbf{f}}\rangle h(R)}{(1 + R)(Rg'-g)}. \tag{3.12}$$

According to (3.5), $Rg' - g \neq 0$ for $0 \leqslant R \leqslant 1$, so the quotient $2\langle \mathbf{x}, \tilde{\mathbf{f}}\rangle h(R)/((1 + R) \times (Rg' - g))$ on the right-hand side of (3.12) is bounded from above and from below as long as $R \leqslant 1$. By integration,

$$\left(1 - R^2(\tau)\right) = \left(1 - R^2(0)\right)\exp\left(\int_0^\tau \frac{2\langle \mathbf{x}, \tilde{\mathbf{f}}\rangle h(R)}{(1 + R)(Rg'-g)}\,d\eta\right).$$

This verifies that $1 - R^2(\tau) > 0$ for every τ. It is clear from the behaviour of Eq. (3.12) that the solution $\mathbf{x}(\tau)$ does not terminate for any finite value of τ inside U.

Let now $\|\mathbf{x}_0\| = 1$. Consider now (3.12) as a differential equation for $u(\tau) = 1 - R^2(\tau)$. Since $R = (1 - u)^{1/2}$, Eq. (3.12) is smooth in u near $u = 0$, so the initial value problem $u(0) = 1 - R^2(0) = 0$ has the unique solution $u(\tau) \equiv 0$. Consequently $\|\mathbf{x}(\tau)\| \equiv 1$, and the trajectory stays on ∂U. If $\mathbf{x}_0$ is not a critical point of (3.11) and the right-hand side of (3.11) does not vanish at $\mathbf{x}_0$, we have $\mathbf{x}(\tau) \not\equiv \mathbf{x}_0$. □

Remark. The trajectories that satisfy $R(\tau) \equiv 1$ correspond to new ideal objects associated with (1.1). They could be interpreted as a manifestation of solutions $\mathbf{y}(t) \equiv \infty$ of (1.1) by virtue of (3.3) where the denominator is identically 0.

The type of a critical point at infinity in the direction of $\mathbf{x}$ is determined by the Jacobian matrix of the right-hand side of (3.11) at the corresponding point, provided that the Jacobian is not the zero matrix. Its evaluation is of interest as seen in Section 4. For a radial compactification this is easily calculated.

Proposition 3.2. *The Jacobian of* (3.11) *at* $\mathbf{x} \in \partial U$, *which corresponds to a critical point at infinity in the direction* $\mathbf{x}$, *depends only on* $\mathbf{p}_L(\mathbf{x})$ *and* $\mathbf{p}_{L-1}(\mathbf{x})$. *More precisely, its terms are*

$$J_{ij} = \frac{\partial p_{L,i}}{\partial x_j} - x_i \left\langle \mathbf{x}, \frac{\partial \mathbf{p}_L}{\partial x_j} \right\rangle - \langle \mathbf{x}, \mathbf{p}_L \rangle \delta_{ij} + g'(1)\left[p_{L-1,i} - x_i \langle \mathbf{x}, \mathbf{p}_{L-1} \rangle\right] x_j. \tag{3.13}$$

Proof. For short we write the right-hand side of (3.11) as

$$\mathbf{F}(\mathbf{x}) = \tilde{\mathbf{f}}(\mathbf{x}, 1/g) - A(R)\langle \mathbf{x}, \tilde{\mathbf{f}} \rangle \mathbf{x}, \quad A(R) = g'(R)/R(Rg' - g).$$

As $\partial R/\partial x_j = x_j/R$, the (i, j)th term of the Jacobian matrix is

$$\begin{aligned} \frac{\partial F_i}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\tilde{f}_i - A(R)\left(\sum_{k=1}^n x_k \tilde{f}_k\right) x_i\right] \\ &= \frac{\partial \tilde{f}_i}{\partial x_j} - A'(R)\frac{x_j}{R}\langle \mathbf{x}, \tilde{\mathbf{f}} \rangle x_i - A(R)\tilde{f}_j x_i - A(R)\left(\sum_{k=1}^n x_k \frac{\partial \tilde{f}_k}{\partial x_j}\right) x_i \\ &\quad - A(R)\langle \mathbf{x}, \tilde{\mathbf{f}} \rangle \delta_{ij}. \end{aligned}$$

Here

$$\begin{aligned} \frac{\partial \tilde{f}_i}{\partial x_j} &= \frac{\partial}{\partial x_j}\left[\sum_{k=0}^L g^{L-k}(R) p_{k,i}(\mathbf{x})\right] \\ &= \sum_{k=0}^L \left[(L-k) g^{L-k-1}(R) g'(R) \frac{x_j}{R} p_{k,i} + g^{L-k}(R) \frac{\partial p_{k,i}}{\partial x_j}\right]. \end{aligned}$$

At $R = 1$ we have $g(1) = 0$, so $\tilde{f}_i|_{R=1} = p_{L,i}$ and

$$\begin{aligned} \left.\frac{\partial \tilde{f}_i}{\partial x_j}\right|_{R=1} &= g'(1) x_j p_{L-1,i}(\mathbf{x}) + \frac{\partial p_{L,i}}{\partial x_j}, \\ \sum_{k=1}^n x_k \left.\frac{\partial \tilde{f}_k}{\partial x_j}\right|_{R=1} &= g'(1) x_j \langle \mathbf{x}, \tilde{\mathbf{p}}_{L-1} \rangle + \left\langle \mathbf{x}, \frac{\partial \mathbf{p}_L}{\partial x_j} \right\rangle. \end{aligned}$$

In addition $A(1) = 1$ by (3.6) and by a straightforward calculation also $A'(1) = -1$. So

$$\begin{aligned} \left.\frac{\partial F_i}{\partial x_j}\right|_{R=1} &= \left(g'(1) x_j p_{L-1,i}(\mathbf{x}) + \frac{\partial p_{L,i}}{\partial x_j}\right) - x_i \left(g'(1) x_j \langle \mathbf{x}, \tilde{\mathbf{p}}_{L-1} \rangle + \left\langle \mathbf{x}, \frac{\partial \mathbf{p}_L}{\partial x_j} \right\rangle\right) \\ &\quad + x_i \left(x_j \langle \mathbf{x}, \tilde{\mathbf{p}}_L \rangle - p_{L,j}(\mathbf{x})\right) - \langle \mathbf{x}, \tilde{\mathbf{p}}_L \rangle \delta_{ij}. \end{aligned}$$

However, by the nonlinear eigenvalue problem (2.19), $x_j \langle \mathbf{x}, \tilde{\mathbf{p}}_L \rangle - p_{L,j}(\mathbf{x}) = 0$, so precisely the required (3.13) remains.

The term J_{ij} may be written also as

$$J_{ij} = \left[\frac{\partial p_{L,i}}{\partial x_j} + g'(1) x_j p_{L-1,i} \right] - x_i \left\langle \mathbf{x}, \frac{\partial \mathbf{p}_L}{\partial x_j} + g'(1) x_j \mathbf{p}_{L-1} \right\rangle - \langle \mathbf{x}, \mathbf{p}_L \rangle \delta_{ij}. \qquad \square$$

We note that even though $\mathbf{p}$ and $-\mathbf{p}$ are both critical points of (3.11) on ∂U, the corresponding Jacobians at $\mathbf{p}$ and $-\mathbf{p}$ could be different. The Example 2.7, that is widely discussed in the literature is atypical, as in this example $\mathbf{p}_{L-1}(\mathbf{y}) \equiv 0$.

Example 3.3 *([2] Projection on a parabolic bowl).* The parabolic surface $x_{n+1} = x_1^2 + \cdots + x_n^2$ is an efficient instrument for radial compactification. With this choice the mapping (3.3) between $\mathbf{y} \in R^n$ to $\mathbf{x} \in U$ becomes

$$y_j = \frac{x_j}{1 - R^2}, \quad R^2 = \sum x_k^2, \; j = 1, \ldots, n, \tag{3.14}$$

and its inverse in $R < 1$ is given by the branch

$$x_i = \frac{2y_i}{1 + \sqrt{1 + 4 \sum y_k^2}}, \quad i = 1, \ldots, n. \tag{3.15}$$

To Eqs. (2.13) and (3.10) there corresponds

$$\frac{d\mathbf{x}}{dt} = \left(1 - R^2\right)^{1-L} \left[\tilde{\mathbf{f}}(\mathbf{x}) - \frac{2\langle \mathbf{x}, \tilde{\mathbf{f}} \rangle}{1 + R^2} \mathbf{x} \right], \tag{3.16}$$

where $\tilde{\mathbf{f}}(x_1, \ldots, x_n) = (1 - R^2)^L \mathbf{f}(x_1/(1 - R^2), \ldots, x_n/(1 - R^2))$. The change of independent variable

$$\frac{d\tau}{dt} = \frac{1}{(1 - R^2(t))^{L-1}(1 + R^2(t))} \tag{3.17}$$

or

$$\tau = \int^t \frac{dt}{(1 - R^2(t))^{L-1}(1 + R^2(t))} \tag{3.18}$$

takes (3.16) into the equation

$$\frac{d\mathbf{x}}{d\tau} = \left(1 + R^2\right) \tilde{\mathbf{f}}(\mathbf{x}) - 2\langle \mathbf{x}, \tilde{\mathbf{f}} \rangle \mathbf{x}. \tag{3.19}$$

We note that right-hand side of (3.19) is a polynomial. Now we get a simplified version of (3.12):

$$\frac{d}{d\tau}\left(1 - R^2\right) = -2\left[\left(1 + R^2\right)\langle \tilde{\mathbf{f}}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \tilde{\mathbf{f}} \rangle \langle \mathbf{x}, \mathbf{x} \rangle\right] = -2\langle \tilde{\mathbf{f}}, \mathbf{x} \rangle \left(1 - R^2\right). \tag{3.20}$$

Another family of radial compactification is obtained when we project $\mathbf{y} = (y_1, \ldots, y_n)$ into the unit sphere centered at $(0, \ldots, 0, 1)$ through the point $(0, \ldots, 0, \gamma)$, $\gamma \neq 1$, different from its center. For details, see [3]. For a family of compactifications with radial symmetry originating from certain parabolic bowls, see [2].

4. The rate of blow up

The representation of solutions of (1.1) yields quantitative information about the rate of blow up of solutions which tend to critical points at infinity. We formulate a prototype theorem.

Theorem 4.1. *Let system* (1.1) *have a critical point at infinity in the direction* $\mathbf{p}$. *Suppose that all eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of the Jacobian* (3.13) *at* $\mathbf{p}$ *have negative real parts and they do not satisfy any resonance relation* $\lambda_\nu = \sum_{i=1}^{k} m_i \lambda_i$, m_i *nonnegative integers. Then there exists a* n*-parameter family of solutions of* (1.1) *such that each of them tends to infinity in the direction of* $\mathbf{p}$ *so that*

$$\|\mathbf{y}(t)\| \sim c(t_{\max} - t)^{-1/(L-1)} \quad \text{as } t \to t_{\max}, \tag{4.1}$$

where L *is the degree of the polynomial* $\mathbf{f}(\mathbf{y})$. *Moreover, there exists another* n*-parameter family of solutions of* (1.1) *such that each of them tends to infinity in the direction of* $-\mathbf{p}$ *and they satisfy* (4.1) *as well.*

Proof. For the sake of simplicity we use here the transformation (3.14), the projection on a parabolic bowl, as a compactification. Since we want to study the behaviour of the transformed equation (3.19) at a critical point $\mathbf{p}$ on the boundary of the unit ball U, it will be useful to consider the differential equations on a complete neighborhood of $\mathbf{p}$, part of which will be in U^c, the complement of U.

Recall that (3.15) is one branch of (3.14). However (3.14) has another branch,

$$x_i = \frac{2y_i}{1 - \sqrt{1 + 4\sum y_k^2}}, \quad i = 1, \ldots, n, \tag{4.2}$$

which maps $\mathbf{y} \in R^n - \{0\}$ to $\mathbf{x} \in U^c$. It is easy to verify that (4.2) maps the neighborhood of the point at infinity in the direction $\mathbf{p}$ in the $\mathbf{y}$-space to the neighborhood of the antipodal $-\mathbf{p}$ outside of U in the $\mathbf{x}$-space.

For the mapping (4.2), Eqs. (3.16) and (3.19) remain unaltered. Hence they are defined now not only in U but for all $\mathbf{x} \in R^n$. However, as $R > 1$, $d\tau/dt$ may be now either positive or negative, according to the parity of L.

Summarizing the last statements, we see that if $\mathbf{p}$ is a critical point of Eq. (3.19), its neighborhood inside U describes the critical point of (1.1) at infinity in the direction $\mathbf{p}$, while its neighborhood outside U describes the critical point of (1.1) at infinity in the direction $-\mathbf{p}$.

Let us linearize the transformed equation (3.19) at $\mathbf{p}$:

$$\frac{d(\mathbf{x} - \mathbf{p})}{d\tau} = \frac{d\mathbf{x}}{d\tau} = A(\mathbf{x} - \mathbf{p}) + g(\mathbf{x} - \mathbf{p}),$$

i.e.,

$$\frac{d\mathbf{u}}{d\tau} = A\mathbf{u} + g(\mathbf{u}),$$

where $g(\mathbf{u}) = o(\mathbf{u})$ as $\mathbf{u} = \mathbf{x} - \mathbf{p} \to 0$. By standard results about asymptotic stability equation (3.19) has an n-parameter family of solutions such that each of them tends to $\mathbf{p}$ as $\tau \to \infty$. Moreover, by the Lyapunov expansion theorem we have

$$\mathbf{u}(\tau) = \sum_{r=1}^{\infty} \sum_{r_1+\cdots+r_k=r} \mathbf{q}^{(r)}(\tau) e^{(r_1\lambda_1+\cdots+r_k\lambda_k)\tau}. \tag{4.3}$$

Here the summation is carried out over all decompositions of r into nonnegative integer summands $r_1, \ldots, r_k$ and $\mathbf{q}^{(r)}(\tau)$ are polynomials. See [4, Theorem 67.1]. In particular, in our case when the eigenvalues do not satisfy any resonance relation, the $\mathbf{q}^{(r)}$'s are constants. For each individual solution of the n-parameter family, the representation (4.3) starts with a certain dominant exponential term. Consequently

$$\mathbf{x} - \mathbf{p} = \mathbf{u} = c_1 e^{-m\tau}\bigl(1 + o(1)\bigr) \quad \text{as } \tau \to \infty$$

with $-m \leqslant \max\{\operatorname{Re}\lambda_1, \ldots, \operatorname{Re}\lambda_n\} < 0$ and some constant c_1.

Let us turn to the quantity $1 - R^2$. Near $\mathbf{p}$, $\|\mathbf{p}\| = 1$,

$$\begin{aligned} 1 - R^2(\tau) &= 1 - \sum_{i=1}^{n} (x_i - p_i + p_i)^2 = -\sum_{i=1}^{n} \bigl[2p_i(x_i - p_i) + (x_i - p_i)^2\bigr] \\ &= c_2 e^{-\ell\tau}\bigl(1 + o(1)\bigr) \end{aligned} \tag{4.4}$$

for some $-\ell \leqslant -m < 0$. Recall that by inverting (3.18) we have for a certain initial point t_1

$$\begin{aligned} t - t_1 &= \int_0^{\tau} \bigl(1 - R^2\bigr)^{L-1}\bigl(1 + R^2\bigr)\, d\eta \\ &= \int_0^{\infty} \bigl(1 - R^2\bigr)^{L-1}\bigl(1 + R^2\bigr)\, d\eta - \int_{\tau}^{\infty} \bigl(1 - R^2\bigr)^{L-1}\bigl(1 + R^2\bigr)\, d\eta, \end{aligned}$$

and the integrals converge due to (4.4). Thus

$$t_{\max} - t = \int_{\tau}^{\infty} \bigl(1 - R^2\bigr)^{L-1}\bigl(1 + R^2\bigr)\, d\eta = c_3 e^{-(L-1)\ell\tau}\bigl(1 + o(1)\bigr).$$

Since $dt/d\tau > 0$, this relation is solvable for τ and together with (4.4) it yields that

$$1 - R^2(\tau) \sim c_4 (t_{\max} - t)^{1/(L-1)}$$

and

$$\bigl\|\mathbf{y}(t)\bigr\| = \frac{\|\mathbf{x}\|}{1 - R^2} \sim c(t_{\max} - t)^{-1/(L-1)}$$

as $t \to t_{\max}$, where c_2, c_3, c_4 and c are certain constants. □

While until here it was assumed that $n > 1$, our final remarks are about the application of compactification for one-dimensional differential equation. Consider the scalar polynomial equation

$$\frac{dy}{dt} = f(y) = a_0 + a_1 y + \cdots + a_L y^L, \quad L > 1,\ a_L \neq 0, \tag{4.5}$$

and transform it by the analogue of the mapping (3.14), $y = x/(1-x^2)$ from $y \in R$ to $x \in (-1, 1)$. The transformed equation is

$$\frac{dx}{dt} = \frac{dy}{dt} \Big/ \frac{dy}{dx} = \frac{(1-x^2)^2}{1+x^2} f(y) = \frac{(1-x^2)^2}{1+x^2} f\left(\frac{x}{1-x^2}\right) = \frac{(1-x^2)^{2-L}}{1+x^2} \tilde{f}(x), \tag{4.6}$$

with $\tilde{f}(x) = \sum_{i=0}^{L} a_i (1-x^2)^{L-i} x^i$.

The scalar equation (4.6) differs from its n-dimensional, $n > 1$, analogue (3.16) by the singular term $(1-x^2)^{2-L}$, while (3.10) and (3.16) contain the singular factors $g^{1-L}(R)$ and $(1-R^2)^{1-L}$, respectively. This is the reason that throughout this paper we assumed until now that $n > 1$.

Our approach is fruitful even in the scalar case. A change of variables, analogous to (3.19),

$$\frac{dt}{d\tau} = \left(1-x^2\right)^{L-2}\left(1+x^2\right)$$

takes (4.6) into

$$\frac{dx}{d\tau} = \tilde{f}(x) \tag{4.7}$$

with a continuous right-hand side for $-1 \leqslant x \leqslant 1$. According to the definition of critical points at infinity (4.5) has no critical points at $y = +\infty$ and at $y = -\infty$ since $\tilde{f}(\pm 1) = \pm a_L \neq 0$. A solution of (4.5) at $y = \infty$ (or $y = -\infty$) corresponds to a solution of (4.7) with initial value condition $x(0) = 1$ (or $x(0) = -1$). Recall that near $x = 1$ Eq. (4.7) is $dx/d\tau = \tilde{f}(x) = a_L(1 + O(x-1))$. Once the solution $x(\tau)$ of this initial value problem is available, it is translated back to the original variable t.

The rate of blow up of solutions of the scalar equation (4.5) is compatible with the n-dimensional case (4.1). As $y \to +\infty$, we have $f(y) > 0$ for $y \geqslant y_0$ and

$$t - t_1 = \int_{y_0}^{y} \frac{d\eta}{f(\eta)} = d - \int_{y}^{\infty} \frac{d\eta}{f(\eta)}.$$

Since $f(y) \sim a_L y^L$, $L > 1$, the last integral exist and

$$t_{\max} - t = \int_{y}^{\infty} \frac{d\eta}{f(\eta)} \sim c_1 y^{-L+1}.$$

Consequently $y(t)$ blows up at the rate $y(t) \sim c(t_{\max} - t)^{-1/(L-1)}$.

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